Final

Exam Location: In Person

PRINT your student ID: ____________________

PRINT AND SIGN your name: ____________, ____________ ____________

(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ____________________

Row Number: ____________ Seat Number: ____________

Name and SID of the person to your left: ____________________

Name and SID of the person to your right: ____________________

Name and SID of the person in front of you: ____________________

Name and SID of the person behind you: ____________________

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

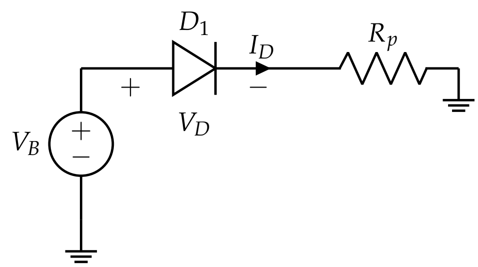
2. **(EXAM EXTRA CREDIT) What is your favorite topic of the course? (2 pts.)**

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

3. **Diode System ID (10 pts.)**

Note: This problem does not require the use of your knowledge of circuits or diodes. The problem is about system ID.

Suppose that we want to characterize a diode D_1 with parasitic resistance R_p, represented by the following model:



The current through the diode (forward-biased) is modeled as

$$I_D = I_0 e^{\frac{V_D}{V_T}} \tag{1}$$

where V_D is the voltage across the diode (as shown in the diagram), I_0 is a parameter associated with the diode, and V_T is the thermal voltage, a temperature dependent variable.

With some analysis, the system equation can be simplified to:

$$I_D = I_0 e^{\frac{V_B - I_D R_p}{V_T}} \tag{2}$$

To collect data, we will vary the temperature to vary V_T and measure the current I_D through the diode.

(a) (2 pts.) **Use the provided system equation to show that**

$$I_D R_p - V_T \ln(I_0) = V_B - V_T \ln(I_D) \tag{3}$$

is a valid equation to represent the system. *(HINT: Use the logarithm identity* $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$*.)*

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(b) (4 pts.) Suppose we use $V_B = v_B$V and collect the following data points:

- For $V_T = v_{T1}$V, $I_D = i_{D1}$A.
- For $V_T = v_{T2}$V, $I_D = i_{D2}$A.
- For $V_T = v_{T3}$V, $I_D = i_{D3}$A.

Find the matrix D and vector $\vec{s}$ to set up a least-squares problem $D\vec{p} \approx \vec{s}$ where $\vec{p} = \begin{bmatrix} \ln(I_0) \\ R_p \end{bmatrix}$.

(c) (4 pts.) **Explain how you would use the least-squares system from the previous part to find model parameters I_0 and R_p.** You do not need to use actual values for this part; you explanation can just use D and $\vec{s}$ as known variables to show what calculations you would do to find I_0 and R_p.

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4. Feedback Control of an Aircraft (8 pts.)

Let the current state of an aircraft be described by its height above ground, vertical speed, and pitch (in order). Let the inputs to the system be the aircraft's engine thrust and the angle of the elevators (which control pitch). Moreover, assume that the aircraft's instruments are perfect so that there is no noise. The system is, in reality, quite complex, so you approximate it by linearization such that, the state system looks like:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$$

where $A \in \mathbb{R}^{3\times 3}, B \in \mathbb{R}^{3\times 2}$, and

$$\vec{x}[i] = \begin{bmatrix} h[i] \\ s[i] \\ p[i] \end{bmatrix} \in \mathbb{R}^3$$

$$\vec{u}[i] = \begin{bmatrix} u_1[i] \\ u_2[i] \end{bmatrix} \in \mathbb{R}^2$$

To simplify the system further, you run Schur Decomposition on A to upper-triangularize it. This results in the system below:

$$\vec{\tilde{x}}[i+1] = \begin{bmatrix} 0.8 & 5 & 2 \\ 0 & -0.5+j & -0.1 \\ 0 & 0 & -0.5-j \end{bmatrix} \vec{\tilde{x}}[i] + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{u}[i]$$

where $\vec{\tilde{x}}[i] = \begin{bmatrix} \tilde{h}[i] & \tilde{s}[i] & \tilde{p}[i] \end{bmatrix}^T$

(a) (2 pts.) **Is the original system BIBO stable? Why or why not?**

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(b) (6 pts.) Now, we want to apply closed-loop feedback to the system such that:

$$\tilde{s}[i+1] = \tilde{s}[i] \text{ and } \tilde{p}[i+1] = \tilde{p}[i]$$

Using these equations, **apply a closed loop feedback and solve for the coefficients f_1, f_2, and f_3 in**

$$F = \begin{bmatrix} 0 & f_1 & f_2 \\ 0 & 0 & f_3 \end{bmatrix}$$

(c) (4 pts.) **(EXAM EXTRA CREDIT)** For this part, in addition to the closed-loop feedback from the previous part, we also add the open loop input vector

$$\vec{u}_{OL}[i] = \begin{bmatrix} u_{1,OL}[i] \\ u_{2,OL}[i] \end{bmatrix} \in \mathbb{R}^2$$

Regardless of your answers for the previous part, suppose that you found, $f_1 = -j, f_2 = 1.1, f_3 = j$. **Is the system with closed-loop feedback and open loop input controllable? Why or why not?** *(HINT: First calculate $A_{CL}\tilde{B}$ and $\tilde{B}$. Do you need to calculate $A_{CL}^2\tilde{B}$ as well?)*

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5. Designing a Controllable Aircraft (8 pts.)

Continuing from the previous problem, we have a similar setup where an aircraft's state can be described by its vertical speed and pitch only. Therefore, the new system will look like this:

$$\begin{bmatrix} s[i+1] \\ p[i+1] \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} s[i] \\ p[i] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i]$$

You are working with aeronautical engineers to come up with a design for an aircraft that can always recover from a stall.

(a) (3 pts.) Suppose that these are the designed system parameters.

$$\begin{bmatrix} s[i+1] \\ p[i+1] \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} s[i] \\ p[i] \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u[i]$$

Is the system controllable?

(b) (3 pts.) Based on your answer for the previous part, **is it possible for the aircraft to recover from a stall (ie. an underisable state)? Why or why not?** Note: Recovering means that the aircraft leaves an undesirable state and gets steered to (or eventually reaches) a desirable state.

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(c) (2 pts.) Regardless of the previous parts, you found that the system was actually not controllable. The engineers re-design the parts and you find the following system parameters.

$$\begin{bmatrix} s[i+1] \\ p[i+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} s[i] \\ p[i] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$$

Given an initial state where the aircraft has stalled (an undesirable state), does there exist a sequence of inputs that can recover the aircraft? Why or why not? Note: Recall that recovering means that the aircraft leaves an undesirable state and gets steered to (or eventually reaches) a desirable state.

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6. SVD (30 pts.)

Consider the following matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. In this problem, we will be finding the SVD of the matrix A.

(a) (4 pts.) **First, find the eigenvalues and eigenvectors of AA^T.**

(b) (2 pts.) **Now, what are the singular values σ_1 and σ_2 as well as the corresponding columns $\vec{u}_1$ and $\vec{u}_2$ of U (note: $\sigma_1 >= \sigma_2$) in the SVD of A?**

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(c) (4 pts.) **Using your answer to the previous question, find $\vec{v}_1$ and $\vec{v}_2$, the first two columns of V in SVD. Show all your work.**

(HINT: If we know that $V_r^T = \Sigma_r^{-1} U_r^T A$, how can we write this matrix expression for the compact SVD in terms of the individual vectors $\vec{v}_i^T$ (and thus solving for $\vec{v}_i$)?)

(d) (5 pts.) Regardless of your answer to the previous question, suppose $\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}$. **Find $\vec{v}_3$ using Gram-Schmidt basis extension with the standard basis vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Show all your work.**

(e) (6 pts.) Regardless of your answers to all previous parts, suppose $\sigma_1 = 4$, $\sigma_2 = 2$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$, and $\vec{v}_1$ and $\vec{v}_2$ are defined as in part (d). **Using these definitions, write out the SVD of A *numerically* in compact form.**

(f) (3 pts.) In the next several parts, consider the compact SVD of the same wide A matrix *symbolically* (i.e. $A = U_{\mathrm{r}}\Sigma_{\mathrm{r}}V_{\mathrm{r}}^T$). Suppose you are told to solve the problem $A\vec{x} = \vec{b}$. **If $\vec{b} \in \mathrm{Col}(A)$, are there zero, one, or many solutions for $\vec{x}$? Explain your answer briefly in one to two sentences referring to the $\mathrm{Col}(A)$ and the $\mathrm{Null}(A)$.**

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(g) (3 pts.) **Symbollically, solve for $\vec{x}$ using the SVD of A. Your answer should only contain the terms U_r, Σ_r, and/or V_r and any necessary transposes or inverses. In one sentence or less, what is special about this solution to the equation?**

(h) (3 pts.) **Simplify the quantity $AA^\dagger A$ as much as possible using the compact SVD of A ($A^\dagger$ is the psuedo-inverse of A). Your final answer should only be in terms of A. Show your work.**

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7. PCA and LS Prediction (12 pts.)

In this problem, you will be helping to analyze some clinical data for patients using PCA and least squares. Suppose the following is a data matrix consisting of scaled *zero-mean* patient data. The patients (i.e. data points) are the *columns*, while the rows from top to bottom represent scores for the height, weight, and blood pressure of the 4 patients.

$$A = \begin{bmatrix} -3 & 0 & 2 & 1 \\ -1 & -2 & 0 & 3 \\ -1 & -1 & 1 & 1 \end{bmatrix} \tag{4}$$

Using this information, we would like to predict a risk index for heart disease for each patient (negative if low-risk and positive if high risk). But unfortunately, there appears to be some noise mixed into the data. Let's try to mitigate this by *performing PCA to embed the training data into a lower dimension first*.

(a) (4 pts.) Let us perform PCA on A to embed the data into a single dimension. Suppose you know that $\sigma_1 = \sqrt{8}$ and $\vec{v}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ from the first element of Σ and first column of V for the SVD of A.

What is the first principal component $\vec{u}_1$ of A? Show all your work.

(b) (4 pts.) Regardless of your answer to the previous question, suppose $\vec{u}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$. **Project your three-dimensional datapoints from A into a one-dimensional data. Express your answer as the vector $\vec{z} \in \mathbb{R}^{1\times 4}$, and show all your work.**

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(c) (4 pts.) Now, suppose that for this training data we have known risk index scores already (with a scalar index for each patient):

$$\vec{b} = \begin{bmatrix} -3 & -1 & 4 & 6 \end{bmatrix} \tag{5}$$

We would like to have a way to predict risk scores. Let's try to use least squares on our scenario. In one dimension, we can set up our least squares problem using data $\vec{d} = \vec{z}^T$ and target $\vec{s} = \vec{b}^T = \begin{bmatrix} -3 \\ -1 \\ 4 \\ 6 \end{bmatrix}$ to estimate a scalar parameter p_{LS}.

Perform least squares on the system $\vec{d}p_{\text{LS}} = \vec{s}$ to estimate p_{LS}, and show your work.

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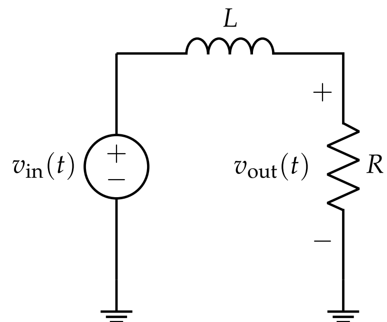
8. Pulse Response (12 pts.)

In this problem, we will explore the pulse response associated with RL circuits.

Suppose you have the following circuit:



(a) (4 pts.) **Write the differential equation for $v_{\text{out}}(t)$ in terms of $v_{\text{in}}(t)$, R, and L.**

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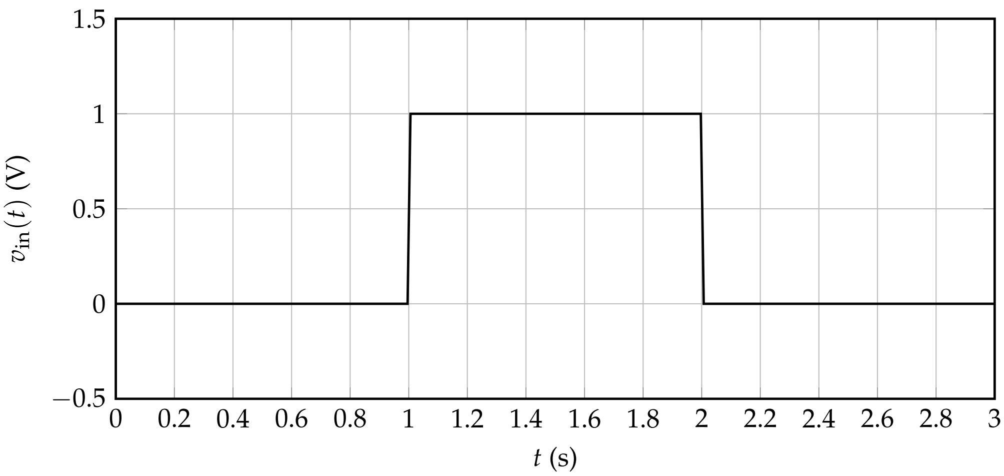
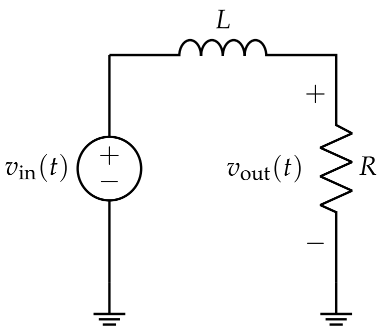
(b) (2 pts.) **What is the time constant τ of the circuit in terms of R and L?** Recall that the time constant will show up in the exponential term $e^{-\frac{t}{\tau}}$ that is characteristic of first order differential equations as the one obtained in the previous part.

(c) (8 pts.) Suppose the plot of input $v_{\text{in}}(t)$ is a pulse as shown below. Assume $v_R(0) = 0$.

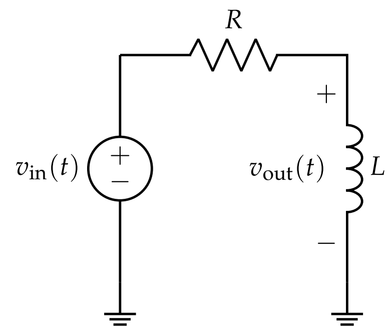


For this part, we will try and plot the output waveform of this circuit and the same circuit if the resistor and inductor were swapped with varying time constants.

Here are the two circuits for reference:

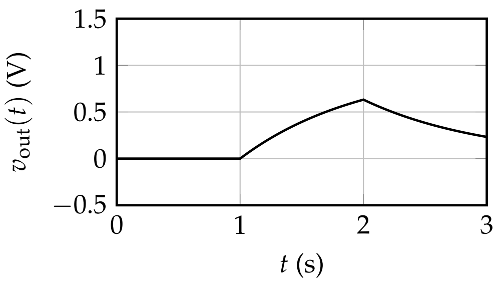


(a) Circuit 1

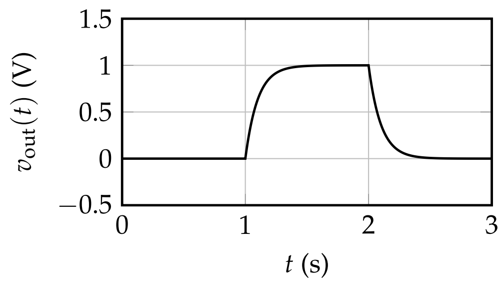


(b) Circuit 2

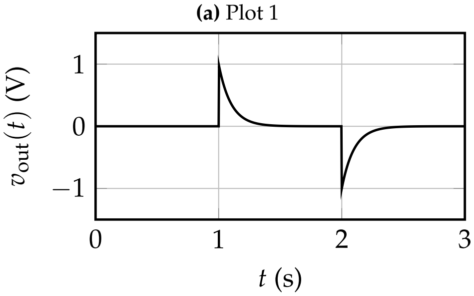
The following four plots will be used for this part:



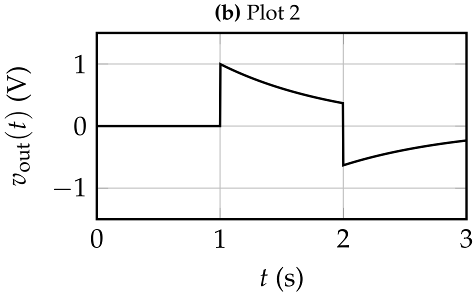
(a) Plot 1



(b) Plot 2



(c) Plot 3



(d) Plot 4

Fill out the provided table to match each combination of circuit and time constant to one of the provided plots. You should have one answer per row, but each column could have multiple selections.

	Plot 1	Plot 2	Plot 3	Plot 4
Circuit 1, $\tau = 0.1\,\text{s}$	○	○	○	○
Circuit 1, $\tau = 1\,\text{s}$	○	○	○	○
Circuit 2, $\tau = 0.1\,\text{s}$	○	○	○	○
Circuit 2, $\tau = 1\,\text{s}$	○	○	○	○

(HINT: Use your knowledge of the differential equation for Circuit 1. For Circuit 2, it may be helpful to use the KVL equation $v_{\text{in}}(t) = v_L(t) + v_R(t)$. You should also think about the steady state voltage for each circuit.)

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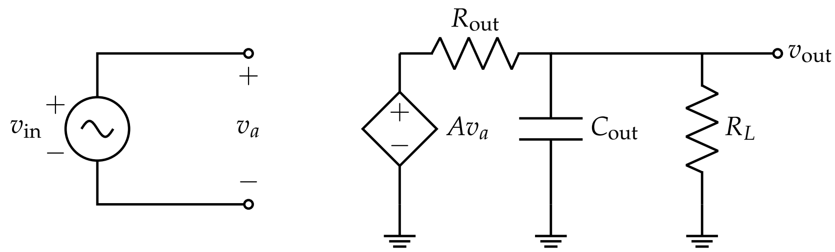
9. Amplifier Analysis (10 pts.)

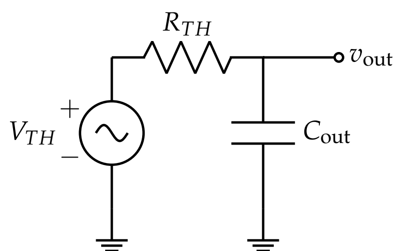
For this problem, we will examine the transfer function for an amplifier.

Suppose that we have the following circuit:



We want to find the transfer function $H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ and analyze its behavior.

(a) (6 pts.) To simplify the analysis, we will find the Thevenin equivalent circuit as seen by the capacitor C_{out} as follows:



Find $V_{TH} = A_0 v_{\text{in}}$ **and** R_{TH}**, where** A_0 **and** R_{TH} **are in terms of** A **and the resistances** R_{out} **and** R_L**.** *(HINT: For this scenario,* V_{TH} *will be voltage at the load (the capacitor) when the load (the capacitor) is replaced with an open circuit, and* R_{TH} *will be the equivalent resistance seen from the load (the capacitor) perspective with independent sources set to* 0*, which will be easiest found with series and parallel equivalence.)*

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(b) (4 pts.) **Find** $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ in terms of A_0, R_{TH}, and C_{out}.

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10. Stability Analysis with Bode Plots (13 pts.)

For this problem, we will use Bode plots to examine the stability of an amplifier.

Suppose that we analyze our amplifier circuit and are able to approximate the overall transfer function as a two-pole system:

$$T(\mathrm{j}\omega) = \frac{T_0}{(1+\mathrm{j}\frac{\omega}{\omega_{p1}})(1+\mathrm{j}\frac{\omega}{\omega_{p2}})} \tag{6}$$

We will assume that $\omega_{p1} \ll \omega_{p2}$ for the entire problem.

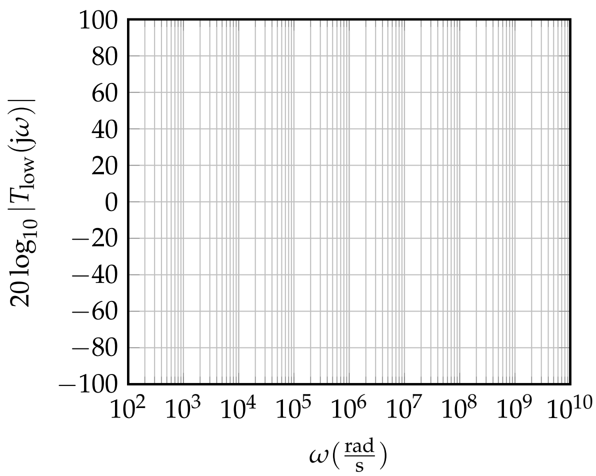
Important: Your answers should be made with the Bode straight-line approximation in place; with this approximation, your answers should all be relatively simple.

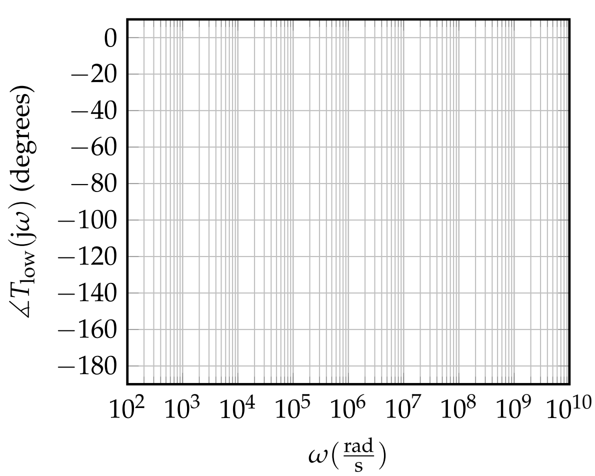
(a) (5 pts.) For $\omega \ll \omega_{p2}$, we can approximate the transfer function as a one-pole system with pole ω_{p1}:

$$T(\mathrm{j}\omega) \approx T_{\text{low}}(\mathrm{j}\omega) = \frac{T_0}{1+\mathrm{j}\frac{\omega}{\omega_{p1}}} \tag{7}$$

Let $T_0 = 10^3$ and $\omega_{p1} = 10^4 \frac{\text{rad}}{\text{s}}$. **Plot the Bode plots for $|T_{\text{low}}(\mathrm{j}\omega)|$ and $\measuredangle T_{\text{low}}(\mathrm{j}\omega)$.**

Based on the Bode plots, what is ω_u, defined as the frequency such that $|T_{\text{low}}(\mathrm{j}\omega_u)| = 1$?





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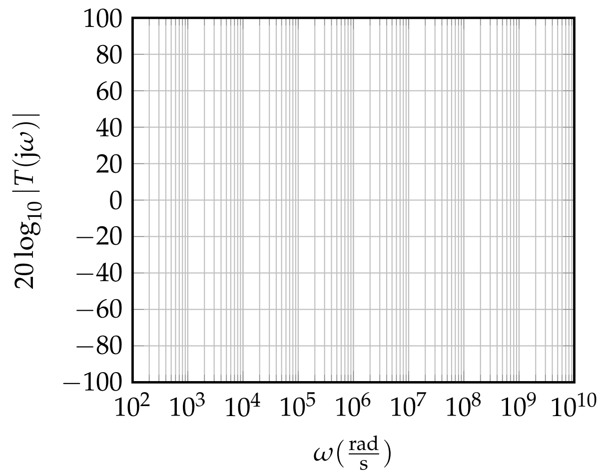
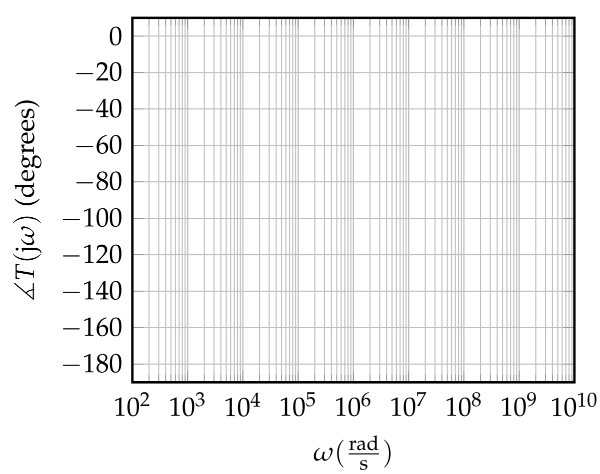
(b) (8 pts.) Now, we will work with the overall two-pole transfer function $T(\mathrm{j}\omega)$ (assume T_0 and ω_{p1} are the same as from the previous part).

Suppose that we are able to choose what frequency we want the second pole ω_{p2} at. For stability purposes, we want $\measuredangle T(\mathrm{j}\omega_u)$ to be reasonably above -180°, where ω_u is the same frequency as the one found in the previous problem; for this problem, we will choose $\measuredangle T(\mathrm{j}\omega_u) = -135^\circ$ to be the target phase.

Find the pole frequency ω_{p2} that causes $\measuredangle T(\mathrm{j}\omega_u) = -135^\circ$. Justify your answer by plotting the Bode plots for the two-pole transfer function $T(\mathrm{j}\omega) = \frac{T_0}{(1+\mathrm{j}\frac{\omega}{\omega_{p1}})(1+\mathrm{j}\frac{\omega}{\omega_{p2}})}$, with ω_{p2} as your chosen second pole. (Use your answer for the previous part for ω_u; even if your answer to that part is incorrect, you can still receive credit for this part if your answer is consistent with ω_u from the previous part.) *(HINT: You should have found that $10\omega_{p1} < \omega_u$ from the previous part, which means that the first pole ω_{p1} will have already contributed its -90° of phase by ω_u, which means that to have a total phase of -135° at ω_u, the second pole ω_{p2} must contribute -45° phase at ω_u. Think about at what frequency a pole contributes -45° phase; if you are unsure, draw the Bode plot, check to see if you pole frequency works, and adjust as necessary.)*





[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.

If needed, you can also use this space to work on problems. But if you want the work on this page to be graded, make sure you tell us on the problem's main page.]

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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