Exam Location: Draft

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ________________

Row Number: ________________ Seat Number: ________________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

2. **What are you planning to do during your summer break? (2 pts.)**

3. **What is the happiest moment of your semester so far? (2 pts.)**

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

4. Orthonormality, Gram-Schmidt, and SVD Potpourri (21 pts.)

(a) (2 pts.) Suppose $Q \in \mathbb{R}^{m \times n}$ is tall (i.e. $m \geq n$) matrix and has orthonormal columns, which of the following is true:

◯	$Q^\top Q = I_n$
◯	$QQ^\top = I_m$
◯	neither
◯	both

(b) (2 pts.) Suppose $Q \in \mathbb{R}^{n \times n}$ is a square, orthonormal matrix, which of the following is true:

◯	$Q^\top Q = I_n$
◯	$QQ^\top = I_n$
◯	neither
◯	both

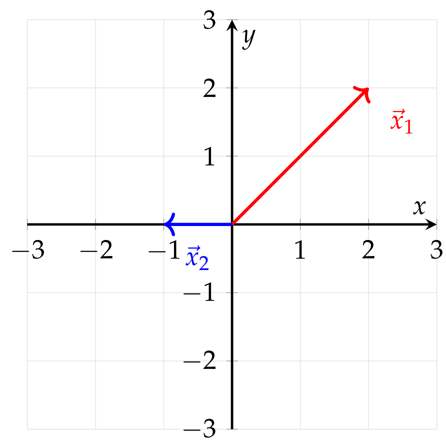
(c) (2 pts.) Suppose $Q \in \mathbb{R}^{m \times n}$ is a wide (i.e. $m \leq n$) matrix and has orthonormal columns, which of the following is true:

◯	$Q^\top Q = I_n$
◯	$QQ^\top = I_m$
◯	neither
◯	both

(d) (5 pts.) **Using Gram-Schmidt, find an orthonormal basis for $\mathbb{R}^2$ starting with the vectors $\vec{q}_1 = \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}$ and $\vec{q}_2 = \begin{bmatrix} -\sqrt{3} \\ 3 \end{bmatrix}$. Let $\vec{v}_1$ be the corresponding orthonormal basis vector for $\vec{q}_1$ and $\vec{v}_2$ be the corresponding orthonormal basis vector for $\vec{q}_2$.**

Suppose $\vec{v}_1$ and $\vec{v}_2$ are actually the columns of $V^\top$ from the SVD of some arbitrary matrix A (i.e. $A = U\Sigma V^\top$). In the next several problem parts, we will evaluate how this matrix A transforms two vectors $\vec{x}_1$ and $\vec{x}_2$.

(e) (5 pts.) The following is a graph of $\vec{x}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.



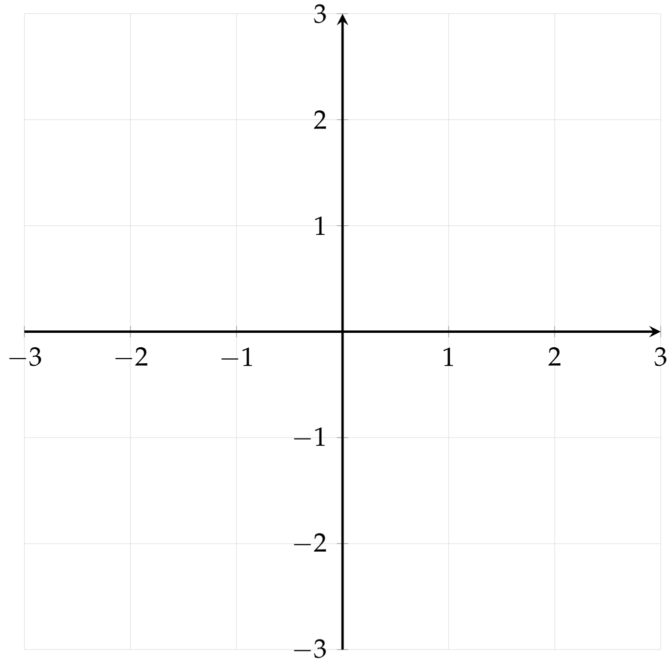
As a first step, draw $V^\top \vec{x}_1$ and $V^\top \vec{x}_2$ on the following graph.

Use the fact that the rotation matrix for a counterclockwise rotation of θ (in degrees or radians) is given by $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

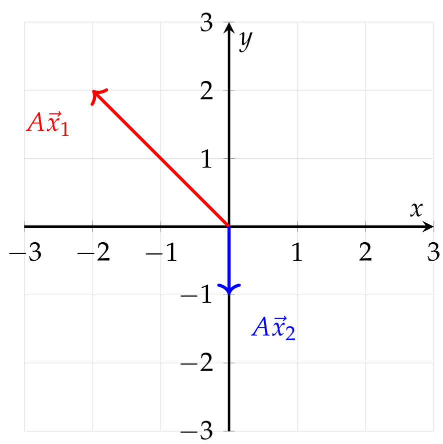
The following trigonometric values may also be of use:

	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$

A blank plot is provided for you on the next page. *Note: In your graphs, indicate the angle of your vectors with respect to the x-axis.*



(f) (5 pts.) The following graph represent how the A-matrix affects the vectors $\vec{x}_1$ and $\vec{x}_2$ where $A\vec{x}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $A\vec{x}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

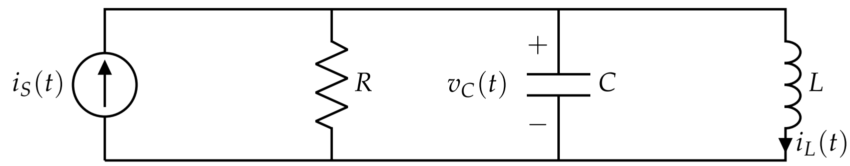


Find Σ and U.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

5. Critically Damped Parallel RLC Circuit (15 pts.)

In this problem, we will use vector differential equations to analyze the following circuit:



This circuit can have different behavior depending on the values of the elements. In this case, we will examine the critically damped case, where the state variables ($v_C(t)$ and $i_L(t)$) decay over time without oscillation, just like the overdamped case, but with repeated eigenvalues, which could result in a system that cannot be diagonalized.

(a) (6 pts.) **Solve for** A **and** $\vec{b}$ in terms of R, L, and C such that the differential equation for $\vec{x}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ can be written in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}i_S(t) \tag{1}$$

(b) (3 pts.) Suppose the system from the previous part has a repeated eigenvalue with only one eigenvector and is thus not diagonalizable. We will use upper triangularization to solve it.

You are provided an orthonormal change of basis matrix U such that $\vec{x}(t) = U\vec{\widetilde{x}}(t)$ which allows for upper triangularization of the vector differential equation with $\vec{x}$.

Solve for $\widetilde{A}$ **and** $\vec{\widetilde{b}}$ in terms of A, U, and $\vec{b}$ such that the differential equation for $\vec{\widetilde{x}}(t) = \begin{bmatrix} \widetilde{v}_C(t) \\ \widetilde{i}_L(t) \end{bmatrix}$ can be written in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t) + \vec{\widetilde{b}}i_S(t) \tag{2}$$

such that $\widetilde{A}$ is an upper triangular matrix. **You are NOT allowed to use U^{-1} in your answer.**

For the rest of the problem, we will let $i_S(t) = 0$.

(c) (3 pts.) Regardless of your answers to the previous parts, assume that we are able to reach the following vector differential equation (again, with $i_S(t) = 0$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \begin{bmatrix} -5 & -10 \\ 0 & -5 \end{bmatrix}\vec{\widetilde{x}}(t) \tag{3}$$

Solve for $\widetilde{i}_L(t)$ with the initial condition $\widetilde{i}_L(0) = 2$. Remember that $\vec{\widetilde{x}}(t) = \begin{bmatrix} \widetilde{v}_C(t) \\ \widetilde{i}_L(t) \end{bmatrix}$.

(d) (3 pts.) Using your answer from the previous part, **set up a first order differential equation for** $\widetilde{v}_C(t)$. You DO NOT have to solve it.

6. Using PCA to Detect Fraudulent Transactions (15 pts.)

PCA has many different uses when applied to real-world data. One potential application is making classification of data much easier.

Suppose we are given some data, where each datapoint represents a transaction. Each one is labeled either normal or fraudulent. We will utilize PCA to develop a useful classifier.

We plot the data in two dimensions, where each dimension is some unspecified feature that will aid us in classifying the points:

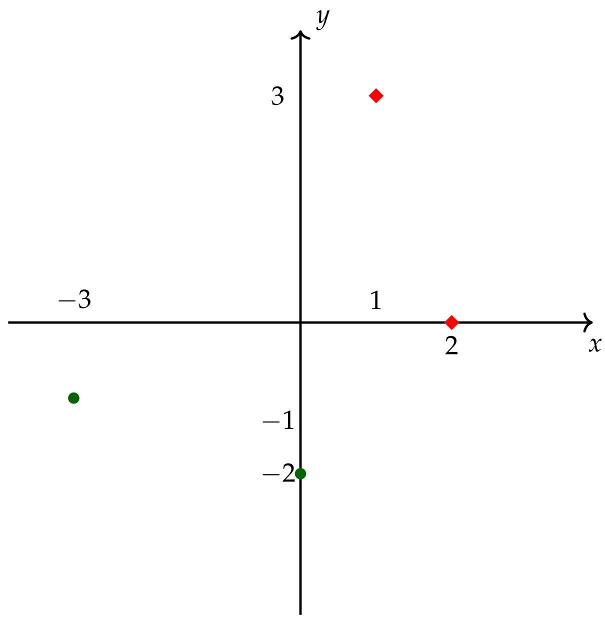


Figure 1: Plot of Transactions in 2-D

Thus, we have a total of 4 transactions, $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ are normal, while $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are fraudulent.

(a) (3 pts.) Suppose we now construct a data matrix, where the *data points are columns*.

$$X = \begin{bmatrix} -3 & 0 & 2 & 1 \\ -1 & -2 & 0 & 3 \end{bmatrix} \tag{4}$$

Using this data matrix, **calculate its first principal component $\vec{u}_1$.**

(HINT:

i. You may also make use of the fact that XX^T is given by:

$$XX^\top = \begin{bmatrix} -3 & 0 & 2 & 1 \\ -1 & -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & 0 & 2 & 1 \\ -1 & -2 & 0 & 3 \end{bmatrix}^\top \tag{5}$$

$$= \begin{bmatrix} 14 & 6 \\ 6 & 14 \end{bmatrix} \tag{6}$$

ii. *You may also make use of the characteristic polynomial of XX^T:*

$$\lambda^2 - 28\lambda + 160 = (\lambda - 20)(\lambda - 8) = 0 \tag{7}$$

)

(HINT: Remember that your principal component should be of unit norm.)

(b) (3 pts.) It's difficult to come up with a useful classifier in two dimensions. Let's use PCA dimensionality reduction to help.

Using your answer in part (a), **project your two-dimensional data points onto one dimension. Express your answer as the vector $\vec{z} \in \mathbb{R}^{1\times 4}$.**

(c) (2 pts.) **Now, plot each of these points, on the line below. Indicate the value and label (normal as circle and fraudulent as diamond) for each point.**

Note: your plot doesn't have to be to scale.

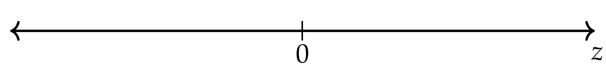


Figure 2: Plot of Transactions in 1-D

(d) (1 pts.) Suppose you are given some transaction datapoint, $\vec{x}_i$ and you project it to be one-dimensional, i.e. z_i. **Based on your plot from part (c), come up with an inequality in terms of z_i to identify if that transaction is fraudulent.**

Note: There can be more than one answer, but please only give one.

(e) (3 pts.) It can often be informative to compare our original data with its PCA reconstructions. **Using PCA, reconstruct $\vec{z}$ back into 2-D as the matrix $\widetilde{X} \in \mathbb{R}^{2\times 4}$.**

(f) (3 pts.) Finally, we visualize our PCA reconstruction. **On the graph below, draw your first principal component direction (extend it as a solid line in both directions). Draw your reconstructioned points from $\widetilde{X}$, with dotted lines connecting them with the corresponding point in X.** You may draw on the plot on the next page.

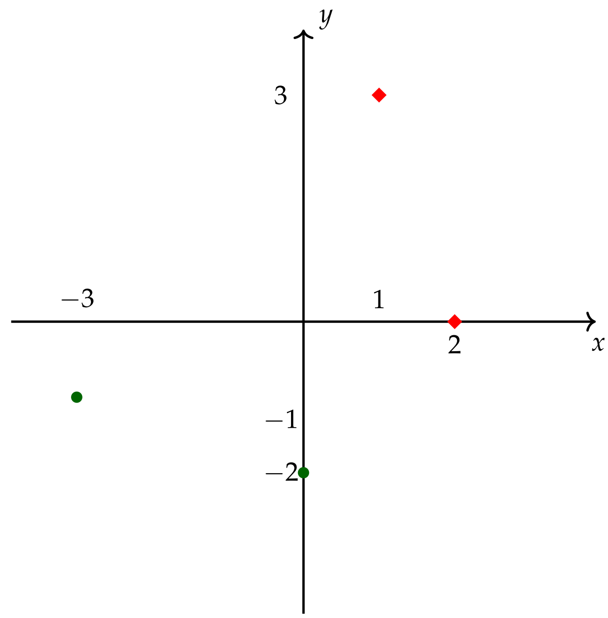
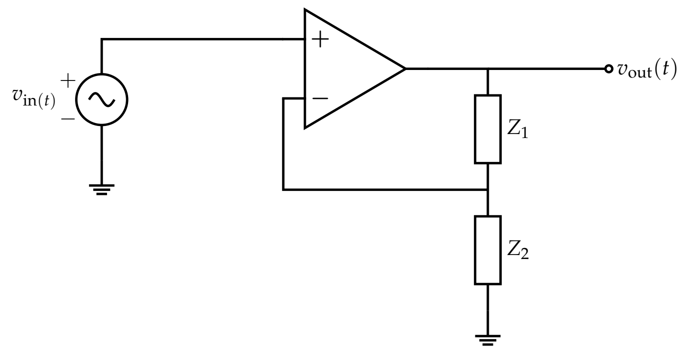


Figure 3: Plot of Transactions in 2-D

7. Op-Amp Transfer Function (25 pts.)

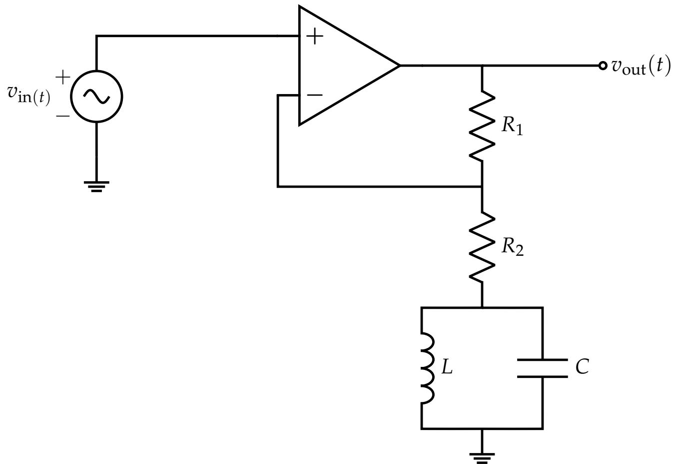
In this problem, we will analyze the transfer function of a non-inverting amplifier implemented with an op-amp. For the entire problem, you can assume that all of the op-amp Golden Rules apply.

(a) (7 pts.) Consider the following op-amp non-inverting amplifier, with generic impedances Z_1 and Z_2 present on the feedback loop:



Show that the transfer function for the circuit above is $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = 1 + \frac{Z_1}{Z_2}$.

For the remaining parts, we will consider the following non-inverting amplifier, which is a specific instance of the amplifier from part (a).



(b) (4 pts.) **Draw an equivalent circuit if the input frequency is $\omega = 0$.** Then, using the equation for $H(\mathrm{j}\omega)$ from part (a), **find a simplified expression for** $H(\mathrm{j}0)$ in terms of R_1, R_2, L, and C.

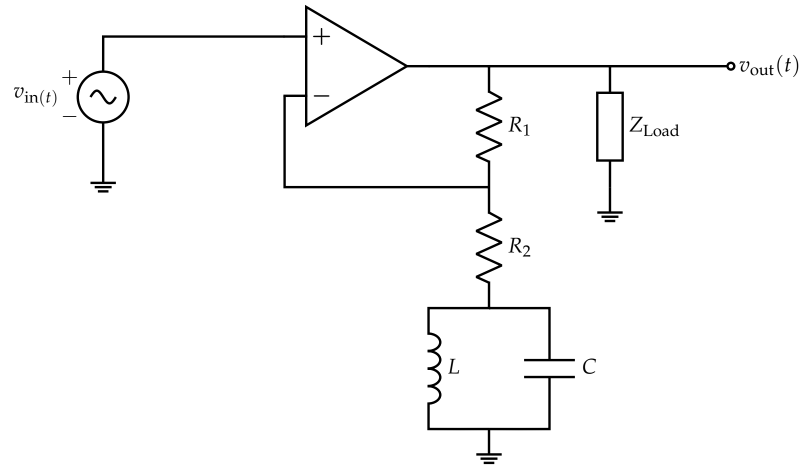
(HINT: To draw the equivalent circuit, think about the impedance of inductors and capacitors at $\omega = 0$ (DC).)

(c) (4 pts.) **Draw an equivalent circuit if the input frequency is $\omega = \infty$.** Then, the equation for $H(\mathrm{j}\omega)$ from part (a), **find a simplified expression for $H(\mathrm{j}\infty)$** in terms of R_1, R_2, L, and C.

(d) (2 pts.) **Find an expression for ω_r, the resonant frequency**, in terms of R_1, R_2, L, and C. For this circuit, the resonant frequency is defined as the frequency at which the equivalent impedance of the inductor and capacitor is $Z_{\text{eq}} = \infty$.

(e) (2 pts.) Now that we have found the resonant frequency, let's analyze the circuit for $\omega = \omega_r$. **Draw an equivalent circuit if the input frequency is $\omega = \omega_r$.** Then, using the equation for $H(\mathrm{j}\omega)$ from part (a), **find a simplified expression for $H(\mathrm{j}\omega_r)$** in terms of R_1, R_2, L, and C.

(f) (6 pts.) Suppose we have a load impedance $Z_{\text{Load}} = |Z_{\text{Load}}|e^{j\frac{\pi}{4}}$ at the output of this circuit:



Suppose the input phasor is $\widetilde{V}_{\text{in}}$, and the input is at a cutoff frequency ω_c. **Find the average power P_{avg} delivered to the load impedance Z_{Load}** in terms of $|\widetilde{V}_{\text{in}}|$, $|H(j\omega)|_{\text{max}}$ (the maximum magnitude of the transfer function), and the provided element variables.

(HINT: You DO NOT need to solve for $H(j\omega)$. Recall that $|H(j\omega_c)| = \frac{|H(j\omega)|_{\text{max}}}{\sqrt{2}}$.)

8. Controllability (10 pts.)

(a) (3 pts.) **Is the following, discrete-time system controllable? Explain your answer.**

$$\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \tag{8}$$

(b) (3 pts.) **Is the following, discrete-time system controllable? Explain your answer.**

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \vec{u}[i] \tag{9}$$

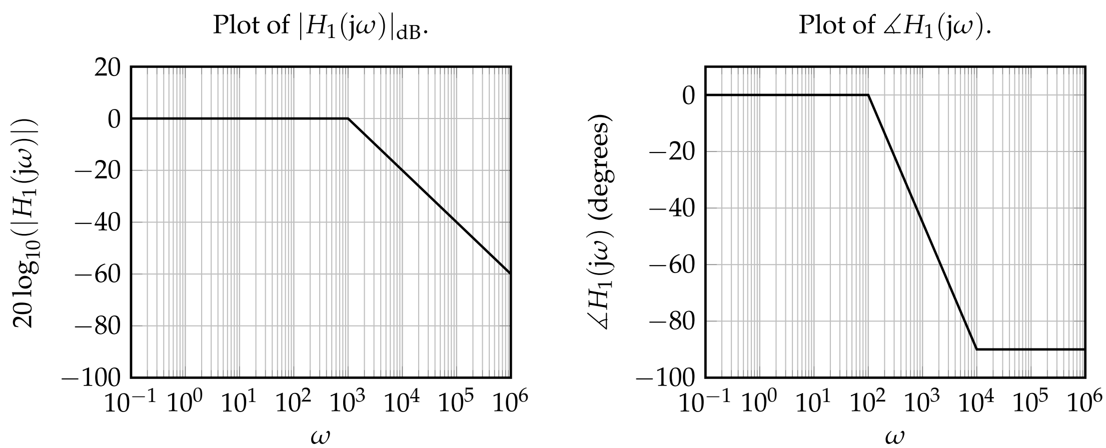
(c) (4 pts.) **Prove or disprove: if the matrix A in the discrete-time system $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$ is full rank, then the system is controllable.** Assume $\vec{b} \neq \vec{0}$. If you believe the statement is TRUE, then prove it by showing that the controllability matrix is always full rank for any full-rank matrix $A \in \mathbb{R}^{n \times n}$. If you believe the statement is FALSE, give a counterexample with a specific matrix A and $\vec{b}$ that produces a rank deficient controllability matrix.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

9. Bode Plot Analysis (20 pts.)

In this problem, we will analyze various Bode plots and implement them.

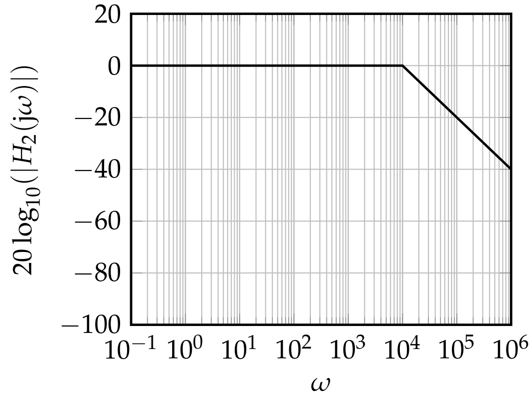
(a) (2 pts.) Here are the Bode magnitude and phase plots (straight line approximations) for transfer function $H_1(j\omega)$:



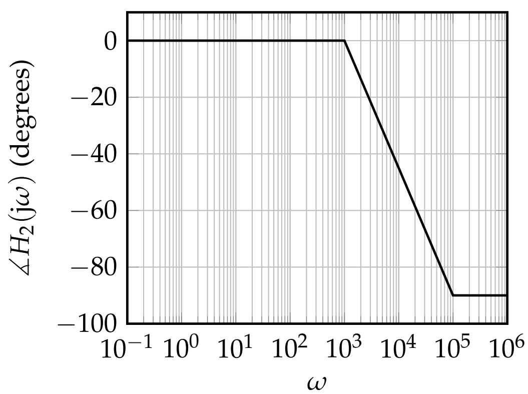
Find an expression for $H_1(j\omega)$. What type of filter is H_1?

(b) (2 pts.) Here are the Bode magnitude and phase plots (straight line approximations) for transfer function $H_2(j\omega)$:

Plot of $|H_2(j\omega)|_{\text{dB}}$.



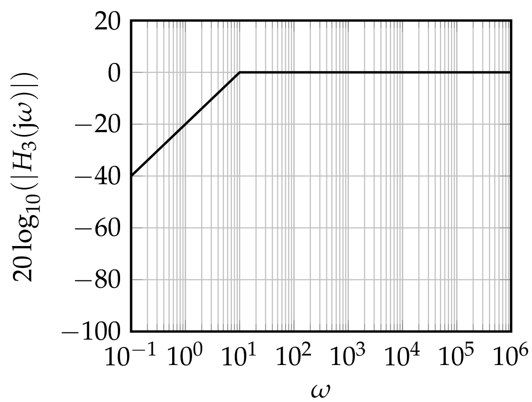
Plot of $\measuredangle H_2(j\omega)$.



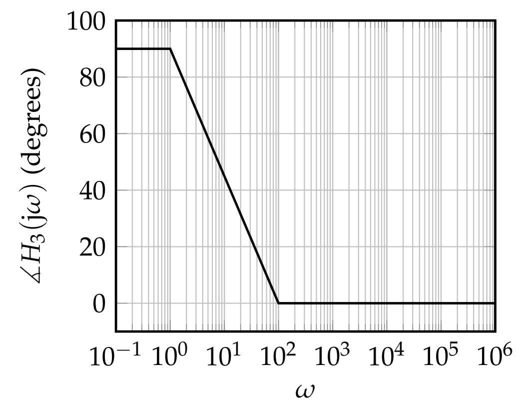
Find an expression for $H_2(j\omega)$. What type of filter is H_2?

(c) (2 pts.) Here are the Bode magnitude and phase plots (straight line approximations) for transfer function $H_3(j\omega)$:

Plot of $|H_3(j\omega)|_{\text{dB}}$.



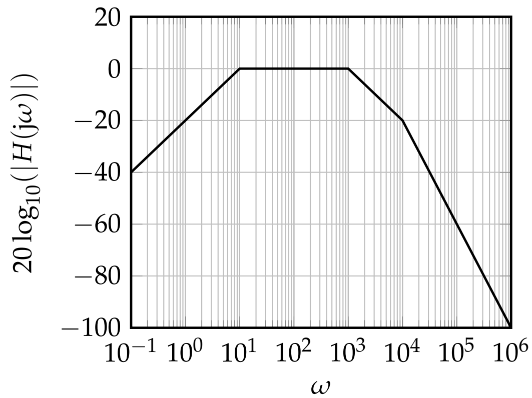
Plot of $\measuredangle H_3(j\omega)$.



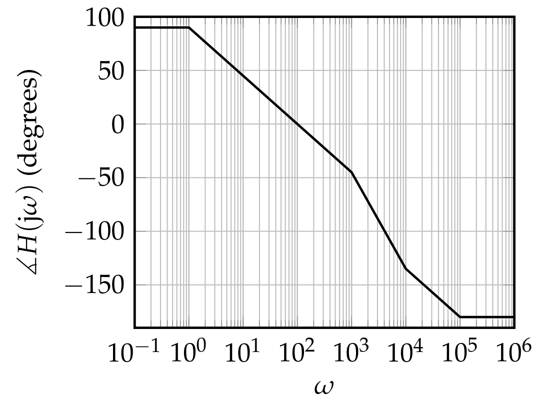
Find an expression for $H_3(j\omega)$. What type of filter is H_3?

(d) (4 pts.) Here are the Bode magnitude and phase plots (straight line approximations) for transfer function $H(\mathrm{j}\omega)$:

Plot of $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$.



Plot of $\measuredangle H(\mathrm{j}\omega)$.



Find an expression for $H(\mathrm{j}\omega)$. What type of filter is H?

(HINT: Consider the plots from the previous parts.)

(e) (5 pts.) Suppose we have the following input into the filter with transfer function $H(\mathrm{j}\omega)$:

$$v_{\mathrm{in}}(t) = 1 + \cos\left(10^{2}t\right) + \sin\left(10^{4}t\right) + \cos\left(10^{10}t\right) \tag{10}$$

Find an expression for $v_{\mathrm{out}}(t)$, the output of the filter for this input. You can approximate any values you use as part of this calculation.

(HINT: You can use the plot of $H(\mathrm{j}\omega)$ from the previous part.)

(f) (5 pts.) Now, suppose we have the following transfer function, independent of the previous parts of this problem:

$$F(\mathrm{j}\omega) = \frac{(\mathrm{j}\frac{\omega}{10^3})(\mathrm{j}\frac{\omega}{10^5})}{(1+\mathrm{j}\frac{\omega}{10^3})(1+\mathrm{j}\frac{\omega}{10^5})(1+\mathrm{j}\frac{\omega}{10^6})} \tag{11}$$

Now, suppose we want to build a filter that implements $F(\mathrm{j}\omega)$. You are provided with 3 $1\,\mathrm{k}\Omega$ resistors, 2 ideal op-amps, and 3 capacitors whose values you can choose as necessary. **Draw a filter that implements $F(\mathrm{j}\omega)$. Make sure to label the values of any capacitors you use.**

(HINT: Try to break up the filter $F(\mathrm{j}\omega)$ into multiple first order low pass / high pass filters.)

10. Nonlinear System ID and Feedback (20 pts.)

Suppose we know that a scalar, discrete-time system follows some nonlinear dynamics equation of the form

$$x[i+1] = f(x[i], u[i]) \tag{12}$$

for some nonlinear function $f(\cdot, \cdot)$ that is unknown. In this problem, we will make guesses about the form of f and use system ID to estimate the exact function from data.

(a) (3 pts.) Your friend Vik believes that f has the following form:

$$f(x[i], u[i]) = a_3(x[i])^3 + a_2(x[i])^2 + a_1 x[i] + a_0 + u[i] \tag{13}$$

He doesn't know the values of a_3, a_2, a_1, a_0, but he wants to estimate them using system ID. He provides a sequence of control inputs $u[0], u[1], \ldots, u[n-1]$ to the system, and he observes $x[0], x[1], x[2], \ldots, x[n]$. **Set up a system of linear equations of the form $D\vec{p} = \vec{s}$ to estimate $\vec{p} = \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$.**

(HINT: Isolate all the known quantities to one side of the equation, and the unknowns to the other.)

(b) (3 pts.) Your other friend Prakash believes that f has the following form:

$$f(x[i], u[i]) = \mathrm{e}^{a_1 x[i] + a_0} \cdot u[i] \tag{14}$$

He doesn't know the values of a_1, a_0, but he wants to estimate them using system ID. He provides a sequence of positive-valued control inputs $u[0], u[1], \ldots, u[n-1]$ to the system and he observes $x[0], x[1], x[2], \ldots, x[n]$. **Set up a system of linear equations of the form $D\vec{p} = \vec{s}$ to estimate $\vec{p} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$.**

(HINT: Consider writing $\ln(x[i+1]) = \ln(f(x[i], u[i]))$*.)*

(c) (6 pts.) After all their testing, Vik and Prakash conclude that Prakash's guess is correct. In fact, the system follows the dynamics equation given by

$$x[i+1] = \mathrm{e}^{-x[i]} \cdot u[i] \tag{15}$$

This system is not BIBO stable.

i. **Construct a sequence of non-zero control inputs such that $x[i]$ remains bounded. Explain what value $x[i]$ is bounded by (i.e., find a α such that $|x[i]| \leq \alpha$).**

ii. **Construct a sequence of non-zero control inputs such that $x[i]$ becomes unbounded. Explain your answer intuitively.**

Assume that $x[0] = 0$.

(d) (8 pts.) Consider again the system in the previous part and the feedback law $u[i] = \mathrm{e}^{kx[i]}$ where k is some constant. In this part, we will prove that the system will only be stable if and only if $k \leq 1$. To do this, accomplish the following:

i. **Assume $k > 1$. Show that there exists an N such that, for all $i \geq N$, $x[i] < x[i+1]$.**
(HINT: It is the case that there exists an N such that, for all $i \geq N$, $\ln(x[i]) < \alpha x[i]$ for $\alpha > 0$.)

ii. **Assume $k \leq 1$. Find a constant ε such that $|x[i]| \leq \varepsilon$ for every $i \geq 0$.**
(HINT: Remember that $\mathrm{e}^{\text{anything non-positive}}$ is bounded between 0 and 1.)

Again, assume $x[0] = 0$.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

11. Motivating Ridge Regression From Least Squares (30 pts.)

Suppose we have the linear system:

$$A\vec{x} = \vec{y} \tag{16}$$

As we have seen in this class, Least Squares (LS) is a great tool for performing System ID to determine $\vec{x}$ in such a system. However, there are a few problems and limitations with the method.

1 When the singular values of A are small, our answers can be "unstable" (we'll see what this means soon).

2 A may not always be full column rank.

In this problem, we will introduce and motivate **Ridge Regression**, which will hopefully improve upon Least Squares by solving the above issues. For the sake of simplicity, assume $A \in R^{n\times n}$, $\vec{x} \in R^{n\times 1}$, and $\vec{y} \in R^{n\times 1}$.

(a) (6 pts.) Suppose the compact SVD of A is given by $A = U_r \Sigma_r V_r^\top$. **Rewrite the least squares solution $\vec{x}^*_{\text{LS}}$ in its simplest form, replacing A with its SVD.**

(b) (4 pts.) In order to see how small singular values can make $\vec{x}^*_{\text{LS}}$ unstable, let us simplify and consider the scalar case. Let us suppose our SVD is now completely scalar (i.e. $U_r = u$, $\Sigma_r = \sigma$, and $V^\top = v$).

After rewriting x^*_{LS} in the scalar case, what does $\|x^*_{\text{LS}}\|$ approach as $\sigma \to 0$?

Oh no! It seems that our solution $\vec{x}^*_{\text{LS}}$ can really get unstable when the singular values are small. Let us take a look at Ridge Regression, and see if it is any better.

One way to think of Ridge Regression is to see it as exactly like Least Squares, except with a slight modification to A and $\vec{y}$. Thus, we formulate Ridge Regression as:

$$A_{\text{ridge}}\vec{x} = \vec{y}_{\text{ridge}} \tag{17}$$

where $A_{\text{ridge}} = \begin{bmatrix} A \\ I_{n\times n} \end{bmatrix} \in \mathbb{R}^{2n\times n}$, $\vec{x} \in \mathbb{R}^{n\times 1}$, and $\vec{y}_{\text{ridge}} = \begin{bmatrix} \vec{y} \\ \vec{0}_n \end{bmatrix} \in \mathbb{R}^{2n\times 1}$.

Well, for starters, we've already solved Problem 2. With one small change, we've ensured that our data matrix, A_{ridge} will always be full column rank. In the following parts, we'll try and see if we've solved Problem 1.

For the rest of the problem, assume that A **is full row rank.**

(c) (8 pts.) **Show that the solution to the Ridge Regression problem $\vec{x}^*_{\text{ridge}}$ is given by $\vec{x}^*_{\text{ridge}} = (A^\top A + I)^{-1}A^\top\vec{y}$**

(d) (8 pts.) **Now, just like in part A, show that the Ridge Regression solution can be written as $\vec{x}^*_{\text{ridge}} = V_r(\Sigma_r^2 + I)^{-1}\Sigma_r U_r^\top \vec{y}$ by replacing A in terms of its SVD $A = U_r \Sigma_r V_r^\top$.**

(HINT: How can you rewrite I in the Ridge Regresssion solution as a product of two convenient orthonormal matrices?)

(HINT: For arbitrary matrices, A, B, C, and D $ABC + ADC = A(B + D)C$.)

(e) (4 pts.) Once again, to understand the effect of A having small singular values, let us simplify and consider the scalar case. Suppose once again that the SVD of A is now completely scalar (i.e. $U = u$, $\Sigma = \sigma$, and $V^\top = v$).

After rewriting x^*_{ridge} in the scalar case, what does $\|x^*_{\text{ridge}}\|$ approach as $\sigma \to 0$?

12. Bounding Linearization Error (20 pts.)

For the entire problem, we will consider the nonlinear, continuous-time system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), u(t)) = \sin(x(t)) + u(t) \tag{18}$$

where $f(x, u) := \sin(x) + u$. We can find an approximate linear system to represent the original nonlinear system, using a process called linearization. We choose a linearization point such that $f(x^\star, u^\star) = 0$. As a concrete example, let's choose $x^\star = 0$ and $u^\star = 0$. When we linearize about this point, we obtain the following linear system approximation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \widehat{f}(x(t), u(t)) = x(t) + u(t) \tag{19}$$

where $\widehat{f}(x, u) = x + u$. In this problem, we will derive a control law that bounds the approximation error.

(a) (8 pts.) Recall the discretization formula for a one-dimensional system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x(t) + bu(t) \tag{20}$$

$$x_d[i+1] = \mathrm{e}^{\lambda\Delta} x_d[i] + b \cdot \frac{\mathrm{e}^{\lambda\Delta} - 1}{\lambda} u_d[i] \tag{21}$$

where Δ is the discretization sampling period. **Discretize the linearized system in eq. (19) for a given sampling period $\Delta > 0$. Is the discretized system stable?**

(b) (7 pts.) Regardless of your answer to the previous part, suppose we have the following discrete-time system

$$x_d[i+1] = ax_d[i] + bu_d[i] \tag{22}$$

where $b \neq 0$. Suppose we want $|x_d[i+1] - x_d[i]| \leq \delta$ for some given value of δ. In order to achieve this, we impose the following condition on $u_d[i]$:

$$\frac{-\delta + (1-a)x_d[i]}{b} \leq u_d[i] \leq \frac{\delta + (1-a)x_d[i]}{b} \tag{23}$$

Generally, we want to choose a single value for $u_d[i]$ in our feedback law. Given the condition above, we may want to choose the $u_d[i]$ with the lowest energy (i.e., minimize $(u_d[i])^2$ subject to the constraint above). **For each of the following cases, give a single expression for $u_d[i]$ that minimizes $(u_d[i])^2$ subject to the constraint above:**

i. $\frac{-\delta+(1-a)x_d[i]}{b} \leq 0$ and $\frac{\delta+(1-a)x_d[i]}{b} \geq 0$

ii. $\frac{-\delta+(1-a)x_d[i]}{b} \geq 0$ and $\frac{\delta+(1-a)x_d[i]}{b} \geq 0$

iii. $\frac{-\delta+(1-a)x_d[i]}{b} \leq 0$ and $\frac{\delta+(1-a)x_d[i]}{b} \leq 0$

You may not use min or max in your final answer.

(c) (5 pts.) Now, we can return to the nonlinear system in eq. (18). Suppose we want to impose a condition on linearization error as follows:

$$\left| f(x(t), u(t)) - \widehat{f}(x(t), u(t)) \right| \leq \pi \tag{24}$$

where $f(x(t), u(t)) = \sin(x(t)) + u(t)$ from eq. (18) and $\widehat{f}(x(t), u(t)) = x(t) + u(t)$ from eq. (19). **Find the least restrictive condition on $x(t)$, of the form $x_\ell \leq x(t) \leq x_u$, for concrete values of $x_\ell, x_u \in \mathbb{R}$, such that the condition on linearization error in eq. (24) is satisfied.**

(HINT: In this case, $\left| f(x(t), u(t)) - \widehat{f}(x(t), u(t)) \right|$ increases monotonically as $x(t)$ moves further from the origin.)

(HINT: Your expression for $\left| f(x(t), u(t)) - \widehat{f}(x(t), u(t)) \right|$ shouldn't depend on $u(t)$.)

(HINT: What is $\sin(\pi)$?)

From here, you can set δ from part (b) to $\delta = \min(x_d[i] - x_\ell, x_u - x_d[i])$. This gives you the optimal control law to ensure your linearization error is bounded.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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