Note 8: Vector Differential Equations

1 Introduction to Vector Differential Equations

1.1 Motivation

In the first part of the class we studied how to solve first order linear differential equations. We analyzed in particular scalar ODE, meaning that we solved for just one variable, for example in the capacitor ODE we usually solved for $v_c(t)$. But what if there are multiple unknowns we want to solve for at the same time? Another reason we are teaching you this concept is to understand how computers solve hard circuits using linear algebra.

Now, we are going to expand our tool set and learn how to solve multivariable differential equations. (You will see in homeworks that these exact same ideas will apply to solve higher-order differential equations where we have second and third derivatives involved.) Let's motivate the need to understand two dimensional systems with the circuit in fig. 1.

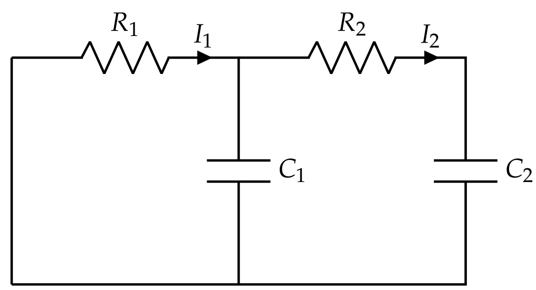

Figure 1: Circuit forming a two-dimensional system.

Suppose we want to model the voltage across both C_1 and C_2 simultaneously. As you may notice (through KCL and NVA), it is not possible to model the two capacitors independently. Let $V_1(t)$ denote the voltage across capacitor C_1 and $V_2(t)$ denote the voltage across capacitor C_2. Applying NVA and KCL, we obtain the following two differential equations:

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_1 + \frac{1}{R_2C_1}V_2 \tag{1}$$

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = \frac{1}{R_2C_2}V_1 - \frac{1}{R_2C_2}V_2 \tag{2}$$

Concept Check: Derive these differential equations using KCL and NVA. Ensure to simplify your equations to only include V_1, V_2, R_1, and R_2.

The differential equation for V_1 includes both V_1 and V_2 terms, and likewise for the differential equation for V_2[1]. We cannot use techniques we have previously covered to solve these kinds of differential equations, so we will have to derive new techniques.

[1]We typically refer to this as *coupled* differential equations

1.2 Matrix Form for Differential Equations

We will refer to a collection of differential equations, similar to eq. (1) and eq. (2), as a *system* of differential equations. The terminology is analogous to systems of linear equations.

Definition 1 (Vector Differential Equations)

Suppose we are given a system of linear differential equations of the form

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) \tag{3}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t) \tag{4}$$

$$\vdots \tag{5}$$

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t) \tag{6}$$

$$\tag{7}$$

with initial conditions $x_1(t_0) = k_1, x_2(t_0) = k_2, \ldots, x_n(t_0) = k_n$ for some constants $a_{ij} \in \mathbb{R}$ and $k_i \in \mathbb{R}$. We may combine them into a single *vector differential equation* of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) \tag{8}$$

with initial condition $\vec{x}(t_0) = \vec{k}$ where

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{9}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \tag{10}$$

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \tag{11}$$

We refer to $\vec{x}(t)$ as our "state".

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_n(t)}{\mathrm{d}t} \end{bmatrix} \tag{12}$$

Example:
We may turn the system of differential equations in eq. (1) and eq. (2) into a vector differential equation by pattern matching $x_1(t) := V_1(t)$, $x_2 := V_2(t)$, $a_{11} := -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)$, $a_{12} := \frac{1}{R_2C_1}$, $a_{21} := \frac{1}{R_2C_2}$, and $a_{22} := -\frac{1}{R_2C_2}$. Applying the symbolic simplifications in Definition 1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\vec{x}(t)} \tag{13}$$

In the next section, we will go over how to solve these kinds of vector differential equations, for a special case where A is *diagonalizable*.

1.3 State Space Representation

State Space Representation is a method of modeling dynamical systems by a set of input, output, and state variables described by first-order differential equations. In this framework, the system's state at any given time is represented by a vector within a multidimensional space, where each dimension corresponds to one state variable. The state space model consists of sets of equations, the state equations, which describe how the state of the system evolves over time.

By focusing on solving VDEs within the state space representation, we emphasize a structured approach to modeling and analyzing the dynamics of complex systems, providing a clear path from inputs through system states to outputs, all governed by the the vector differential equations.

2 Solving Diagonalizable Vector Differential Equations

2.1 Motivation

Suppose that, from the previous section, we had

$$A := \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix} \tag{14}$$

for some constants $a_i \in \mathbb{R}$. Indeed, this would make our differential equation easier to solve since we would have

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) \tag{15}$$

$$\implies \begin{bmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} a_1x_1(t) \\ a_2x_2(t) \\ \vdots \\ a_nx_n(t) \end{bmatrix} \tag{16}$$

which we could solve row-by-row. That is, we can solve $\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_1x_1(t)$ first, then $\frac{\mathrm{d}x_2}{\mathrm{d}t} = a_2x_2(t)$, etc. We will use the help of diagonalization to obtain this end goal.

2.2 Diagonalizing Matrices

Definition 2 (Diagonalizability)
A square matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable (i.e., it can be diagonalized) if it admits n linearly independent eigenvectors[a]. If A is diagonalizable, it can be diagonalized as follows:

$$A = V\Lambda V^{-1} \tag{17}$$

where

$$V := \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \tag{18}$$

$$\Lambda := \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{19}$$

[a]Note that it is not always the case for a matrix to have linearly independent eigenvectors. Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example.

Theorem 3 (Diagonalization)
We will use the definition of V and Λ from Definition 2. If $\vec{v}_1, \ldots, \vec{v}_n$ are the eigenvectors of A, then $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues.

Proof. We can show that, if the columns of V are the eigenvectors of A, then Λ will be a diagonal matrix with the corresponding eigenvalues along the diagonal. We know that we can write $A = V\Lambda V^{-1}$ from Definition 2. This is equivalent to writing $\Lambda = V^{-1}AV$. Let ℓ_i be the eigenvalue of A corresponding to $\vec{v}_i$. Now,

$$\Lambda = V^{-1}AV \tag{20}$$

$$= V^{-1}A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \tag{21}$$

$$= V^{-1}\begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix} \tag{22}$$

$$= V^{-1}\begin{bmatrix} \ell_1\vec{v}_1 & \ell_2\vec{v}_2 & \cdots & \ell_n\vec{v}_n \end{bmatrix} \tag{23}$$

$$= V^{-1}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \ell_1 & 0 & 0 & \cdots & 0 \\ 0 & \ell_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \ell_n \end{bmatrix} \tag{24}$$

$$= V^{-1}V\begin{bmatrix} \ell_1 & 0 & 0 & \cdots & 0 \\ 0 & \ell_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \ell_n \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} \ell_1 & 0 & 0 & \cdots & 0 \\ 0 & \ell_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \ell_n \end{bmatrix} \tag{26}$$

and hence, $\lambda_i = \ell_i$. Note that in steps eq. (22) and eq. (24), we use facts of the matrix multiplication algorithm to distribute the A inside V and to split up the algebraic multiplication into matrix multiplication, respectively. □

Note: This result also holds if λ_i and $\vec{v}_i$ are complex, i.e., $\lambda_i \in \mathbb{C}$ and $\vec{v}_i \in \mathbb{C}^n$.

Example:
Consider again the circuit in Figure 1. Let $C_1 = C_2 = 1\,\mu\text{F}$, $R_1 = \frac{1}{3}\text{M}\Omega$ and $R_2 = \frac{1}{2}\text{M}\Omega$.[2] This means that $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. We can find its eigenvalues by solving for λ in $\det\{A - \lambda I_2\} = 0$. With I_2 being the identity matrix, and λ the unknown we want to solve for.

From this, we obtain $\lambda_1 = -6$ and $\lambda_2 = -1$. To find $\vec{v}_1$, the corresponding eigenvector for λ_1, we can find a basis for $\text{Null}(A - \lambda_1 I_2)$. Being $\text{Null}()$ the Null space of that matrix. Doing this yields $\vec{v}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$. Applying a similar process for λ_2, we obtain $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. These eigenvectors are linearly independent, so we can diagonalize this matrix. Hence,

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \tag{27}$$

$$\Lambda = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \tag{28}$$

Notice that, with this definition of Λ, we can try to use it to obtain the goal mentioned in Section 2.1. For this idea to work, we need to introduce a *change of basis*.

2.3 Change of Basis

Key Idea 4 (Bases and Basis Coefficients of Vectors)
Any vector in $\mathbb{R}^n$ is fundamentally written in a certain basis that spans $\mathbb{R}^n$. Typically, we implicitly use the *standard basis*. If we have $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, this is equivalent to writing

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n \tag{29}$$

[2] The SI prefixes 'M' and 'μ' stand for "mega" and "micro," and correspond to the decimal multiples of 10^6 and 10^{-6} respectively.

where $\vec{e}_i$ is a vector of all zeros, except for a 1 in the ith entry. We also call $\vec{e}_i$ the ith standard basis vector.

Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ represent a basis for $\mathbb{R}^n$, with each $\vec{v}_i \in \mathbb{R}^n$. This means that we can represent any vector (written in standard basis) $\vec{x} \in \mathbb{R}^n$ as

$$\vec{x} = z_1\vec{v}_1 + z_2\vec{v}_2 + \ldots + z_n\vec{v}_n \tag{30}$$

for some constants $z_i \in \mathbb{R}$. These constants are known as the *coefficients* of $\vec{x}$ in the basis given by $\{\vec{v}_1, \ldots, \vec{v}_n\}$. That is, $\vec{z} := \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ is the representation of $\vec{x}$ in this basis.

Theorem 5 (Change of Basis)
Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ represent a basis for $\mathbb{R}^n$, with each $\vec{v}_i \in \mathbb{R}^n$. Define this as V-basis. Suppose we are given $\vec{x}$ in standard basis, and further suppose we wish to find $\vec{z}$, the representation of $\vec{x}$ in V-basis. We can accomplish this by computing

$$\vec{z} = V^{-1}\vec{x} \tag{31}$$

where $V := \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Proof. It first needs to be shown that V is invertible. We know that $\vec{v}_1, \ldots, \vec{v}_n$ must be a linearly independent collection of vectors, since they span $\mathbb{R}^n$. Hence, $V := \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \in \mathbb{R}^{n \times n}$ is a square matrix with linearly independent columns, meaning it is invertible. Now, we can equivalently show that $\vec{x} = V\vec{z}$:

$$\vec{x} = V\vec{z} \tag{32}$$

$$= \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}_{\vec{z}} \tag{33}$$

$$= z_1\vec{v}_1 + z_2\vec{v}_2 + \ldots + z_n\vec{v}_n \tag{34}$$

which is exactly what we have in eq. (30). □

Example:
Let V be given as in eq. (27). Consider the basis of $\mathbb{R}^2$ that is spanned by the columns of V (recall that V has linearly independent columns). Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and suppose we want to find the coefficients of $\vec{x}$ in V-basis,

namely $\vec{z}$. Using the matrix inversion formula for a 2×2 matrix, we find that $V^{-1} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$. Hence,

$$\vec{z} = V^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix} \tag{35}$$

Now we can use these results to derive a process to solve vector differential equations, where A is diagonalizable.

2.4 Solving Vector Differential Equations

Theorem 6 (Diagonalizing a Vector Differential Equation)
Suppose we are given a vector differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) \tag{36}$$

with initial condition $\vec{x}(t_0) = \vec{k}$. Further suppose $A \in \mathbb{R}^{n \times n}$ is a diagonalizable matrix which can be written as $A = V\Lambda V^{-1}$. The solution to eq. (36) is given by $\vec{x}(t) = V\vec{z}(t)$, where $\vec{z}(t)$ is the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t) \tag{37}$$

with initial condition $\vec{z}(t_0) = V^{-1}\vec{k}$

Proof. First, we can rewrite eq. (36) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) \tag{38}$$

We can also notice that, since the derivative operator is linear,

$$M\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(M\vec{x}(t)) \tag{39}$$

for any matrix $M \in \mathbb{R}^{m \times n}$. Applying this fact,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) \tag{40}$$

$$V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \Lambda V^{-1}\vec{x}(t) \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(V^{-1}\vec{x}(t)\right) = \Lambda V^{-1}\vec{x}(t) \tag{42}$$

Notice that $\vec{z}(t) := V^{-1}\vec{x}(t)$ gives the coefficients of $\vec{x}(t)$ in V-basis, as covered in Section 2.3. Hence, we may further simplify as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(V^{-1}\vec{x}(t)\right) = \Lambda V^{-1}\vec{x}(t) \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t) \tag{44}$$

This is exactly the vector differential equation in eq. (37). To find the initial condition, we know that $\vec{x}(t_0) = \vec{k}$. We find the coefficients of this vector in V-basis and obtain $\vec{z}(t_0) = V^{-1}\vec{x}(t_0) = V^{-1}\vec{k}$ as desired. To recover $\vec{x}(t)$ from $\vec{z}(t)$, we undo the change of basis, i.e., $\vec{x}(t) = V\vec{z}(t)$. □

Key Idea 7 (Diagonalizing to Solve an Easier Vector Differential Equation)

Notice that the differential equation given in eq. (37) is a diagonal system, in that it matches the desired form in Section 2.1. That is, we can write out the differential equation more explicitly as

$$\frac{\mathrm{d}}{\mathrm{d}t}z_1(t) = \lambda_1 z_1(t) \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z_2(t) = \lambda_2 z_2(t) \tag{46}$$

$$\vdots \tag{47}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z_n(t) = \lambda_n z_n(t) \tag{48}$$

with initial conditions $z_1(t_0) = \widetilde{k}_1, z_2(t_0) = \widetilde{k}_2, \ldots, z_n(t_0) = \widetilde{k}_n$, where $\widetilde{\vec{k}} := V^{-1}\vec{k}$ for $\vec{k}$ as defined in Theorem 6. A graphical representation of this strategy to solve a vector differential equation is visualized in Figure 2.

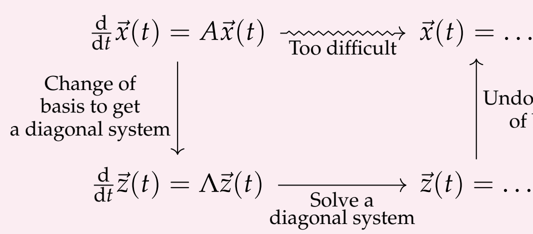

Figure 2: A Strategy to Solve eq. (36)

Example:

Using this strategy, we can solve the circuit example in Figure 1, with initial condition $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Using the V and Λ from eq. (27) and eq. (28) respectively, we have the following differential equation for $\vec{z}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}\vec{z}(t) \tag{49}$$

with initial condition $\vec{z}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix}$, which is taken from eq. (35). We can explicitly write this out as

$$\frac{\mathrm{d}}{\mathrm{d}t}z_1(t) = -6z_1(t) \tag{50}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z_2(t) = -z_2(t) \tag{51}$$

with initial conditions $z_1(0) = -\frac{1}{\sqrt{5}}$ and $z_2(0) = \frac{3}{\sqrt{5}}$. Notice that, unlike in eq. (1) and eq. (2), the system here is un-coupled, so we can solve it directly. Using techniques covered previously, we have

$$z_1(t) = -\frac{1}{\sqrt{5}}\mathrm{e}^{-6t} \tag{52}$$

$$z_2(t) = \frac{3}{\sqrt{5}}e^{-t} \tag{53}$$

so $\vec{z}(t) = \begin{bmatrix} -\frac{1}{\sqrt{5}}e^{-6t} \\ \frac{3}{\sqrt{5}}e^{-t} \end{bmatrix}$. Now, to undo the change of basis and recover $\vec{x}(t)$, we compute

$$\vec{x}(t) = V\vec{z}(t) = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}}e^{-6t} \\ \frac{3}{\sqrt{5}}e^{-t} \end{bmatrix} \tag{54}$$

$$= \begin{bmatrix} \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t} \end{bmatrix} \tag{55}$$

The voltage curves are plotted below in Figure 3.

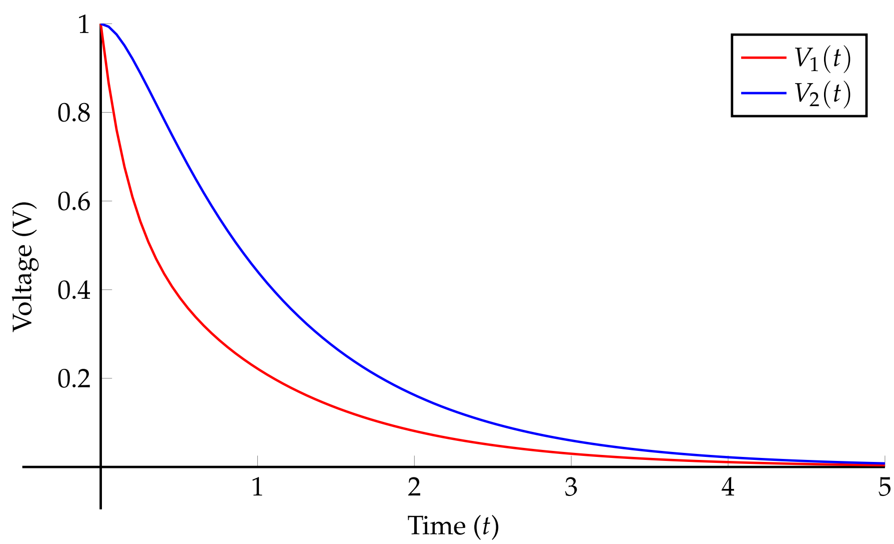

Figure 3: Initial Conditions: $V_1(0) = 1\,\mathrm{V}$ and $V_2(0) = 1\,\mathrm{V}$.

Corollary 8 (Non-homogeneous Vector Differential Equations)
Suppose we are given a vector differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{56}$$

with initial condition $\vec{x}(t_0) = \vec{k}$ and with $\vec{u}(t) : \mathbb{R}_+ \to \mathbb{R}^n$. Further suppose $A \in \mathbb{R}^{n\times n}$ is a diagonalizable matrix which can be written as $A = V\Lambda V^{-1}$, and $B \in \mathbb{R}^n$. The solution to eq. (56) is given by $\vec{x}(t) = V\vec{z}(t)$, where $\vec{z}(t)$ is the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t) + V^{-1}B\vec{u}(t) \tag{57}$$

with initial condition $\vec{z}(t_0) = V^{-1}\vec{k}$

Proof. The proof of this is almost exactly the same as the proof of Theorem 6, and the additional algebraic portions have been left as an exercise to the reader. It should be noted that we can "relabel" B where

$\widetilde{B} := V^{-1}B$. In doing this, we can see that we still have a system of un-coupled equations, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}z_1(t) = \lambda_1 z_1(t) + \widetilde{b}_1\vec{u}(t) \tag{58}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z_2(t) = \lambda_2 z_2(t) + \widetilde{b}_2\vec{u}(t) \tag{59}$$

$$\vdots \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z_n(t) = \lambda_n z_n(t) + \widetilde{b}_n\vec{u}(t) \tag{61}$$

$$\tag{62}$$

where $\widetilde{b}_i$ is the ith entry of $\widetilde{B}$, and $\widetilde{b}_1\vec{u}(t)$ is a scalar product operation. We can solve each equation individually, using previous knowledge regarding first order ode. □

3 LC Tank Example Revisited

This section is an extended example to demonstrate how to apply vector differential equations to the LC circuit from the previous note.

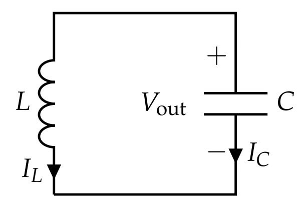

Figure 4: An LC Tank.

We can model $V_{\text{out}}(t)$ using vector differential equations. Suppose that $V_{\text{out}}(0) = 0$ and $I_L(0) = 1\,\text{A}$.

3.1 Deriving the Differential Equations

We will use KCL and NVA to derive the system of differential equations that models this circuit. NVA gives us

$$V_L = V_C = V_{\text{out}} \tag{63}$$

KCL gives us

$$I_L = -I_C = -C\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t} \tag{64}$$

$$\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t} = -\frac{1}{C}I_L \tag{65}$$

and NVA again gives us

$$V_L = V_{\text{out}} = L\frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{66}$$

$$L\frac{\mathrm{d}I_L}{\mathrm{d}t} = V_{\text{out}} \tag{67}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{1}{L}V_{\text{out}} \tag{68}$$

Notice that we have derivatives of $I_L(t)$ and $V_L(t)$, so we can make these state variables. Arranging this as a matrix differential equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix}}_{\vec{x}(t)} \tag{69}$$

3.2 Solving the Matrix Differential Equation

It happens to be the case A is diagonalizable here. We can solve this matrix differential equation using by first diagonalizing, then performing a change of basis, solving a diagonal system, and then undoing the change of basis. We can find the eigenvalues by solving for λ in

$$\det\{A - \lambda I_2\} = 0 \tag{70}$$

which yields $\lambda_1 = \mathrm{j}\frac{1}{\sqrt{LC}}$ and $\lambda_2 = -\mathrm{j}\frac{1}{\sqrt{LC}}$. We can find $\vec{v}_1$, the eigenvector for λ_1, by finding a basis for $\text{Null}(A - \lambda_1 I)$. Computing this gives $\vec{v}_1 = \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$. We perform a similar operation with λ_2 and obtain $\vec{v}_2 = \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$. Hence, we have

$$\Lambda = \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix} \tag{71}$$

$$V = \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} & -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix} \tag{72}$$

$$\implies V^{-1} = \frac{1}{2\mathrm{j}\sqrt{\frac{L}{C}}}\begin{bmatrix} 1 & \mathrm{j}\sqrt{\frac{L}{C}} \\ -1 & \mathrm{j}\sqrt{\frac{L}{C}} \end{bmatrix} \tag{73}$$

The new differential equation for $\vec{z}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\vec{z}(t) \tag{74}$$

with initial condition $\vec{z}(0) = V^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Solving this diagonal system, we see that

$$\vec{z}(t) = \begin{bmatrix} \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} \\ \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}} \end{bmatrix} \tag{75}$$

Undoing the change of variables to find $\vec{x}(t)$, we obtain

$$\vec{x}(t) = V\vec{z}(t) \tag{76}$$

$$= \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} & -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} \\ \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}} \end{bmatrix} \tag{77}$$

$$= \begin{bmatrix} \sqrt{\frac{L}{C}}\left(\frac{\mathrm{j}}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} - \frac{\mathrm{j}}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}}\right) \\ \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}} \end{bmatrix} \tag{78}$$

Using Euler's formula ($\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$), we can simplify the above to obtain

$$\vec{x}(t) = \begin{bmatrix} -\sqrt{\frac{L}{C}}\sin\left(\frac{t}{\sqrt{LC}}\right) \\ \cos\left(\frac{t}{\sqrt{LC}}\right) \end{bmatrix} \tag{79}$$

so we have $V_{\text{out}}(t) = -\sqrt{\frac{L}{C}}\sin\left(\frac{t}{\sqrt{LC}}\right)$ and $I_L(t) = \cos\left(\frac{t}{\sqrt{LC}}\right)$.

3.3 Visualizing $V_{\text{out}}(t)$, $I_L(t)$, and Energy

A plot of $I_L(t)$ and $V_{\text{out}}(t)$ will resemble the graph in Figure 5.

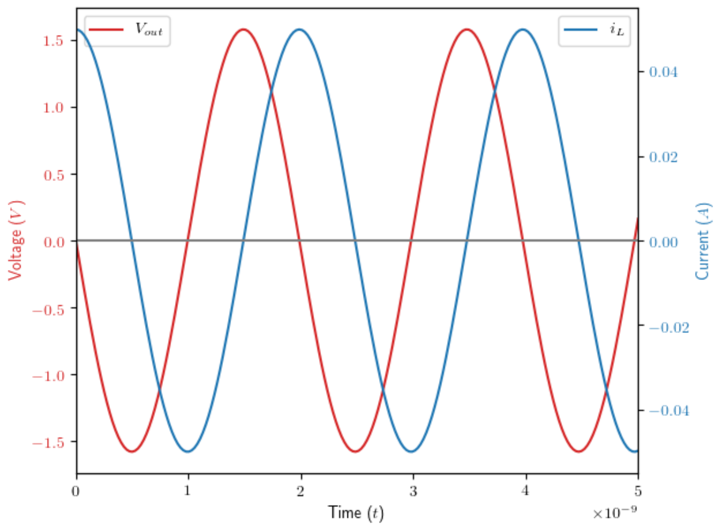

Figure 5: Voltage and Current response of LC Tank

We can find the energy in the capacitor and inductor respectively:

$$E_C = \frac{1}{2}CV_{\text{out}}^2 = \frac{L}{2}\sin^2\left(\frac{t}{\sqrt{LC}}\right) \tag{80}$$

$$E_L = \frac{1}{2}LI_L^2 = \frac{L}{2}\cos^2\left(\frac{t}{\sqrt{LC}}\right) \tag{81}$$

Notice that $E_C + E_L = \frac{L}{2}$, so the energy is constant over time. This is expected, since physics tells us that energy in this closed system should be conserved. A plot of E_C and E_L will resemble Figure 6.

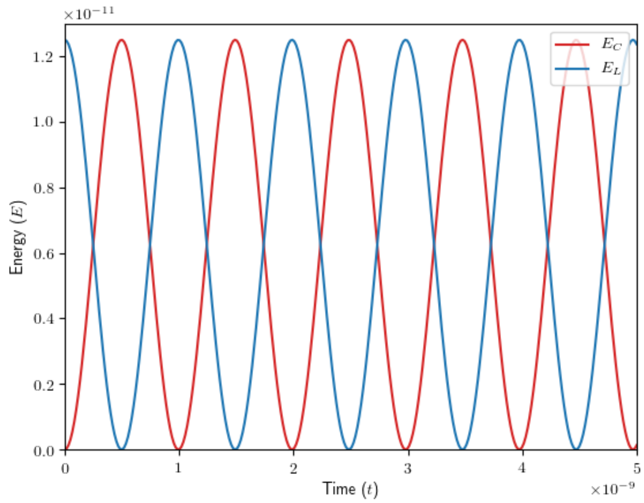

Figure 6: Energy stored in Inductor and Capacitor. Notice the sum is constant.

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