EECS 16B Summer 2020 Final (Form 1)

Instructions

You have 180 minutes. You may access printed, handwritten, and pre-downloaded materials during this exam. You may not consult the Internet or other human beings during this exam. You may not share the contents of this exam by any medium, verbal or digital, at any time before it is posted on the course website.

Justify your answers. A correct result without justification will not receive full credit.

Honor Code

COPY, SIGN, and DATE the UC Berkeley Honor Code by hand. (*If you fail to do so, your exam may not be accepted.* Email the instructors if you are unable.)

> *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*

....

COPY, SIGN, and DATE the following addendum by hand. (*If you fail to do so, your exam may not be accepted.* Email the instructors if you are unable.)

> *Should I commit academic misconduct during this exam, let me receive a failing grade in EECS 16B or dismissal from the University.*

....

1 Abstract singular values

Let $M \in \mathbb{C}^{N\times N}$ be a matrix that satisfies $M^N = I$ and $M^*M = MM^* = I$. Let $\omega = e^{2\pi j/N}$.

a) Show that the eigenvalues of M are powers of ω.[1]

b) Next we will compute the SVD of $A = M - M^2$. Show that $A^*A = 2I - M - M^{N-1}$.

[1] A typo in the original exam was corrected.

c) Show that the eigenvalues of A^*A are $2\left(1 - \cos\left(2\pi n/N\right)\right)$, $n = 0, 1, \ldots, N-1$. Conclude that they are nonnegative.

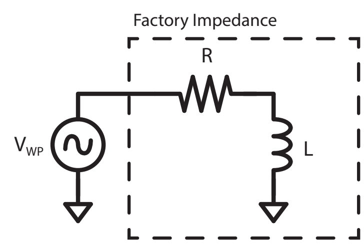
d) What is a minimal singular value of A? What does this say about the rank of A?

e) If N is even, what is a maximal singular value of A?

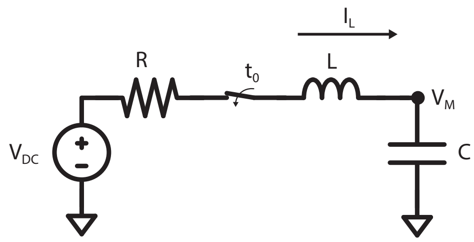
2 Factory Power

A factory has several very large motors with large inductance. The factory has noticed that their power bill is high and they would like to lower it.

a) The factory's power draw is more efficient if their load impedance is purely real at f_{wp}. Let us model the factory's impedance load as shown below. The wall power has frequency $f_{wp} = 60Hz$, $L = 20\,\text{H}$, and $R = 200\,\Omega$. Add a useful device, either in series or in parallel, to their impedance load and solve for it's optimum value.



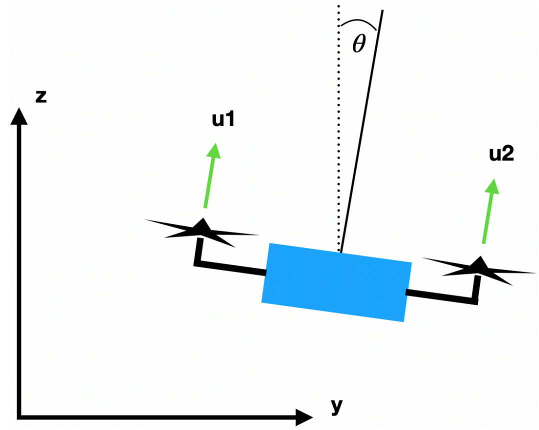
b) The factory has a machine, with a DC power source, that flickers when it turns on.



Write the differential equation for motor voltage $V_M(t)$ in terms of L, C, and R that describes the given circuit in form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_\mathrm{M} \\ I_\mathrm{L} \end{bmatrix} = A \begin{bmatrix} V_\mathrm{M} \\ I_\mathrm{L} \end{bmatrix} + b$$

3 Controlling a Quadrotor to Hover



In this problem you will design a controller which will make a planar quadrotor hover. The quadrotor we will consider is defined by the following state space model:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v_y}(t) \\ \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{z}(t) \\ \dot{v_z}(t) \end{bmatrix} = \begin{bmatrix} v_y(t) \\ \frac{sin(\theta(t))}{m}(u_1(t)+u_2(t)) \\ \omega(t) \\ \alpha(u_1(t)-u_2(t)) \\ v_z(t) \\ \frac{cos(\theta(t))}{m}(u_1(t)+u_2(t))-g \end{bmatrix}$$

Here $y(t)$ denotes lateral position, $z(t)$ the altitude, $v_y(t)$ and $v_z(t)$ the corresponding linear velocities, $\theta(t)$ the roll angle, and $\omega(t)$ the angular velocity. The parameters α and m are positive, real constants. The controls $u_1(t)$ and $u_2(t)$ are the thrusts generated by the left and right propellors.

The thrust of each propellor can be positive or negative.

Define the vectors

$$x(t) := \begin{bmatrix} y(t) \\ v_y(t) \\ \theta(t) \\ \omega(t) \\ z(t) \\ v_z(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

a) An equilibrium point for this system is given by

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \\ 0 \end{bmatrix}, \quad u^* = \begin{bmatrix} \frac{mg}{2} \\ \frac{mg}{2} \end{bmatrix}$$

Here $h > 0$ is a specified altitude.

Do there exist any other equilibrium points for this system which satisfy $y^* = 0$ and $z^* = h$? If so, what are they? If not, explain why not.

b) Consider a linearization of this system, formed by taking the first-order taylor approximation of the system about the equililbrium point given in part (a). This linearized system is given by

$$\dot{\delta x}(t) = A\delta x(t) + B\delta u(t)$$

where $\delta x(t) = (x(t) - x^*)$, and $\delta u(t) = (u(t) - u^*)$. The matrices A and B are given by

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \beta_3 & -\beta_3 \\ 0 & 0 \\ \beta_4 & \beta_4 \end{bmatrix}$$

Find the parameters $\beta_1, \beta_2, \beta_3, \beta_4$.

c) Is the linearized system found in part (b) stable? Explain your answer. Hint: notice that A is an upper-triangular matrix.

d) Does the matrix B have full column-rank? Explain your answer. Here you can use the fact $\beta_3 \neq 0$ and $\beta_4 \neq 0$.

e) Consider the matrix $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. Is C full row-rank? What does this imply about the ability or inability to choose arbitrary closed-loop eigenvalues for this system through use of feedback control? Explain your answer.

f) Define a control law for this system of the form $\delta u(t) := -K\delta x(t)$, where K is defined as

$$K := \begin{bmatrix} 0 & 0 & 0 & \frac{k_1}{2} & 0 & \frac{k_2}{2} \\ 0 & 0 & 0 & -\frac{k_1}{2} & 0 & \frac{k_2}{2} \end{bmatrix}.$$

Find the constants k_1 and k_2 in terms of the parameters β_3 and β_4 so that two of the eigenvalues of the closed-loop system are equal to -1.

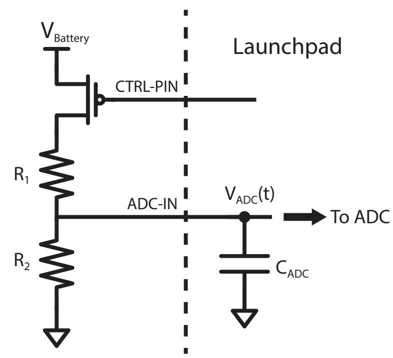
g) Is the closed-loop system found in part(f) stable? Explain your answer. Describe in words how the closed-loop system would respond to the initial condition

$$\delta x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}.$$

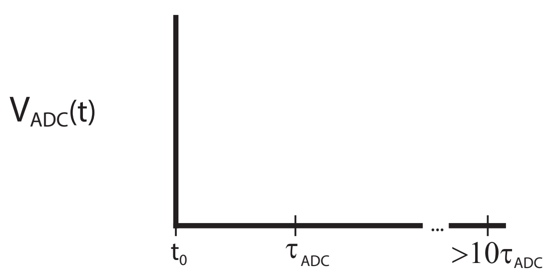
4 Analog to Digital Converter

An ADC (analog to digital converter) is used to convert an analog signal from the real world, to a digital one that a computer can interpret and act on. Let's examine a couple of ADC uses.

a) You'd like to monitor the battery on your 16B car, so that you know when it's low. Your TA gives you the following switched, voltage-divider circuit. C_{ADC} is used by the launchpad's ADC to read and save the input voltage and PMOS threshold voltage $V_{TH} = V_{Battery}/2$.

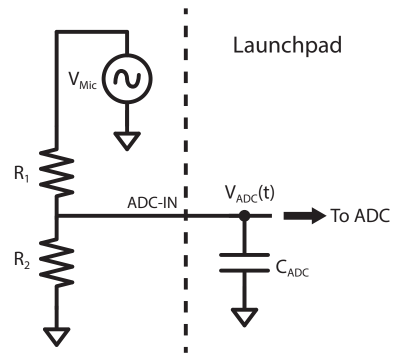


i) $CTRL - PIN$ is switched to GND at $t_0 = 0$, after a long period of being at $V_{Battery}$. Draw $V_{ADC}(t)$. Indicate values on the y-axis in terms of $V_{Battery}$ for each indicated x-axis value for the circuit's time constant τ_{ADC}.



ii) We would like to estimate the battery voltage as quickly as possible to keep from wasting power. You are given that $\tau_{ADC} = C_{ADC}(R_1||R_2)$. If the ADC samples at $t_{sample} = 1 \times 10^{-3}$ and $V_{ADC}(t_{sample}) = 2.1\,\text{V}$, what is the battery voltage $V_{Battery}$? You may use any of the values $C_{ADC} = 100 \times 10^{-9}\,\text{F}$, $R_1 = 20 \times 10^3\,\Omega$, $R_2 = 10 \times 10^3\,\Omega$, PMOS switch resistance $R_p = 1\,\Omega$, and PMOS gate capacitance $C_p = 1 \times 10^{-12}\,\text{F}$.

b) You want to try a fancier microphone than the mic-board to record your audio data with. But the new microphone outputs signals from 0 to 9V! Your lab mate suggests you use the circuit you used in part a to monitor the battery circuit.



i) Symbolically find the factored form transfer function $H(j\omega) = \frac{V_{ADC}}{V_{Mic}}$. What is the corner frequency ω_c of this transfer function in terms of symbols R_1, R_2, and C_{ADC}?

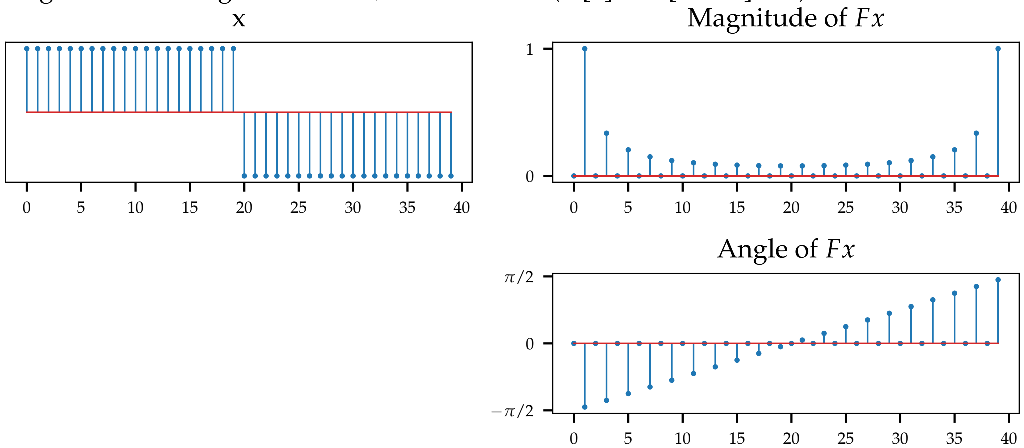
ii) If the highest frequency a human voice can produce is 200Hz, what frequency must the ADC sample at to accurately record your voice?

iii) Your air conditioner is loud and produces an audio tone at $f_{AC} = 500Hz$. You need to attenuate f_{AC} by at least $\frac{1}{10}$ without significantly attenuating your voice. For the values $C_{ADC} = 100 \times 10^{-9}$ F, $R_1 = 20 \times 10^3\ \Omega$, $R_2 = 10 \times 10^3\ \Omega$, show that $H(j\omega)$ from part $b.i.$ is insufficient to meet these requirements.

iv) Propose a new $H'(j\omega)$ in factored form that would meet the requirements in part b.iii. Specify the frequency of the the poles or zeros you use. You do not need to design a schematic to implement the filter.

5 Convolution limit

The vector $x \in \mathbb{R}^{40}$ is a sample of a square wave, as shown below. Shown also are the magnitude and angle of $X = Fx$, the DFT of x. ($X[1] = X[N-1] = 1$)



Let x^r denote the rth convolution of x with itself:

$$x^1 = x, \quad x^2 = x * x, \quad \ldots, \quad x^r = x \underbrace{* x * \ldots * x}_{r-1 \text{ times}}.$$

a) Give a formula for $F(x^r)[k]$, the kth DFT coefficient of x^r, in terms of $X[k]$.

b) Let x^∞ be the normalized limit of x^{4r}:

$$x^\infty = \lim_{r\to\infty} x^{4r}.$$

$X^\infty = F(x^\infty)$ has only two nonzero components, and they are equal. What are they?

c) Argue that x^{∞} has the shape of a sinusoid.