Midterm

Exam Location: In Person

PRINT your student ID: ______________________________

PRINT AND SIGN your name: ______________, ______________ ______________
(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ______________

Row Number: ______________ Seat Number: ______________

Name and SID of the person to your left: ______________________________

Name and SID of the person to your right: ______________________________

Name and SID of the person in front of you: ______________________________

Name and SID of the person behind you: ______________________________

1. **Honor Code (0 pts.)**

We treat all our students with utmost trust and respect, and expect students to return the same trust and respect. In EECS16B we will have ***zero-tolerance*** for academic dishonesty. There will be ***dire consequences*** for students that violate that trust and the Berkeley code of conduct. Both professors are committed to enforcing academic honesty, and ***dishonesty cases will be punished in their fullest -- no excuses or special circumstances will be considered***. Always seek help, never cheat.

I acknowledge: ______________________________ (signature)

Solution: Signature should be present.

2. **What is your favorite topic of the course so far? (0 pts.)**

Solution: Any answer is sufficient.

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

3. Circuit Potpourri (12 pts.)

(a) (3 pts.) Suppose we have voltage $v(t) = V_0 \sin(\omega t)$. Let $\widetilde{V}$ be its phasor representation. **Mark all that are true.**

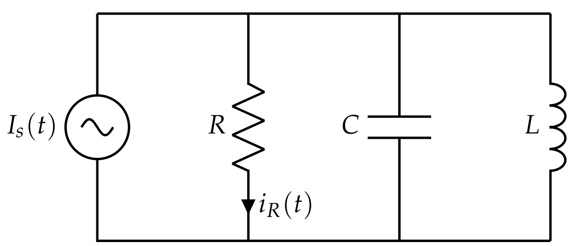
- ☐ $\widetilde{V}$ is a real number.
- ☐ $\widetilde{V}$ is a purely imaginary number.
- ☐ $\mathrm{j}\widetilde{V}\mathrm{e}^{-3}$ would be a valid phasor.

Solution:

- ☐ $\widetilde{V}$ is a real number.
- ■ $\widetilde{V}$ is a purely imaginary number.
- ■ $\mathrm{j}\widetilde{V}\mathrm{e}^{-3}$ would be a valid phasor.

Convert $v(t)$ to its phasor representation. $v(t) = V_0 \sin(\omega t) = V_0 \cos\left(\omega t - \frac{\pi}{2}\right)$ implies that $\widetilde{V} = V_0 e^{-\mathrm{j}\frac{\pi}{2}} = -V_0\mathrm{j}$ which is purely imaginary. The third option essentially scales the magnitude of the phasor by e^{-3} and adds a phase of $\pi/2$ since $j = \mathrm{e}^{\mathrm{j}\pi/2}$, which is a valid phasor, namely, $\mathrm{j}\widetilde{V}\mathrm{e}^{-3} = V_0\mathrm{e}^{-3}$ which is still a valid phasor even though its purely real.

(b) (3 pts.) Suppose we have the following circuit.



Let $I_s(t) = \cos(\omega_0 t)$ where $\omega_0 = \frac{1}{\sqrt{LC}}$. **What is the value of $\widetilde{I}_R$ (the phasor associated with $i_R(t)$)?**

○ 0	○ $\widetilde{I}_s$
○ $\frac{R}{R+(Z_C \| Z_L)}\widetilde{I}_s$	○ $\frac{Z_C+Z_L}{R+Z_C+Z_L}\widetilde{I}_s$

Solution:

○ 0	● $\widetilde{I}_s$
○ $\frac{R}{R+(Z_C \| Z_L)}\widetilde{I}_s$	○ $\frac{Z_C+Z_L}{R+Z_C+Z_L}\widetilde{I}_s$

This is a parallel RLC circuit and its resonant frequency is $\frac{1}{\sqrt{LC}}$. Since the input current is a sinusoid operating at the resonant frequency, the capacitor and inductor in parallel behave as an open circuit in frequency/phasor domain because

$$Z_L || Z_C = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{L/C}{0} = \infty$$

Therefore, the only path that the current can take is through the resistor, which means that $\widetilde{I}_R = \widetilde{I}_S$. Note that options 3 and 4 are incorrect expressions of the current divider. The correct current divider is:

$$\widetilde{I}_R = \frac{Z_C || Z_L}{R + (Z_C || Z_L)}\widetilde{I}_s$$

If we simplify this expression by diving through by $Z_C||Z_L$ we get:

$$\widetilde{I}_R = \frac{1}{\frac{R}{Z_C||Z_L}+1}\widetilde{I}_s = \frac{1}{\frac{R*(Z_L+Z_C)}{Z_C*Z_L}+1}\widetilde{I}_s$$

And since we operate at the resonant frequency,

$$\widetilde{I}_R = \frac{1}{\frac{R*(0)}{L/C}+1}\widetilde{I}_s = \widetilde{I}_s$$

(c) (3 pts.) Suppose you have a 3 bit SAR-ADC with input range $[0,1)$ and $V_{\text{REF}} = 1V$, and that the input analog signal is $v_{\text{in}} = 0.6V$. **Which of the following best describes the relationship between the digitized state of the signal and the analog signal?**

- ◯ The digitized state exactly represents the analog signal.
- ◯ The digitized state under-estimates the analog signal.
- ◯ The digitized state over-estimates the analog signal.
- ◯ None of the above

Solution:

- ◯ The digitized state exactly represents the analog signal.
- ● The digitized state under-estimates the analog signal.
- ◯ The digitized state over-estimates the analog signal.
- ◯ None of the above

A 3 bit SAR-ADC with reference voltage $1V$ can represent the following analog signals exactly:

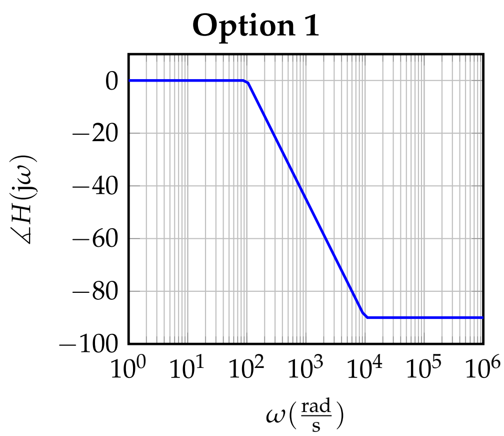
$$\frac{n}{8} * 1V \text{ where } n \in \{0,1,...,7\}$$

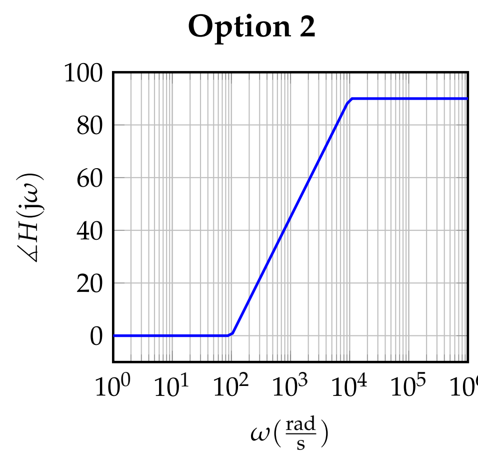
Otherwise, the SAR-ADC algorithm will always underestimate the input signal. Since $0.6V$ is between $0.5V$ and $0.625V$, the digitized state of the signal is underestimated.

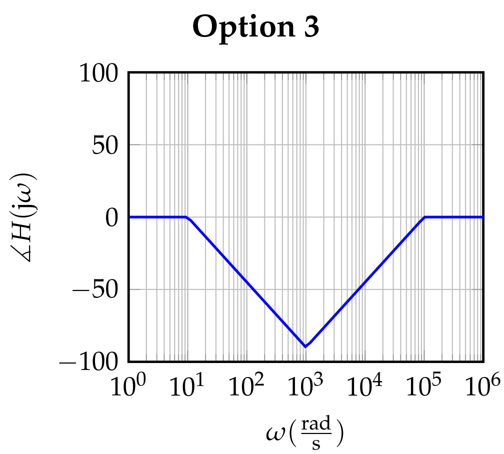
(d) (3 pts.) We are told that the transfer function of some circuit is equal to:

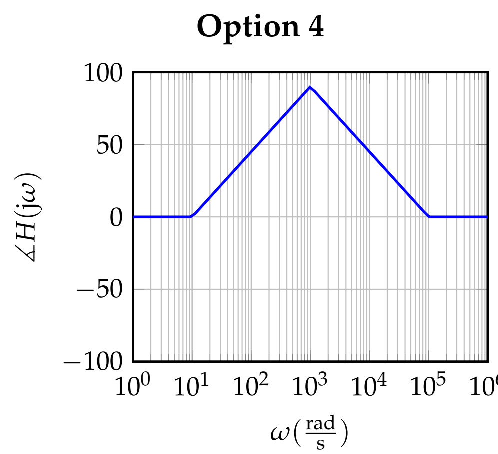
$$H(j\omega) = \frac{1+j\frac{\omega}{10^2}}{1+j\frac{\omega}{10^4}} \tag{1}$$

Which of the following graphs is the phase Bode plot of this transfer function?









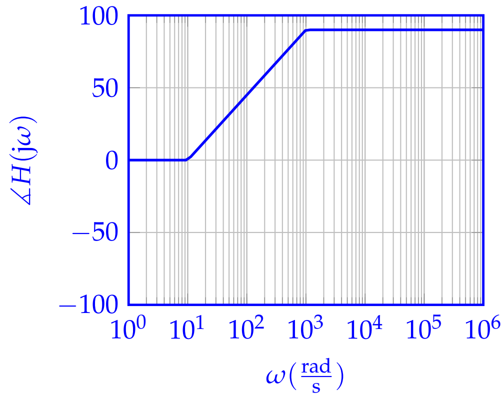
| ◯ Option 1 | ◯ Option 2 | ◯ Option 3 | ◯ Option 4 |

Solution:

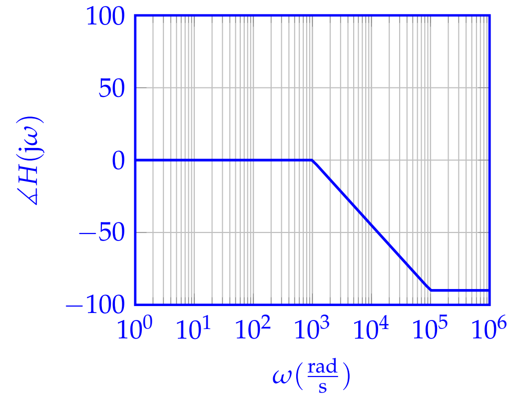
| ◯ Option 1 | ◯ Option 2 | ◯ Option 3 | • Option 4 |

Using the pole and zero frequencies method, this transfer function has one zero frequency at $\omega_z = 10^2$ and one pole frequency at $\omega_p = 10^4$. Thus, the overall phase bode plot is the sum of these two phase bode plots. The bode plot for the zero frequency starts at zero phase, then slopes upward at $\omega = 10$ until reaching $90°$ at $\omega = 10^3$ passing through the point $(10^2, 45°)$, and stays at $90°$. The bode plot for the pole frequency starts at zero phase, then slopes downward at $\omega = 10^3$ until reaching $-90°$ at $\omega = 10^5$ passing through the point $(10^4, -45°)$, and stays at $-90°$. If we sum these two plots, we get the fourth plot.

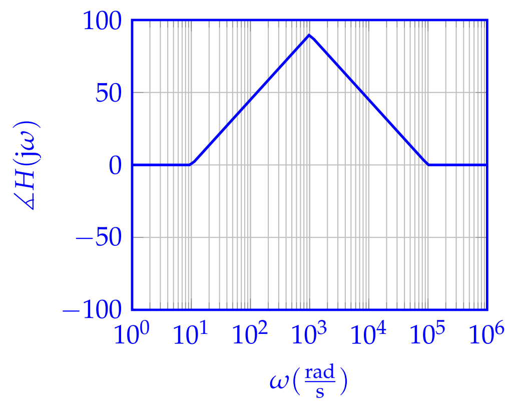
Bode plot for zero frequency



Bode plot for pole frequency

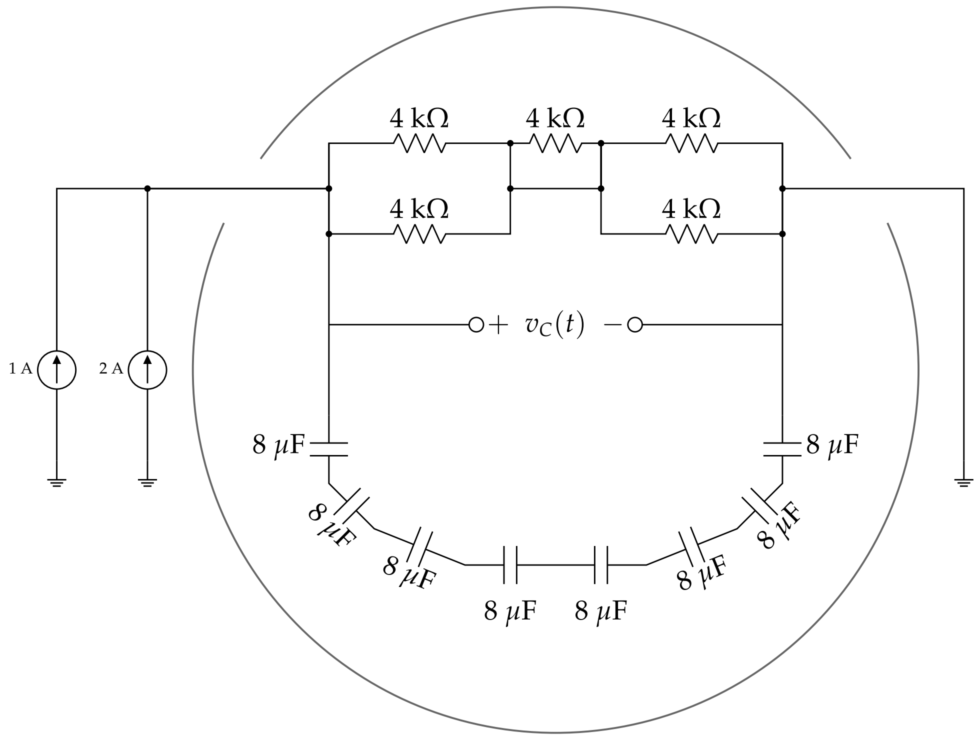


Overall

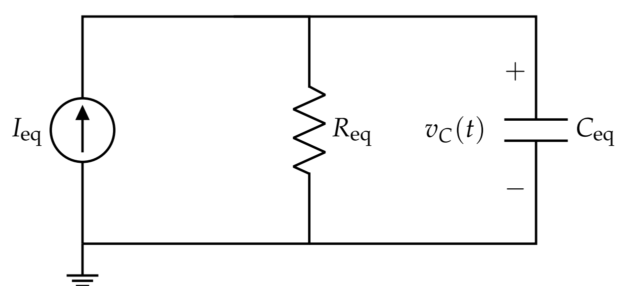


4. The exam is smiling at you. Be happy! (18 pts.)

Keep calm and smile on! Recall *Divide et impera* ("divide and conquer").



The above circuit can be simplified to the circuit below:



(a) (6 pts.) **Find the numerical values of I_{eq}, R_{eq}, and C_{eq} for the equivalent model above.**

Solution: $I_{eq} = 3A$.
The series parallel of the R branch of the circuit is $R_{eq} = 4k\Omega$.
The series of capacitors has capacitance, $C_{eq} = 1\mu F$.
$\tau = 4k\Omega \cdot 1\mu F = 4\ ms$.

(b) (6 pts.) The differential equation for $v_C(t)$ can be written in the following form:

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + av_C(t) = u(t)$$

Find expressions for a and $u(t)$ in terms of I_{eq}, R_{eq}, and C_{eq}.

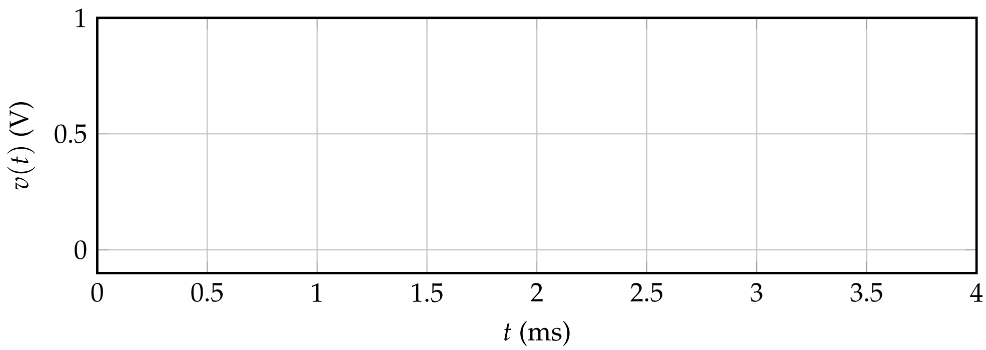
Solution: Using KCL, $I_{eq} = i_R + i_C$.
Using the capacitor law and Ohm law. $I_{eq} = \frac{v_C}{R_{eq}} + C_{eq}\frac{dv_C}{dt}$.

$$\frac{dv_C}{dt} + \frac{v_C}{R_{eq}C_{eq}} = \frac{I_{eq}}{C_{eq}}$$

. Given the previous formulation:
$a = \frac{1}{R_{eq}C_{eq}}$, and $u(t) = \frac{I_{eq}}{C_{eq}}$

(c) (6 pts.) Suppose that $I_{\text{eq}} = 1$ mA, $R_{\text{eq}} = 1$ kΩ, and $C_{\text{eq}} = 1$ μF, and that the capacitor is initially discharged at $t = 0$ (in other words, $v_C(0) = 0$).

Qualitatively plot the waveform of $v_C(t)$. The plot does not need to be exact, but should indicate knowledge of the initial condition, steady state value, and time constant associated with this circuit.

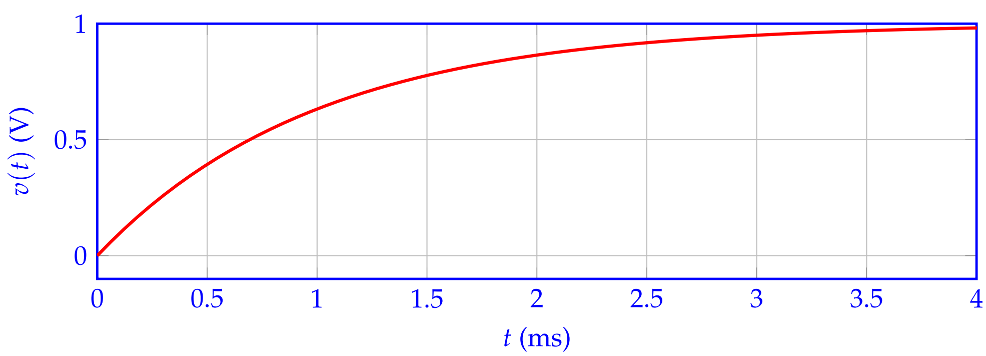


Solution:

There are multiple ways of approaching the qualitative plot. The simplest way is to notice that the steady state solution of v_C is actually $v_C = I_{eq}R_{eq} = 1mA1k\Omega = 1V$ since the capacitor behaves as open circuit. The time constant of the circuit is $\tau = R_{eq}C_{eq} = 1ms$. The initial conditions are given in the exercise, meaning $v_C(t = 0) = 0V$. Instead of rederiving the full solution we can recall that is an exponential like function:

$$v_C(t) = [1 - exp(\frac{-t}{\tau})]V = 1 - exp(\frac{-t}{1ms})V$$

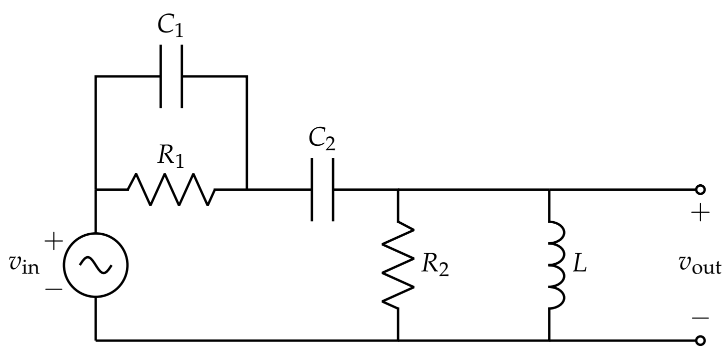
It is relevant that the student understand that the curve does not reach the steady state rapidly. Since the time constant is just $1ms$.



5. Karaoke (20 pts.)

Professor Ana Arias is building a microphone for her homemade karaoke system. She wants to create a filter to attenuate noisy frequencies so the mic only picks up her voice. She decides to build the following circuit.

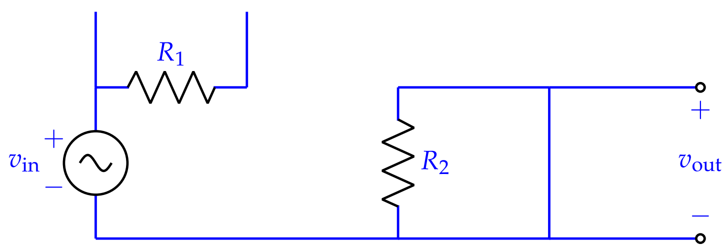
Circuit 1:



(a) (6 pts.) **For Circuit 1: Find the value of the transfer function** $H_1(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ ***only* at** $\underline{\omega = 0}$ ("hint hint").

Solution: For $\omega = 0$ (DC), the inductor has impedance $j(0)L = 0$ and thus acts like a short circuit. The capacitors have impedance $\frac{1}{j(0)C} = \infty$ and thus acts like open circuits.

Here is the equivalent circuit at $\omega = 0$:

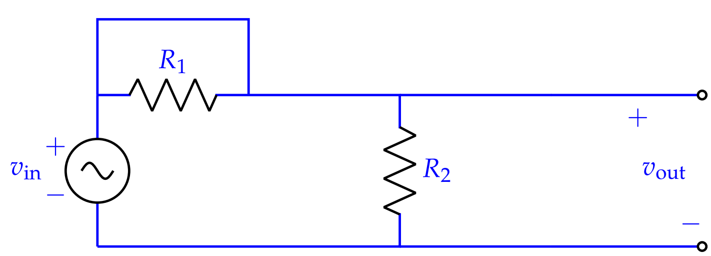


From this equivalent circuit, we can see that $\widetilde{V}_{\text{out}} = 0$. Thus, $H_1(j0) = 0$.

(b) (6 pts.) **For Circuit 1: Find the value of the transfer function** $H_1(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ ***only* at** $\underline{\omega \to \infty}$ ("hint hint").

Solution: For $\omega = \infty$, the inductor has impedance $j(\infty)L = \infty$ and thus acts like an open circuit. The capacitors have impedance $\frac{1}{j(\infty)C} = 0$ and thus acts like short circuits.

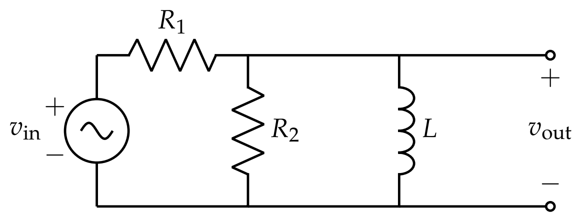
Here is the equivalent circuit at $\omega = \infty$:



From this equivalent circuit, we can see that $\widetilde{V}_{\text{out}} = \widetilde{V}_{\text{in}}$. Thus, $H_1(\text{j}\infty) = 1$.

Professor Arias decides to remove the capacitors and is thus left with the following circuit.

Circuit 2:



(c) (8 pts.) Given that the transfer function for Circuit 2 is of the form

$$H_2(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = K \frac{\text{j}\omega}{1 + \text{j}\frac{\omega}{\omega_c}} \tag{2}$$

where $K \in \mathbb{R}$, **find the values of K and ω_c in terms of the circuit quantities R_1, R_2, and L.**

Solution: We can find the equivalent impedance of R_2 and L_1 using the parallel operator and utilize the voltage divider topology to get

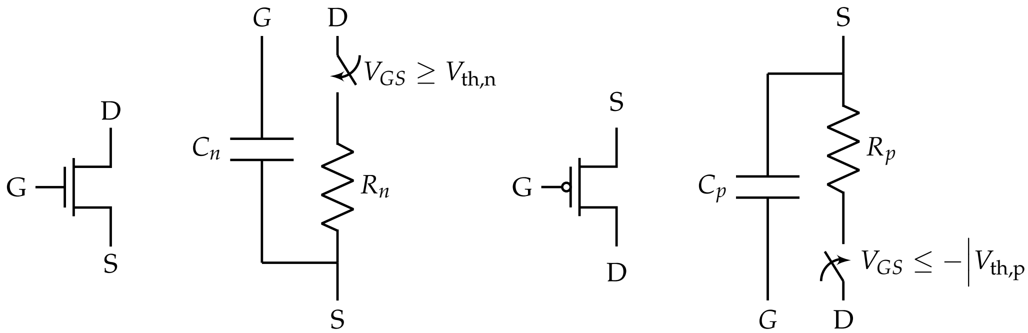
$$\begin{aligned}
H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} &= \frac{R_2||\text{j}\omega L}{R_1 + R_2||\text{j}\omega L} \\
&= \frac{\frac{\text{j}\omega L R_2}{\text{j}\omega L + R_2}}{R_1 + \frac{\text{j}\omega L R_2}{\text{j}\omega L + R_2}} \\
&= \frac{j\omega L R_2}{R_1(j\omega L + R_2) + j\omega L R_2} \\
&= \frac{j\omega L R_2}{R_1 R_2 + j\omega L(R_1 + R_2)} \\
&= \frac{L}{R_1} \frac{\text{j}\omega}{1 + \text{j}\omega\frac{L(R_1+R_2)}{R_1 R_2}}
\end{aligned}$$

Pattern matching to the form given in the question, we can see that $K = \frac{L}{R_1}$ and $\omega_c = \frac{R_1 R_2}{L(R_1+R_2)}$.

6. Transistor Time (10 pts.)

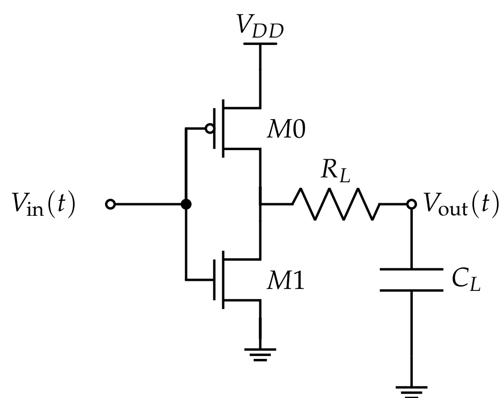
For this problem, assume that V_{DD} is greater than the NMOS threshold $V_{\text{th,n}}$ and the PMOS threshold $|V_{\text{th,p}}|$.

Also assume that we model transistors with the RC models below.



(a) NMOS Resistor-Capacitor Model **(b)** PMOS Resistor-Capacitor Model

Consider the following circuit:



Suppose we have the following input:

$$V_{\text{in}}(t) = \begin{cases} 0 & t < 0 \\ V_{DD} & t \geq 0 \end{cases}$$

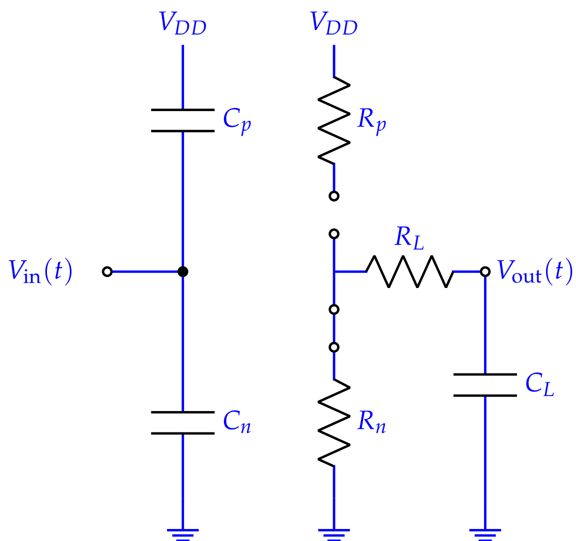
See the next page for questions.

(a) (5 pts.) **Draw the equivalent RC circuit model for $t \geq 0$** with all resistors and capacitors included (specify which switches are open and closed in your equivalent diagram, as shown below):

(a) Open Switch

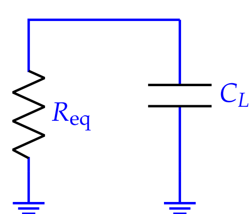
(b) Closed Switch

Solution: The equivalent circuit is shown below. Since $V_{in} = V_{DD}$, the $M1$ switch is closed and the $M0$ switch is open.



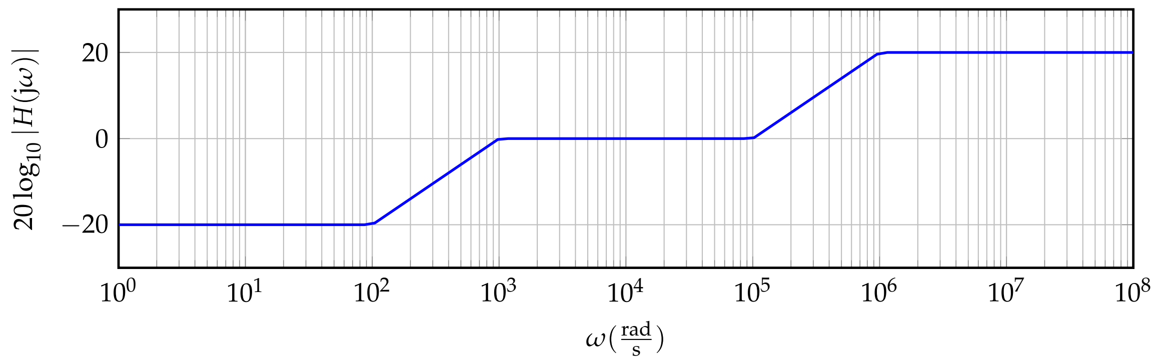
(b) (5 pts.) Let $C_p = C_n = 5\,\mu\text{F}$, $R_p = 20\,\Omega$, $R_n = 10\,\Omega$, $R_L = 40\,\Omega$, and $C_L = 100\,\mu\text{F}$.
With these values, and for the same input $V_{\text{in}}(t)$, **calculate the RC time constant τ of the circuit for $t \geq 0$.**

Solution: The active part of the circuit from part (a) is shown below, where the two resistors in series are combined into one $R_{\text{eq}} = R_n + R_L$. This is a basic RC circuit, and the time constant is given by $\tau = RC = R_{\text{eq}}C_L = 50\,\Omega \times 100\,\mu\text{F} = 5\,\text{ms}$.



7. Bode Plots (20 pts.)

For this question, we will analyze the provided magnitude Bode plot.



(a) (5 pts.) **What are the pole and zero frequencies of this Bode plot?** Make sure to specify which frequencies are pole frequencies and which frequencies are zero frequencies.

Solution: For the magnitude Bode plot, the pole frequencies are where the slope decreases by 20 $\frac{\text{dB}}{\text{dec}}$, while the zero frequencies are where the slope increases by 20 $\frac{\text{dB}}{\text{dec}}$.

Based on the plot, the slope starts off at 0 $\frac{\text{dB}}{\text{dec}}$, increases to +20 $\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^2\ \frac{\text{rad}}{\text{s}}$, decreases to 0 $\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^3\ \frac{\text{rad}}{\text{s}}$, increases to +20 $\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^5\ \frac{\text{rad}}{\text{s}}$, and then finally decreases to 0 $\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^6\ \frac{\text{rad}}{\text{s}}$.

Thus, the pole and zero frequencies are:

- Pole Frequencies: $\omega_{p1} = 10^3\ \frac{\text{rad}}{\text{s}}$, $\omega_{p2} = 10^6\ \frac{\text{rad}}{\text{s}}$
- Zero Frequencies: $\omega_{z1} = 10^2\ \frac{\text{rad}}{\text{s}}$, $\omega_{z2} = 10^5\ \frac{\text{rad}}{\text{s}}$

(b) (5 pts.) **Write a possible transfer function** $H(j\omega)$ **that would produce this Bode plot** (there is more than one possible answer, you only need to write down one possible answer).

Solution: We can use our knowledge of poles and zeros to directly correspond each of them to factors of the transfer function in the numerator (for zeros) and denominator (for poles), where each factor takes the form of $1 \pm j\frac{\omega}{\omega_{zp}}$ with ω_{zp} as the zero or pole frequency (you can choose either plus or minus for each factor as this will not impact the magnitude Bode plot).

We also know we will need a factor of $\pm\frac{1}{10}$ in front because the Bode plot starts at −20 dB or $\frac{1}{10}$ and there are no poles or zeros at $\omega = 0$ based on the flat slope the plot starts with.

Thus, the transfer function will be:

$$H(j\omega) = \pm\frac{1}{10}\frac{(1 \pm j\frac{\omega}{10^2})(1 \pm j\frac{\omega}{10^5})}{(1 \pm j\frac{\omega}{10^3})(1 \pm j\frac{\omega}{10^6})} \tag{3}$$

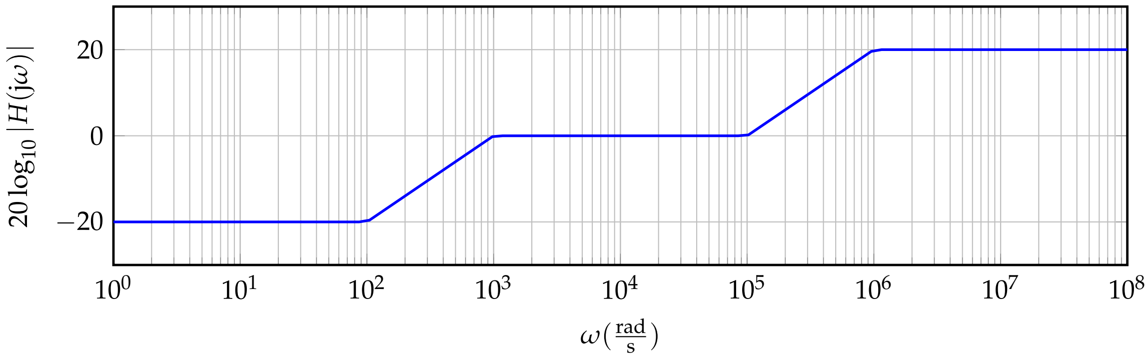
(c) (10 pts.) Suppose that the input voltage into the system with transfer function $H(j\omega)$ (represented by the provided Bode plot) is:

$$v_{\text{in}}(t) = 1 + 2\cos\left(10^4 t + \frac{\pi}{3}\right) + 3\cos\left(10^8 t\right) \tag{4}$$

You are provided that $\measuredangle H(j0) = \measuredangle H(j10^4) = \measuredangle H(j10^8) = 0$.

Find $v_{\text{out}}(t)$ for the input voltage $v_{\text{in}}(t)$.

The plot is provided once more for convenience:



Solution: Remember that for an input sinusoidal function $V_0\cos(\omega t+\phi)$, the associated output will be $|H(j\omega)|V_0\cos(\omega t+\phi+\measuredangle H(j\omega))$.

We will use superposition to calculate the output associated with each term of the input.

- For $v_{\text{in},1}(t)=1$, the frequency is $\omega_1=0$ (DC). The transfer function value for this frequency is $|H(j0)|=-20\text{ dB}=\frac{1}{10}$. Thus, the output will be:

$$v_{\text{out},1}(t)=|H(j0)|(1)=0.1 \tag{5}$$

- For $v_{\text{in},2}(t)=2\cos(10^4t+\frac{\pi}{3})$, the frequency is $\omega_2=10^4$. The transfer function value for this frequency is $|H(j10^4)|=0\text{ dB}=1$. Thus, the output will be:

$$v_{\text{out},2}(t)=|H(j10^4)|2\cos\left(10^4t+\frac{\pi}{3}+\measuredangle H(j10^4)\right)=2\cos\left(10^4t+\frac{\pi}{3}\right) \tag{6}$$

- For $v_{\text{in},3}(t)=3\cos(10^8t)$, the frequency is $\omega_3=10^8$. The transfer function value for this frequency is $|H(j10^8)|=20\text{ dB}=10$. Thus, the output will be:

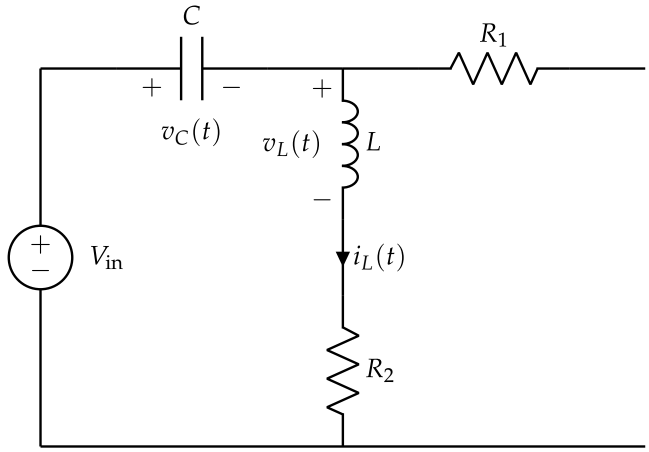
$$v_{\text{out},3}(t)=|H(j10^8)|3\cos\left(10^8t+\measuredangle H(j10^8)\right)=30\cos\left(10^8t\right) \tag{7}$$

Thus, the total output is:

$$v_{\text{out}}(t)=v_{\text{out},1}(t)+v_{\text{out},2}(t)+v_{\text{out},3}(t)=0.1+2\cos\left(10^4t+\frac{\pi}{3}\right)+30\cos\left(10^8t\right) \tag{8}$$

8. 2nd Order Superposition (20 pts.)

Suppose we have the following circuit:



(a) (8 pts.) The differential equation for $i_L(t)$ can be written in the following form:

$$\frac{\mathrm{d}^2 i_L(t)}{\mathrm{d}t^2} + 2\alpha \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + \omega_0^2 i_L(t) = u(t)$$

Solve for the constants α, ω_0, and $u(t)$ in terms of R_1, R_2, L, C, V_{in}.

Solution:

We realize that after zeroing out the current source, our circuit is a series RLC!

Using KVL:

$$\begin{aligned} v_L(t) + v_C(t) + v_R(t) &= v_{\text{in}}(t) \\ L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + v_C(t) + i_{R_2}(t)R_2 &= v_{\text{in}}(t) \\ L\frac{\mathrm{d}^2 i_L(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + R_2\frac{\mathrm{d}i_{R_2}(t)}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}^2 i_L(t)}{\mathrm{d}t^2} + \frac{R_2}{L}\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + \frac{1}{LC}i_L(t) &= 0 \end{aligned}$$

where $i_{R_2}(t) = i_L(t)$ as they are in series.

Thus $\alpha = \frac{R_2}{2L}$, $\omega_0 = \frac{1}{\sqrt{LC}}$, and $u(t) = 0$.

(b) (3 pts.) Now, suppose that you are told $\alpha = \frac{3}{2}$, $\omega_0 = \sqrt{2}$, and $u(t) = 0$. Also, suppose that the following information about the circuit is true:

$$i_L(0) = 3\,\text{A},\ v_L(0) = 1\,\text{V},\ v_C(0) = 0\,\text{V},\ R_1 = R_2 = 6\,\Omega,\ L = 2\,\text{H},\ C = 0.25\,\text{F}$$

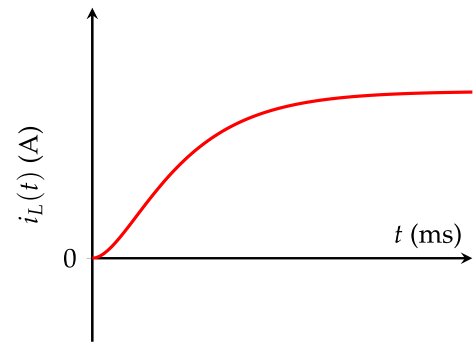
Find $\left.\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}\right|_{t=0}$ (one of the initial conditions necessary for solving $i_L(t)$).

Solution:

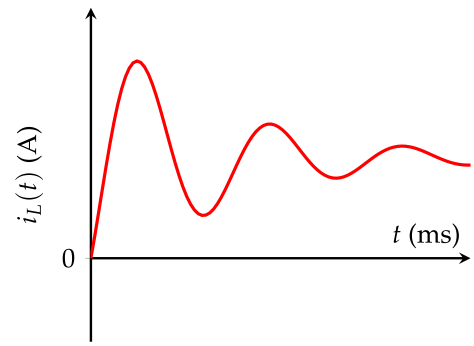
We are already told that $i_L(0) = 3$. Since $v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$, $\left.\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}\right|_{t=0} = \frac{1}{L}v_L(0) = 0.5$.

(c) (5 pts.)

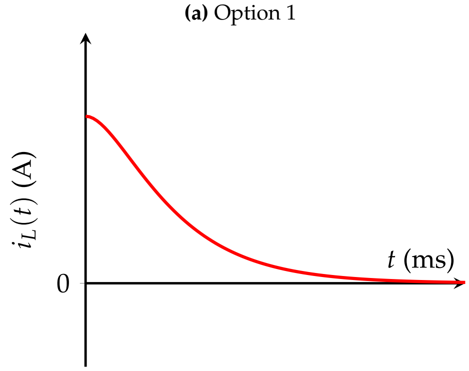
Using the values from the previous part, **state the option below that best describes** $i_L(t)$ *and* **briefly justify your choice.**



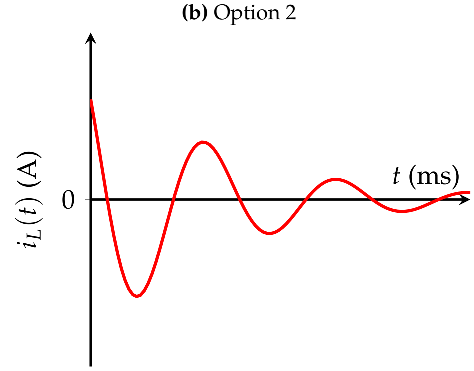
(a) Option 1



(b) Option 2



(c) Option 3



(d) Option 4

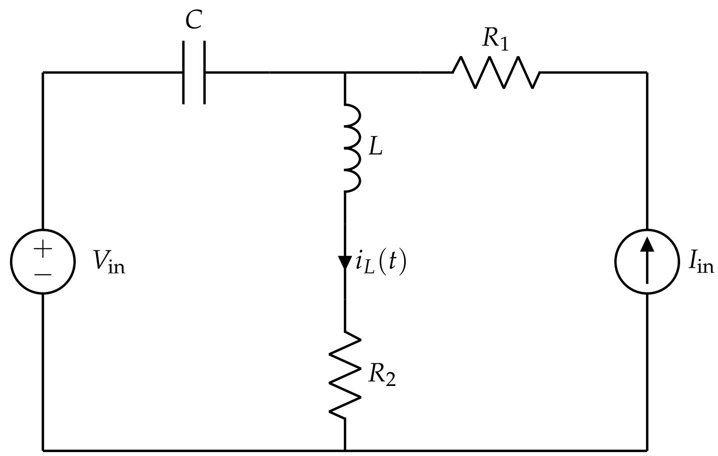
Solution: To determine the correct plot, we first have to find the damping ratio $\zeta = \frac{\alpha}{\omega_0}$.

$$\alpha = \frac{R_2}{2L} = 1.5 \tag{9}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{2} \tag{10}$$

Thus, $\zeta = \frac{3}{2\sqrt{2}} > 1$. This means the behavior is overdamped and we also know the initial condition to be $i_L(0) = 3A$. Thus, our answer is **Option 3**.

(d) (4 pts.) Now, we add a current source to the circuit.



What is the steady state current $i_{L,SS}$ (the current through the inductor as $t \to \infty$) of this circuit? Express in terms of I_{in}, V_{in}, L, C, R_1, R_2. (Note that V_{in} and I_{in} are DC sources.)

Solution: We proceed via superposition. First, we consider the case of zeroing out the current source.

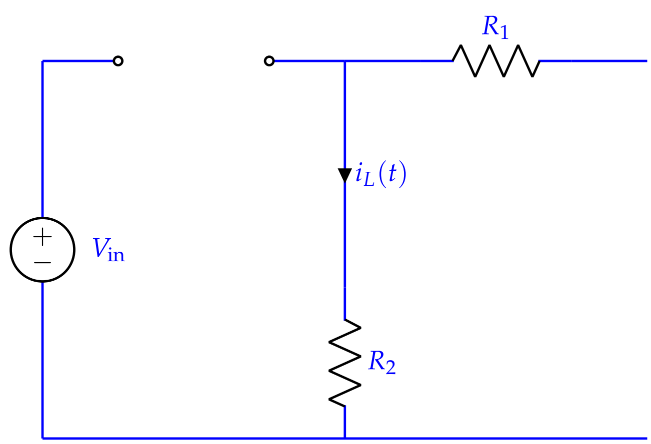


Figure 6: Circuit diagram with the current source replaced by an open circuit.

Since a capacitor is an open circuit and an inductor is a wire in steady-state, we see that the $i_{L,SS} = 0$ when the current source is zeroed out.

Now, we zero the voltage source, replacing it with a short.

Again, if the capacitor is an open circuit and the inductor is a short in steady-state, the current through the inductor in steady-state is fully determined by the current source.

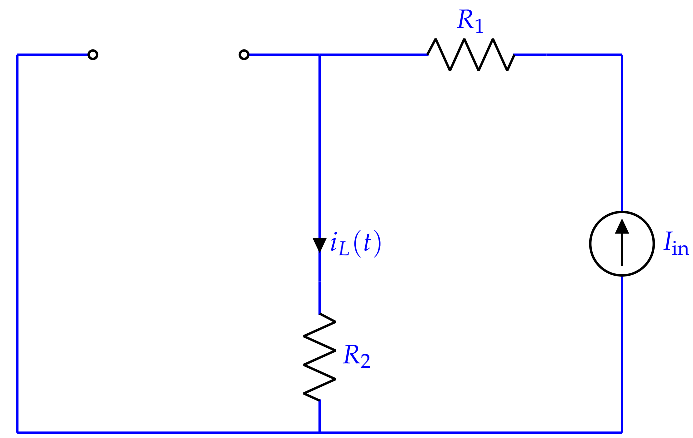


Figure 7: Circuit diagram with the voltage source replaced by a short.

This tells us that $i_{L,\text{SS}} = I_{\text{in}}$ when the voltage source is zeroed out.

Thus, our final answer is the sum of our two steady-state currents (i.e. $i_{L,\text{SS}} = 0 + I_{\text{in}} = I_{\text{in}}$).