EECS 16B Designing Information Systems and Devices II UC Berkeley Spring 2022

Final

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________

(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ____________________

Instructions

Remember, you are supposed to write your answers for every part of every question on a fresh sheet of paper. Because you are taking this exam remotely, keep the following in mind:

(a) Whenever you see a bubble to fill in, write down the answers you would have bubbled on your sheet instead.

(b) If the problem asks you to mark something on an existing plot, just write where you would have marked on your sheet instead.

(c) If the problem asks you to sketch a plot on a template, then just sketch the plot on your sheet by drawing and labeling the axes yourself. You don't have to draw all the gridlines on a log-scale plot -— just the axes are enough to draw along with labeled tick-marks on the axes sufficient to understand your plot sketch.

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

2. **What are you planning to do during your summer break? (2 pts.)**

3. **What's your favorite thing about Berkeley so far? (2 pts.)**

4. SVD and the fundamental subspaces (8 pts.)

Consider a matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$. The compact SVD of A is given by $A = U_r \Sigma_r V_r^\top$ where

$$U_r = \begin{bmatrix} \vec{u}_1 \cdots \vec{u}_r \end{bmatrix} \in \mathbb{R}^{m \times r}, \quad \Sigma_r = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \in \mathbb{R}^{r \times r}, \quad V_r = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_r \end{bmatrix} \in \mathbb{R}^{n \times r}$$

with $\sigma_1 \geq \cdots \geq \sigma_r > 0$ being the singular values of A.

(a) (2 pts.) **Which one of the following sets is always guaranteed to form an orthonormal basis for** $\text{Col}(A)$**?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$
ii. $\{\sigma_1 \vec{u}_1, \ldots, \sigma_r \vec{u}_r\}$
iii. $\{\vec{v}_1, \ldots, \vec{v}_r\}$
iv. $\{\sigma_1 \vec{v}_1, \ldots, \sigma_r \vec{v}_r\}$

Option	i	ii	iii	iv
Answer	◯	◯	◯	◯

(b) (2 pts.) **Which one of the following sets is always guaranteed to form an orthonormal basis for** $\text{Col}(A^\top)$**?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$
ii. $\{\sigma_1 \vec{u}_1, \ldots, \sigma_r \vec{u}_r\}$
iii. $\{\vec{v}_1, \ldots, \vec{v}_r\}$
iv. $\{\sigma_1 \vec{v}_1, \ldots, \sigma_r \vec{v}_r\}$

Option	i	ii	iii	iv
Answer	◯	◯	◯	◯

Now suppose that the considered A matrix has the following compact SVD components:

$$U_r = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma_r = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(c) (2 pts.) **Using the given compact SVD, state α, where α is the tightest upper bound $\|A\vec{x}\| \leq \alpha$ for any $\vec{x}$ such that $\|\vec{x}\| \leq 1$.**

(d) (2 pts.) **Given the compact SVD, which of the following provides a valid full SVD for $A = U\Sigma V^\top$?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

iv. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Option	i	ii	iii	iv
Answer	◯	◯	◯	◯

5. SVD of a matrix with orthogonal columns (4 pts.)

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ where $\vec{a}_i^\top \vec{a}_j = 0$ for all $1 \leq i, j \leq n$ such that $i \neq j$, and $\vec{a}_i^\top \vec{a}_i \neq 0$ for all $i = 1, \ldots, n$. **What is the set of singular values of** A ***for all such matrices*** A**?**

(Please fill in one of the circles for the options below. You will only be graded on your final answer.)

(a) $\{0\}$ (all zero)

(b) $\{\sqrt{\|\vec{a}_1\|}, \ldots, \sqrt{\|\vec{a}_n\|}\}$

(c) $\{\|\vec{a}_1\|, \ldots, \|\vec{a}_n\|\}$

(d) $\{\|\vec{a}_1\|^2, \ldots, \|\vec{a}_n\|^2\}$

(e) $\{1\}$ (all one)

Option	a	b	c	d	e
Answer	◯	◯	◯	◯	◯

6. Finding the line that closely fits the data (4 pts.)

Consider the following matrix A that contains three two-dimensional datapoints:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & -1 \end{bmatrix}. \tag{1}$$

The matrix A has two distinct singular values: $\sigma_1 = \sqrt{21}$ and $\sigma_2 = \sqrt{7}$.

Below is a plot of the datapoints in the 2-D plane, where the x-axis corresponds to the first entry and the y-axis to the second entry of each column. We would like to fit the line $y = \alpha x$ that minimizes the squared sum of perpendicular distances to the datapoints as follows:

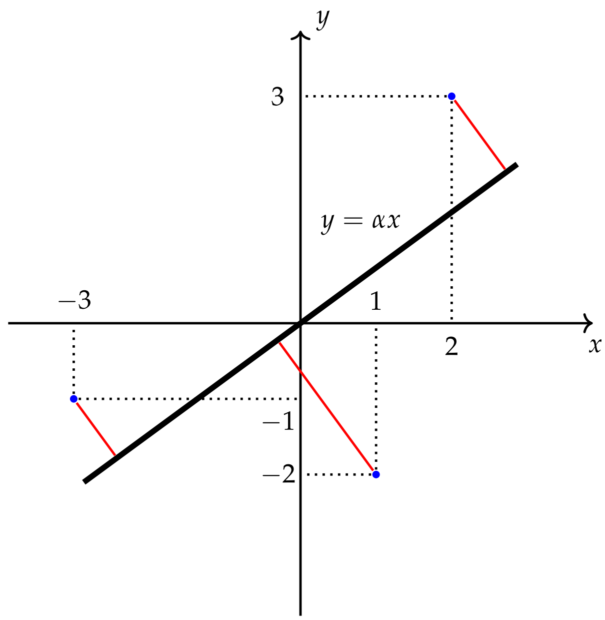


Figure 1: Visualization for perpendicular distance minimization.

Find α, using the left singular vectors of matrix A. Show your work.

(Please fill in one of the circles for the options below.)

(a) $\alpha = \sqrt{3}/2$

(b) $\alpha = 7/6$

(c) $\alpha = 2$

(d) $\alpha = 1$

Option	a	b	c	d
Answer	◯	◯	◯	◯

7. Least squares with repeated columns (4 pts.)

Consider the following matrix A and vector $\vec{b}$:

$$A = \begin{bmatrix} 0 & 0 & 2 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

We want to find a solution for the following least squares problem:

$$\underset{\vec{x} \in \mathbb{R}^2}{\operatorname{argmin}} \left\| A\vec{x} - \vec{b} \right\|^2 \tag{3}$$

However, we cannot use the least squares solution $\vec{x}_{\text{LS}} = (A^\top A)^{-1} A^\top b$, since $A^\top A$ is not invertible due to the repeated columns in A.

We provide a compact SVD of A:

$$A = U_r \Sigma_r V_r^\top = \begin{bmatrix} 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}. \tag{4}$$

Find a solution for the least squares problem using the Moore-Penrose pseudoinverse.

8. Matching systems to time responses (4 pts.)

Consider the following four different 2-D systems $\vec{x}(t)$:

$$\text{System 1:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \vec{x}(t), \tag{5}$$

$$\text{System 2:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{x}(t) \tag{6}$$

$$\text{System 3:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \vec{x}(t) \tag{7}$$

$$\text{System 4:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}(t) \tag{8}$$

where $\vec{x}(t) \in \mathbb{R}^2$ with initial condition $\vec{x}(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

We provide the following possible solutions for $\vec{x}(t)$:

(a) $\begin{bmatrix} \frac{15}{4}\mathrm{e}^{t} + \frac{1}{4}\mathrm{e}^{5t} \\ -\frac{5}{4}\mathrm{e}^{t} + \frac{1}{4}\mathrm{e}^{5t} \end{bmatrix}$

(b) $\begin{bmatrix} 4\mathrm{e}^{t} + 3\sqrt{2}t\mathrm{e}^{t} \\ -\mathrm{e}^{t} - 3\sqrt{2}t\mathrm{e}^{t} \end{bmatrix}$

(c) $\begin{bmatrix} 4\mathrm{e}^{3t} - 3\sqrt{2}t\mathrm{e}^{3t} \\ -e^{3t} + 3\sqrt{2}t\mathrm{e}^{3t} \end{bmatrix}$

(d) $\begin{bmatrix} 7\mathrm{e}^{2t} - 3\mathrm{e}^{3t} \\ -7\mathrm{e}^{2t} + 6\mathrm{e}^{3t} \end{bmatrix}$

Each system has one matching solution from the above choices.

For each system, fill in the circle that matches the correct solution. *(You will only be graded on your final answer.)*

Option	a	b	c	d
System 1	◯	◯	◯	◯

Option	a	b	c	d
System 2	◯	◯	◯	◯

Option	a	b	c	d
System 3	◯	◯	◯	◯

Option	a	b	c	d
System 4	◯	◯	◯	◯

9. Properties of rotation matrices (8 pts.)

Consider the 2×2 matrix

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{9}$$

(a) (2 pts.) **Show that matrix R is orthonormal.**

(b) (2 pts.) Consider some vector $\vec{x} \in \mathbb{R}^2$ with norm $\|\vec{x}\|$. **Show that $\|R\vec{x}\| = \|\vec{x}\|$.**

(c) (4 pts.) Consider arbitrary real vectors $\vec{a}, \vec{b}$ and let β within interval $0 \leq \beta \leq \pi$ be the angle between them. The inner product for this pair of vectors can be related to angle β via the expression $a^\top b = \|\vec{a}\| \left\|\vec{b}\right\| \cos\beta$. Now consider vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ and define θ_1 to be the angle between them. Let $\widetilde{\vec{x}} = R\vec{x}$ and $\widetilde{\vec{y}} = R\vec{y}$ and denote θ_2 to be the angle between $\widetilde{\vec{x}}$ and $\widetilde{\vec{y}}$. **Show that $\theta_1 = \theta_2$, i.e. the angle between two vectors is preserved after an orthonormal transformation.** *(HINT: You may use the inner product expression given at the start of this part as well as results from parts a) and b).)*

10. Inner product with a Hermitian matrix (8 pts.)

Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is *Hermitian* if $A^* = A$, where A^* denotes the conjugate transpose of A. We will show that $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A\vec{y}\rangle$.

(a) (3 pts.) Assume that $A \in \mathbb{C}^{n\times n}$ is Hermitian. **Show that for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A\vec{y}\rangle$.**

(b) (5 pts.) Let $A \in \mathbb{C}^{n\times n}$. Assume that for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A\vec{y}\rangle$. Now we want to show that this implies $A = A^*$. Let a_{ij} and $\widetilde{a}_{ij}$ be the elements of A and A^* respectively. **Pick appropriate vectors $\vec{x}$ and $\vec{y}$ to show that $a_{ij} = \widetilde{a}_{ij}$ for all $1 \leq i, j \leq n$.**

11. Adaptive cruise control (20 pts.)

Consider a vehicle traveling with speed $v(t) > 0$ in a road lane behind a "lead" vehicle traveling with constant speed v_L. Denote the distance to the lead vehicle by $h(t)$ and the torque input to the follower vehicle with $u(t)$, as shown in Fig. 2:

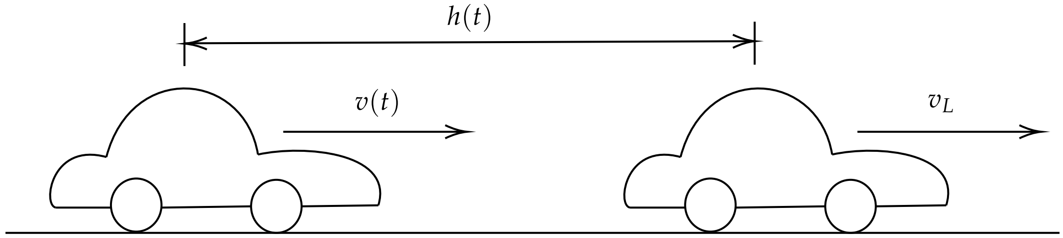


Figure 2: Vehicles for adaptive cruise control.

Then a simplified model for $h(t)$ and $v(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = v_L - v(t) \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = a - bv(t)^2 + cu(t) \tag{11}$$

where a, b, c are the appropriate coefficients with $b > 0$ and $c > 0$. We wish to maintain a given relative distance, $h^* > 0$, between the vehicles.

(a) (3 pts.) **Find the values of v^* and u^* that form a valid operating point.**

(b) (5 pts.) The linearization of the system about (v^*, u^*) takes the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}\delta u(t) \tag{12}$$

where $\delta h(t) := h(t) - h^*$, $\delta v(t) := v(t) - v^*$, and $\delta u(t) := u(t) - u^*$. **Find the values of a_{11}, a_{12}, a_{21}, a_{22}, b_1, and b_2.**

For reference, the system is

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = v_L - v(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = a - bv(t)^2 + cu(t)$$

(c) (12 pts.) Suppose that for some values of a, b, c, and v_L, the linearization about some operating point (v^*, u^*) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} = \begin{bmatrix}0 & -1\\ 0 & -4\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\delta u(t) \tag{13}$$

Suppose we apply the feedback law

$$u(t) = u^* + k_1(h(t) - h^*) + k_2(v(t) - v^*), \tag{14}$$

which means

$$\delta u(t) = k_1\,\delta h(t) + k_2\,\delta v(t) \tag{15}$$

What are the ranges of the feedback gains k_1 and k_2 that *asymptotically stabilize* this linearized model?

12. Minimum-norm input to an RC circuit (20 pts.)

Consider the following circuit in Fig. 3 used to turn on a light-emitting diode (LED):

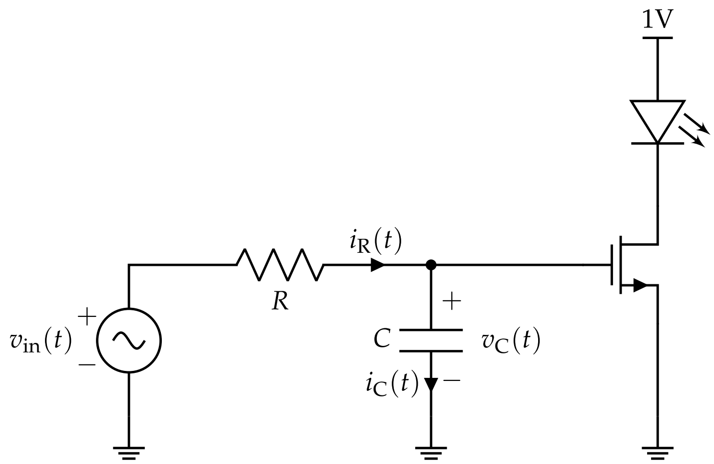


Figure 3: Circuit to turn on LED.

The LED turns on when the NMOS gate voltage $v_C(t)$ sufficiently exceeds its threshold voltage. For this problem, let's say we require $v_C(t) \geq 0.5\,\text{V}$ for both the NMOS and the LED to turn on.

The voltage source can generate any continuous-time function $v_{in}(t)$ you desire. The resistor R models the wire resistance, and the capacitor C models the NMOS gate capacitance.

Assume $v_{in}(t) = 0$ for $t < 0$ and $v_C(t) = 0$ for $t < 0$. Your goal is to turn on the LED while minimizing the norm (energy) of the source voltage $v_{in}(t)$.

(a) (3 pts.) **Use KCL to find the continuous-time differential equation for $v_C(t)$ with an arbitrary input voltage $v_{in}(t)$. Write your answer in the form**

$$\frac{\mathrm{d}}{\mathrm{d}t} v_C(t) = A v_C(t) + B v_{in}(t) \tag{16}$$

and solve for the coefficients A and B.

(b) (5 pts.) Regardless of your previous result, define $x(t) := v_C(t)$ and $u(t) := v_{\text{in}}(t)$ and assume the continuous-time differential equation is of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) - \lambda u(t) \qquad (\lambda < 0) \tag{17}$$

We can discretize this equation assuming a zero-order hold and a uniform sampling period Δ. That is, assuming the discrete forms of $x(t)$ and $u(t)$ are $x_d[i]$ and $u_d[i]$ respectively, where

$$x_d[i] = x(i\Delta) \qquad \text{for } i = 0, 1, 2, ... \tag{18}$$

$$u(t) = u_d[i] \qquad \text{for } t \in [i\Delta, (i+1)\Delta) \tag{19}$$

we can write the discrete-time difference equation as

$$x_d[i+1] = A_d x_d[i] + B_d u_d[i] \tag{20}$$

Solve for A_d and B_d in terms of λ and Δ. Assume that $x_d[0] = 0$.

(c) (10 pts.) Regardless of your previous result, assume that for the rest of this problem our discrete-time difference equation is:

$$x_d[i+1] = A_d x_d[i] + B_d u_d[i] \tag{21}$$

with

$$A_d = \mathrm{e}^{-0.1} \qquad \text{and} \qquad B_d = 0.1 \tag{22}$$

Let us define our target voltage as $X^\star := 0.5\,\mathrm{V}$. We would like $x_d[i]$ to reach $X^\star$ such that our LED turns on in ℓ discrete-time steps using an input sequence $u_d[0], u_d[1], \dots, u_d[\ell-1]$. We can model this as a control problem with a $1\times\ell$ controllability matrix at timestep ℓ as:

$$C_\ell = \begin{bmatrix} A_d^{(\ell-1)} B_d & \cdots & A_d B_d & B_d \end{bmatrix} \tag{23}$$

such that

$$X^\star = \begin{bmatrix} A_d^{(\ell-1)} B_d & \cdots & A_d B_d & B_d \end{bmatrix} \begin{bmatrix} u_d[0] \\ \vdots \\ u_d[\ell-2] \\ u_d[\ell-1] \end{bmatrix} \tag{24}$$

Find the minimum-norm sequence for the input u_d, i.e. find $u_d[i]$ such that $||\vec{u}_d||^2 = \sum_{i=0}^{\ell-1} |u_d[i]|^2$ is minimized.

To simplify your arithmetic, use:

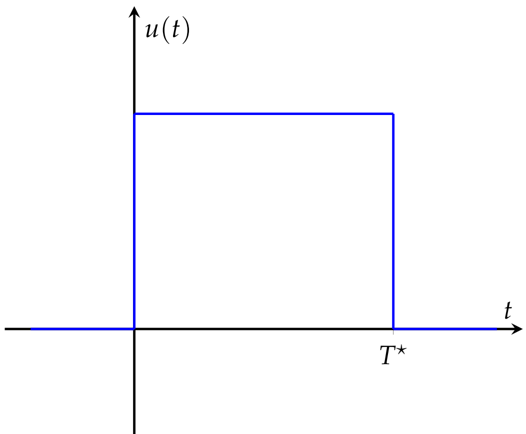
$$C_\ell C_\ell^\top = \frac{1}{20}(1 - \mathrm{e}^{-0.2\ell}) \tag{25}$$

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

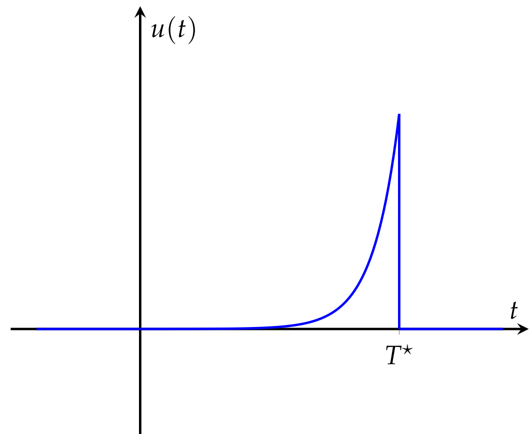
(d) (2 pts.) Assume we want our LED to turn on at target time $t = T^\star$. Beyond that time, the status of the LED is irrelevant, so we turn off our source $u(t)$ for $t > T^\star$.

In the table below, select the closest continuous-time input source voltage $u(t)$ which corresponds to the minimum-norm solution. Assume the axes in the plots are all to the same scale. *You will only be graded on your final answer.*

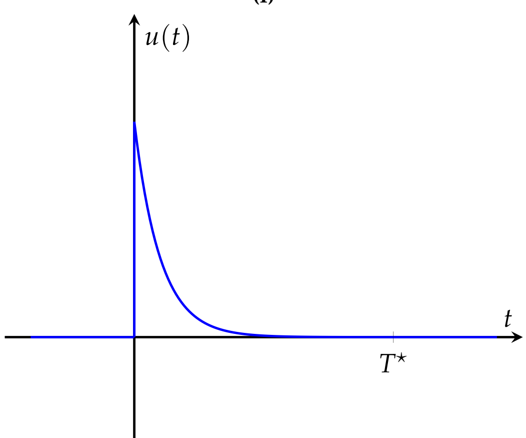
Option	I	II	III	IV
Answer	◯	◯	◯	◯



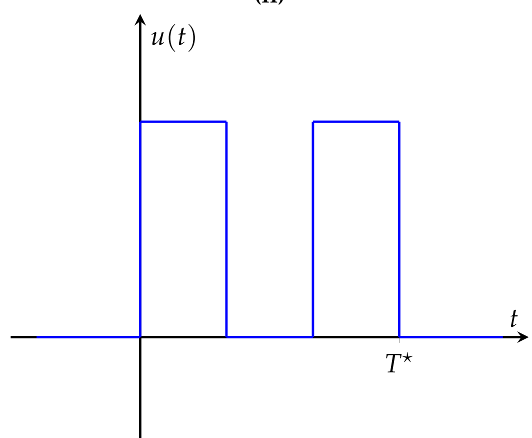
(I)



(II)



(III)



(IV)

13. Polyphase electricity (20 pts.)

Large electric grids such as PG&E generate and distribute not 1 but 3 AC voltages offset by certain phases. This helps efficiently deliver constant power to power-hungry motors in machines of large buildings, such as elevators.

To understand why 3-phase power is used, let's first consider a 1-phase system.

We can model an elevator's motor in a 1-phase system as simply a load resistor R_M. The motor is plugged into a common wall outlet which supplies an AC voltage $v(t) = V_\mathrm{S}\cos(2\pi f t)$ from the generator through the power transmission lines ("hot" and "neutral"), as shown below. Note that ground is defined at the generator, not at the load.

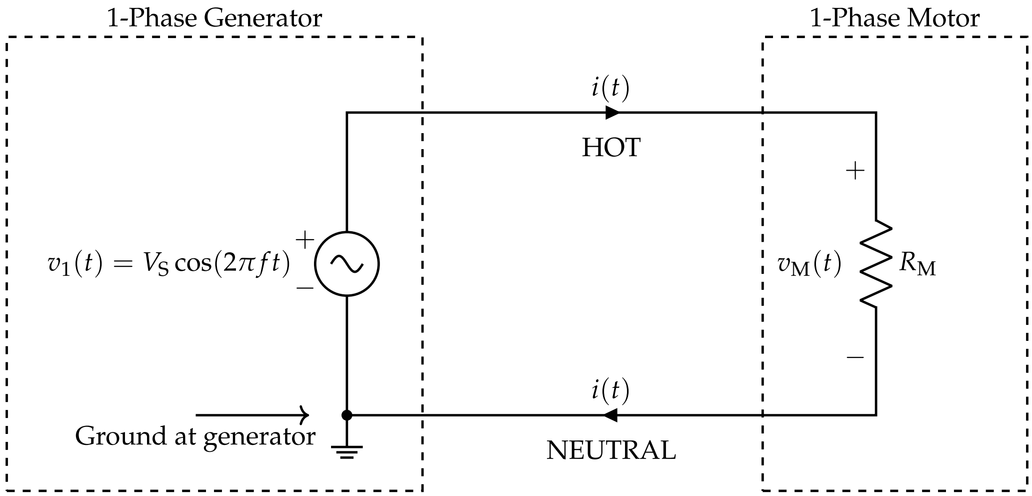


Figure 5: 1-phase power.

(a) (1 pts.) What is the current $i(t)$ flowing through the neutral line as a function of time? Write your answer in terms of V_S, R_M, and f.

(b) (2 pts.) Instantaneous electrical power consumed by the motor, $p_\mathrm{M}(t)$, is defined as:

$$p_\mathrm{M}(t) = v_\mathrm{M}(t)i(t) \tag{26}$$

where $v_\mathrm{M}(t)$ and $i(t)$ are the voltage and current across the motor (following passive sign convention, as drawn).

The instantaneous power $p_\mathrm{M}(t)$ consumed by the 1-phase motor may be written in the form $A + B\cos(2\pi C t)$, where A, B, and C are real numbers. **Find A, B, and C in terms of V_S, R_M, and f.**

(HINT: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$*)*

You should have found that a 1-phase system does not deliver constant power to the elevator motor, resulting in an uneven ride. To fix this, Nikola Tesla was awarded a US patent in 1888 proposing multi-phase power. The simplest case, 2-phase power, is shown below. The voltage sources in a 2-phase generator are phase shifted by 90°, i.e.

$$v_1(t) = V_S \cos(2\pi f t) \tag{27}$$

$$v_2(t) = V_S \cos(2\pi f t - 90^\circ) = V_S \sin(2\pi f t) \tag{28}$$

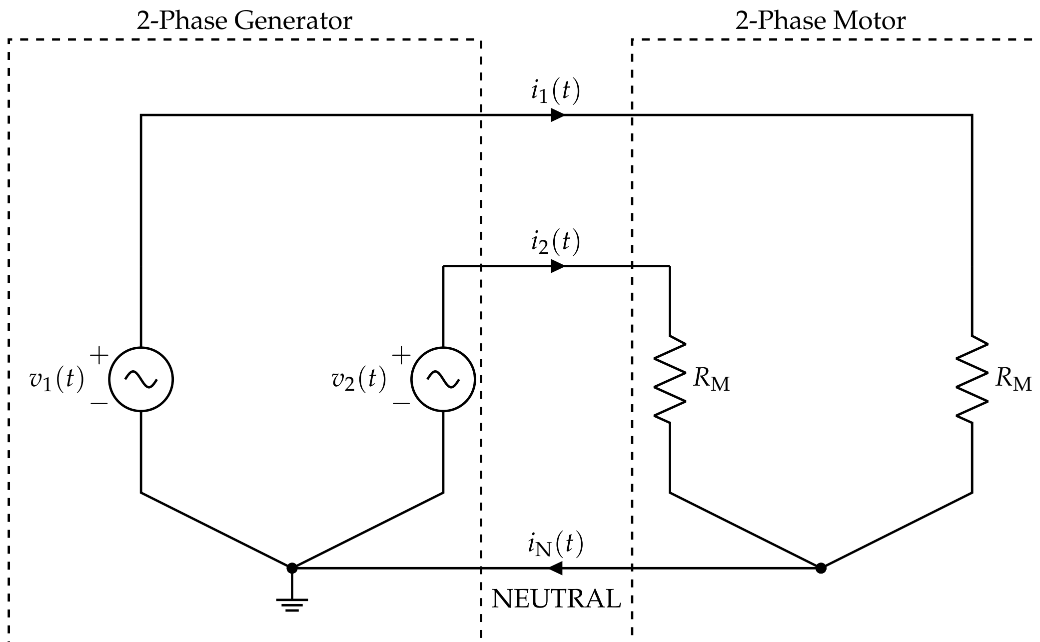


Figure 6: 2-phase power.

(c) (5 pts.) The two-phase motor's instantaneous power is defined as $p(t) = p_1(t) + p_2(t)$, where $p_1(t)$ is the power consumed by branch/phase 1 and $p_2(t)$ is the power consumed by branch/phase 2.

What is the 2-phase motor's total instantaneous consumed power $p(t)$ in terms of V_S, R_M, and f?

(HINT: $\cos^2 x + \sin^2 x = 1$)

(d) (5 pts.) Now let's use phasors to determine the time-domain neutral current $i_{\mathrm{N}}(t)$ for a 2-phase system.

i. **What are the phasors $\widetilde{V}_1$ and $\widetilde{V}_2$ for the voltage sources $v_1(t)$ and $v_2(t)$, respectively? Write your answers in terms of V_{S}.**

ii. **What are the phasor $\widetilde{I}_1$ and $\widetilde{I}_2$ for the currents currents $i_1(t)$ and $i_2(t)$, respectively? Write your answers in terms of V_{S} and R_{M}.**

iii. The neutral current in a 2-phase system can be written in the form $i_{\mathrm{N}}(t) = A\cos(2\pi Bt + C)$. **Find A, B, and C in terms of V_{S}, R_{M}, and f.**

In the early 1900s, electrical engineers at Westinghouse and General Electric proposed a 3-phase distribution method, shown below. One advantage of this 3-phase system is that it delivers 1.5× the constant power compared to a 2-phase system. The 3 generator voltage sources are separated by 120° phase shifts, as defined:

$$v_1(t) = V_S \cos(2\pi f t) \tag{29}$$
$$v_2(t) = V_S \cos(2\pi f t - 120^\circ) \tag{30}$$
$$v_3(t) = V_S \cos(2\pi f t - 240^\circ) \tag{31}$$
$$\tag{32}$$

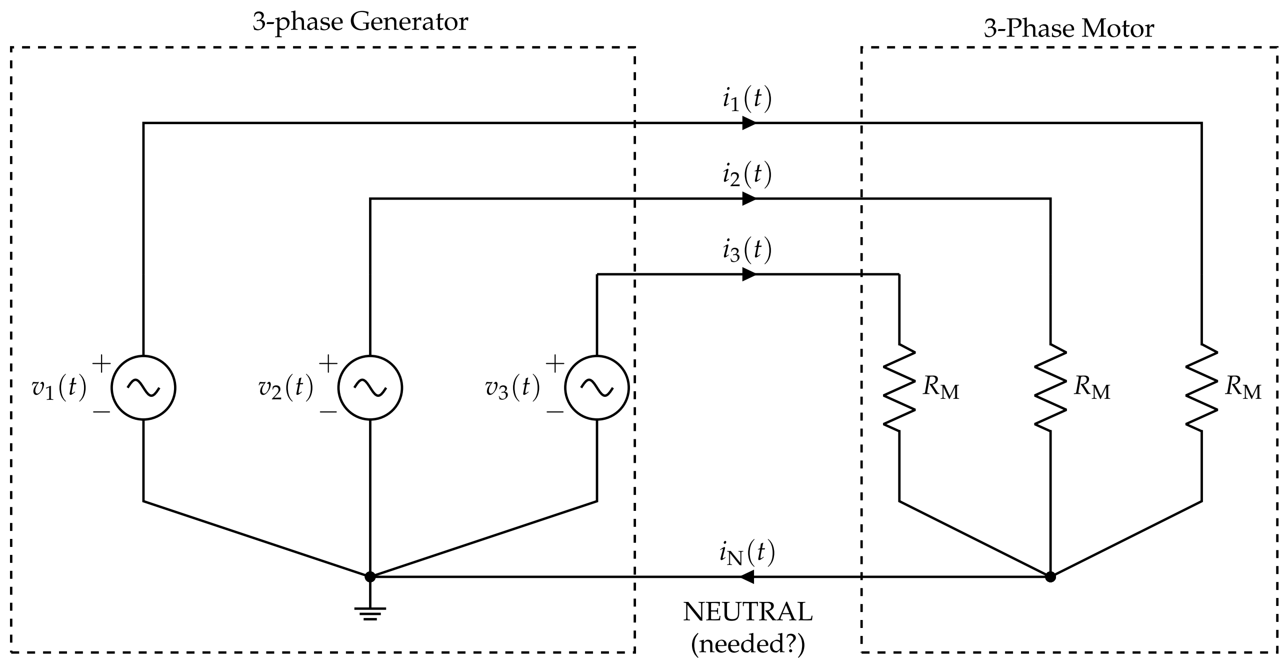


Figure 7: 3-phase power.

(e) (7 pts.) In addition to higher constant power delivery, proponents of the 3-phase system claimed the neutral return wire was not required. This potentially reduces the cost connecting an extra transmission line throughout the power grid. **Is this claim true? Justify your answer by finding $i_N(t)$ as a function of time.**

14. Dynamical system approach to solving Ridge Regression (16 pts.)

In this problem, we will derive a dynamical system based approach to solving a modified version of the least-squares problem, commonly known as "ridge regression". This problem attempts to find the $\vec{x}$ that minimizes $\|A\vec{x} - \vec{y}\|^2 + \lambda \|\vec{x}\|^2$. Here we assume $A \in \mathbb{R}^{m \times n}$ is full column rank and scalar $\lambda \geq 0$.

The solution to the ridge regression problem is

$$\vec{\hat{x}} = (A^\top A + \lambda I)^{-1} A^\top \vec{y}. \tag{33}$$

Note that this solution is quite similar to the solution of least-squares. In many cases, direct computation of the solution to ridge regression is too slow, because it requires computing the matrix inverse $(A^\top A + \lambda I)^{-1}$, which is generally very costly for A with very large dimensions. We will instead solve the problem iteratively by using an update rule which turns this particular problem into an analysis of a particular discrete-time state-space dynamical system.

(a) (2 pts.) We first connect the ridge regression problem to the familiar ordinary least-squares problem. **State the condition on λ in (33) needed to recover the least squares solution.**

(b) (3 pts.) Using (33), **show that $(A^\top A + \lambda I)\vec{\hat{x}} - A^\top \vec{y} = 0$.**

(c) (5 pts.) In iterative optimization schemes, we will get a sequence of estimates for $\vec{\hat{x}}$ at each timestep. Let $\vec{x}[i]$ denote our estimate for $\vec{\hat{x}}$ at timestep i.
In this problem we will consider the following update rule for solving the ridge regression problem:

$$\vec{x}[i+1] = \vec{x}[i] - \alpha\left((A^\top A + \lambda I)\vec{x}[i] - A^\top \vec{y}\right) \tag{34}$$

that gives us an updated estimate $\vec{x}[i+1]$ using the previous one $\vec{x}[i]$. Here α is the "step size" in our update rule which controls how much we update our solution estimate at each time step. For the purposes of this problem, it doesn't matter where we got the update rule, but the important thing to note is that if $\vec{x}[i] = \vec{\hat{x}}$, then by the previous part, $\vec{x}[i+1] = \vec{\hat{x}}$ and the system remains in equilibrium at $\vec{\hat{x}}$ for all time.
To show that $\vec{x}[i] \to \vec{\hat{x}}$, we define a new state variable $\Delta\vec{x}[i] = \vec{x}[i] - \vec{\hat{x}}$. It represents the deviation from where we want to be.
Derive the discrete-time state evolution equation for $\Delta\vec{x}[i]$, and show that it takes the form:

$$\Delta\vec{x}[i+1] = (I - \alpha G)\Delta\vec{x}[i]. \tag{35}$$

What is G?

(d) (3 pts.) We would like to select α such that $\Delta\vec{x}[i]$ converges to 0. In particular, we want to make sure that we have a stable system. To do this, we need to understand the eigenvalues of $I - \alpha G$. **Given that $\lambda_k\{G\}$ are the eigenvalues of G, for $k \in \{1, 2, \ldots, n\}$ what are the eigenvalues of the matrix $I - \alpha G$?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $1 - \alpha\lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

ii. $\alpha\lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

iii. $1 - \lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

iv. $1 + \alpha\lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

Option	i	ii	iii	iv
Answer	◯	◯	◯	◯

(e) (3 pts.) For system (35) to be stable, we need all the eigenvalues of $I - \alpha G$ to have magnitudes that are smaller than 1 (since this is a discrete-time system). **State the condition on α that would ensure that system (35) is stable.** You may assume that $\lambda_k\{G\}$ are real and $\lambda_k\{G\} > 0$ for $k \in \{1, 2, \ldots, n\}$.

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.

If needed, you can also use this space to work on problems. But if you want the work on this page to be graded, make sure you tell us on the problem's main page.]

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]