Section 0: Pre-exam questions (3 points)

1. What classes are you taking this term? (1 pt)

2. Tell us about something that makes you happy (2 pts)
3. High Speed Digital Communication (16 pts)

In this problem, we will analyze a simplified model of a USB communication link and show that there is a limit to how quickly you can transfer data.

As Figure 1 illustrates, the transmitter is a CMOS inverter, whose input is driven with the voltage source $V_{in}$ representing the data to be sent over the link. The USB cable connects the transmitter to the receiver. The link successfully transfers data when the voltage at the output of the USB cable crosses the threshold of the receiver’s inverter, thereby flipping the receiver output voltage $V_{out,Receiver}$.

To simplify, we will model the transmitter’s inverter as a resistance $R_{inv}$ and use $\bar{V}_{in}(t)$ as the digitally flipped voltage that represents the ideal output of the transmitter’s inverter. The receiver inverter is modeled as an input capacitance $C_{inv}$. The cable is modeled as an RC system whose $R_{wire}$ and $C_{wire}$ values grow as the cable length increases. A diagram of the circuit model to be used in this problem is shown in Figure 2.

(a) (2 pts) If $\bar{V}_{in} = 0V$ for all $t < 0$ what is $V_{out}$ at time $t = 0$? This will serve as the initial condition for the rest of the problem.
(b) (5 pts) Write the differential equation for the voltage $V_{out}(t)$ as a function of $V_{in}(t)$ for $t \geq 0$ in terms of $R_{inv}, C_{inv}, R_{wire},$ and $C_{wire}$. 

Figure 3: Simplified circuit model for a USB communication link

By KCL, the current through the resistor must equal the current through the capacitor:

$$V_{R} = IC$$

$$V_{in} = RC\frac{dV_{out}}{dt}$$

$$dV_{out}\frac{dt}{dt} = V_{in} - V_{out}(R_{inv} + R_{wire})(C_{inv} + C_{wire})$$
(c) (5 pts) For the rest of this problem, for ease of computation (since you don’t have calculators), assume that $R_{\text{inv}} = 1\,\text{k}\Omega$, $R_{\text{wire}} = 3\,\text{k}\Omega$, $C_{\text{wire}} = 24\,\text{pF}$, and $C_{\text{inv}} = 1\,\text{pF}$. (Here, pF = $10^{-12}$F and kΩ = $10^3$Ω.) Now assume that $\bar{V}_{\text{in}}$ for time $t \geq 0$ is a piecewise-constant voltage source. $\bar{V}_{\text{in}}$ rises to 1V at $t = 0$ and then falls back down to 0V at some time $t = T_{\text{bit}}$. Write an exact expression for $V_{\text{out}}(t)$ during the time period $0 \leq t < T_{\text{bit}}$. Your expression should be an explicit formula. No integrals are allowed to remain for full credit.

(HINT: Recall that $x_0 e^{\lambda t} + \int_0^t e^{\lambda (t-\tau)} u(\tau) d\tau$ is the unique solution of $\frac{d}{dt} x(t) = \lambda x(t) + u(t)$ with initial condition $x(0) = x_0$. You also might want to draw $\bar{V}_{\text{in}}(t)$ to help yourself.)
(d) (4 pts) Sketch $V_{out}(t)$ during the time period $0 \leq t < T_{bit}$, clearly marking the initial voltage and the final voltage.

4. RLC Circuits (30 pts)

Consider the following circuit:

If we define the state vector $\vec{x}(t) = \begin{bmatrix} V_C \\ I_L \end{bmatrix}$, standard circuit analysis would reveal that this circuit is governed by this system of differential equations:

$$\frac{d}{dt} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = A \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + B \begin{bmatrix} 0 \\ V_{in}(t) \end{bmatrix}.$$
For this problem generally, it may help to recall the standard formula for a $2 \times 2$ matrix inverse:

$$
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
$$

(a) (12 pts) Suppose that we choose to drive the circuit with a $V_{in}(t)$ that is a sinusoidal waveform at the angular frequency $\omega$ radians/sec. Let us consider the voltage $V_{C}(t)$ across the capacitor $C$ as our output voltage $V_{out}(t)$. What is the transfer function of this circuit, namely

$$H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$$

where $\tilde{V}_{in}$ is the input voltage phasor at angular frequency $\omega$ and $\tilde{V}_{out}$ is the output voltage phasor at that same angular frequency $\omega$.

Your answer should be symbolic in terms of the $R_1, L, C$, and $R_2$ along with $j = \sqrt{-1}$ and $\omega$. You don’t have to simplify this to look nice.
[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem’s main page.]
(b) (8 pts) Assume that initially the state is at rest, with the capacitor charged to 1V, so $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Consider the values $C = 1, L = 1, R_1 = 1, R_2 = 1$. For convenience, we have plugged in the values as well as computed eigenvalues and eigenvectors for you. This yields an $A$ matrix with eigenvalues $\lambda_1 = -1 + j$ and $\lambda_2 = -1 - j$.

\[
\frac{d}{dt} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{in}(t), \quad \text{and eigenvectors} \quad \vec{v}_{\lambda_1 = -1+j} = \begin{bmatrix} -j \\ 1 \end{bmatrix}, \vec{v}_{\lambda_2 = -1-j} = \begin{bmatrix} j \\ 1 \end{bmatrix}
\]

For convenience, please note that if $V = [\vec{v}_{\lambda_1 = -1+j}, \vec{v}_{\lambda_2 = -1-j}]$, then $V^{-1} = \begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{j}{2} & \frac{j}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Change coordinates of this differential equation to be in terms of $\vec{\tilde{x}}(t) = V^{-1} \vec{x}(t)$ and the input $V_{in}(t)$.

i.e. give an equation in the form $\frac{d}{dt} \vec{\tilde{x}}(t) = \vec{\tilde{A}} \vec{\tilde{x}}(t) + \vec{\tilde{b}} V_{in}(t)$.

What is the matrix $\vec{\tilde{A}}$, the vector $\vec{\tilde{b}}$ and the intial condition $\vec{\tilde{x}}(0)$?
(c) (10 pts) For the problem as stated in the previous part, solve for the transient solution $\ddot{x}(t)$ given the specified initial condition under the assumption that $V_{in}(t) = 0$ for all $t \geq 0$.

Your solution should be expressed in terms of real functions of time $t$. No $j$’s are permitted in your final answer for full credit.

5. Transfer Functions and Filters (18 pts)

   (a) (6 pts) Identify each of the Bode Plots, circuits, and transfer functions as either a lowpass or highpass filter. Indicate your answer by filling in the appropriate bubble.
Table 1: Table to be filled in for your answers. Fill in bubbles.

<table>
<thead>
<tr>
<th></th>
<th>Lowpass</th>
<th>Highpass</th>
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</thead>
<tbody>
<tr>
<td>Bode Plot A</td>
<td></td>
<td></td>
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<tr>
<td>Bode Plot B</td>
<td></td>
<td></td>
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<tr>
<td>Circuit C</td>
<td></td>
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<tr>
<td>Circuit D</td>
<td></td>
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<tr>
<td>Transfer Fn E</td>
<td></td>
<td></td>
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<tr>
<td>Transfer Fn F</td>
<td></td>
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</tr>
</tbody>
</table>

Bode Plot A ($\omega_c = 10^6$)

![Bode Plot A Graph](image)

Bode Plot B ($\omega_c = 10^6$)

![Bode Plot B Graph](image)

Circuit C: $H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_i}$

Circuit D: $H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_i}$

Transfer function E: $H_E(j\omega) = \frac{j\omega}{1 + j\omega \omega_c}$

Transfer function F: $H_F(j\omega) = \frac{1}{1 + j\omega \omega_c}$
(b) (6 pts) Consider the three filters in cascade below, with unity-gain op-amp buffers in between them:

![Diagram of three filters cascaded via unity-gain op-amp buffers](image)

Figure 5: Three filters cascaded via unity-gain op-amp buffers

Suppose that at some frequency $\omega_0$ radians/sec we know that:

$$ H_1(j\omega_0) = 3e^{j\frac{\pi}{4}} \quad H_2(j\omega_0) = \frac{1}{2}e^{-j\frac{\pi}{2}} \quad H_3(j\omega_0) = 4e^{j\frac{5\pi}{6}} $$

If $V_{in}(t) = 2\sin(\omega_0 t + \frac{\pi}{2})$:

What is the phasor for the input voltage: $\tilde{V}_{in}$?

What is the phasor for the output voltage: $\tilde{V}_{out}$?

What is $V_{out}(t)$?
(c) (6 pts) Suppose there is an interfering signal at 5GHz that you need to get rid of, while passing through your WiFi signal at 2.4GHz. You have access to the following six components (two capacitors, two inductors, and two resistors) which should each only be used exactly once.

\[ C = 66.3\text{pF} \quad C = 31.8\text{pF} \quad L = 31.8\text{pH} \quad L = 66.3\text{pH} \quad R = 1\Omega \quad R = 0.1\Omega \]

Assign each component to the elements \( R_A, R_B, C_A, C_B, L_A, L_B \) in the RLC circuits (‘A’ and ‘B’ below), so that the transfer function of each circuit corresponds to its matching Bode plot. Write values next to components. Hint: the dashed line on the plot is at \( \frac{1}{\sqrt{2}} \). It might be useful to think about what its intersections with the main curve represent.

For your convenience, here are some calculations that may or may not be relevant:

\[
\begin{align*}
\frac{2.4 \times 10^9}{2\pi} & = 382 \times 10^6 & 2.4 \times 10^9 \times 2\pi & = 15.08 \times 10^9 & \frac{5.0 \times 10^9}{2\pi} & = 795 \times 10^6 & 5.0 \times 10^9 \times 2\pi & = 31.4 \times 10^9 \\
\frac{1}{382 \times 10^6} & = 2.62 \times 10^{-9} & \frac{1}{15.08 \times 10^9} & = 6.63 \times 10^{-11} & \frac{1}{795 \times 10^6} & = 1.26 \times 10^{-9} & \frac{1}{31.4 \times 10^9} & = 3.18 \times 10^{-11}
\end{align*}
\]
For the band-pass filter, the peak is located at the frequency where the sum of the impedances of the inductor and capacitor becomes 0. The same is true for the center of the notch-filter.

\[ Z_L + Z_C = 0 \]

\[ j\omega n L - j\omega n C = 0 \]

\[ \omega^2 n L = \frac{1}{C} \]

\[ \omega n = \frac{1}{\sqrt{LC}} \]

Since the notch frequency is slightly more than 2x the band-pass frequency, the \( \sqrt{LC} \) value of the band-pass filter must be around 2x bigger than that of the band pass filter. Thus we will choose the pair of larger \( L \) and \( C \) values for the band-pass filter, each of which are each roughly 2x larger:

\[ L_A = 66.3 \text{ pH} \]

\[ C_A = 66.3 \text{ pF} \]

This can be confirmed by using the provided calculations. Since each component can only be used once, the notch filter must use:

\[ L_B = 31.8 \text{ pH} \]

\[ C_B = 31.8 \text{ pF} \]

From looking at the dotted lines on the plots, it is clear that the notch filter has much larger bandwidth than the band-pass filter. Let's evaluate the transfer function of the band-pass filter to determine the relationship between resistance and the cutoff frequencies. From the voltage divider formula:

\[ H(j\omega) = \frac{R_A}{R_A + j(\omega L - \frac{1}{\omega C})} \]

The magnitude of this transfer function will hit \( 1/\sqrt{2} \) when the magnitude of the imaginary component of the denominator equals \( R_A \). Clearly, a larger \( R_A \) leads to larger cutoff frequency. Alternatively, you could recall that the bandwidth of a band-pass filter is \( \sqrt{R_L} \). Since the band-pass filter has a much smaller bandwidth than the notch-filter, we will choose the smaller resistance \( R_A = 0.1 \Omega \) and \( R_B = 1 \Omega \).
6. Separation of Variables and Uniqueness (20 pts)

Recall that the classic scalar differential equation

$$\frac{dx}{dt} = \lambda x(t)$$  \hspace{1cm} (1)

with initial condition $x(0) = x_0 \neq 0$ has the unique solution $x(T) = x_0 e^{\lambda T}$ for all $T \geq 0$.

(Note: to avoid variable-name confusion here, we are using $T$ as the argument of the solution $x(T)$.)

The separation of variables approach to getting a guess for this problem would proceed as follows:

\[
\frac{dx}{dt} = \lambda x(t) \hspace{1cm} (2) \\
\frac{dx}{dt} = \lambda x \hspace{1cm} (3) \\
\frac{dx}{x} = \lambda dt \hspace{1cm} \text{separating variables to sides} \hspace{1cm} (4) \\
\int_{x_0}^{x(T)} \frac{dx}{x} = \int_0^T \lambda dt \hspace{1cm} \text{integrating both sides from where they start to where they end up} \hspace{1cm} (5) \\
\ln(x(T)) - \ln(x_0) = \lambda T \hspace{1cm} (6) \\
\ln(x(T)) = \ln(x_0) + \lambda T \hspace{1cm} (7) \\
x(T) = x_0 e^{\lambda T} \hspace{1cm} \text{exponentiating both sides} \hspace{1cm} (8)
\]

and in this case it gave a good guess. Of course, this guess needed to be justified by a uniqueness proof, which you did in the homework.

This exam problem asks you to carry out this program for the time-varying differential equation:

$$\frac{dx}{dt} = \lambda(t) x(t)$$  \hspace{1cm} (9)

with initial condition $x(0) = x_0 \neq 0$. You can assume that $\lambda(t)$ is a nice continuously differentiable function of time $t$ that is bounded.
(a) (8 pts) Use the separation of variables approach to get a guess for the solution to the differential equation (9) — namely \( \frac{dx}{dt} = \lambda(t)x(t) \) — with initial condition \( x(0) = x_0 \neq 0 \). **Show work and give a formula for \( x(T) \) for \( T \geq 0 \).**

*(HINT: It is fine if your answer involves a definite integral.)*

*(If you can’t solve this for a general \( \lambda(t) \), for partial credit, feel free to just consider the special case of \( \lambda(t) = -2 - \sin(t) \) and give a guess for that case.)*

*(You can also get full credit if you follow the approach from discussion section of taking a piecewise-constant approximation and then taking a limit, but that might involve more work.)*
(b) (12 pts) Prove the uniqueness of the solution — i.e. that if any function solves differential equation
(9) — namely \( \frac{d}{dt}x(t) = \lambda(t)x(t) \) — with the given initial condition \( x(0) = x_0 \neq 0 \), then it must in
fact be the same as your guessed solution everywhere for \( T \geq 0 \).

(HINT: A ratio-based argument might be useful. You don’t actually need to know the exact form of
your guessed solution to carry out much of this argument, but you do need the fact that it is never zero
and that it solves (9).)
Looking at the derivative of $z(t)$, we have:

$$\frac{d}{dt} z(t) = \frac{d}{dt} y(t)x(t) \quad (20)$$

$$= \left( \frac{d}{dt} y(t) \right)x(t) - y(t) \frac{d}{dt} x(t) \quad (21)$$

$$= \lambda(t)y(t)x(t) - y(t)\lambda(t)x(t) \quad (22)$$

$$= 0. \quad (23)$$

Since $\frac{d}{dt} z(t) = 0$ for all $t \geq 0$, it is not changing, and is therefore a constant. This means that $z(t) = z(0) = 1$ which implies that $y(t)x(t) = 1$ which means that $y(t) = x(t)$, thereby establishing uniqueness for our guessed solution.
[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem’s main page.]