Note 17: Linearization

1 Overview

Thus far, we have covered only *linear* control models. Let's consider the models we have seen in this class (without input for simplicity).

Discrete time:

$$\vec{x}[i+1] = A\vec{x}[i] \tag{1}$$
$$\vec{x}[0] = \vec{x}_0 \tag{2}$$

Or continuous time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) \tag{3}$$
$$\vec{x}(0) = \vec{x}_0. \tag{4}$$

However, many systems in the real world can only be faithfully represented by *nonlinear* models. A very nonexhaustive list of such models follows.

1. Up until now, we have considered transistors to be binary – turning off or on at some voltage differential. But in reality, transistors work continuously, and the governing equations are highly nonlinear.
2. Robotics systems have in general highly nonlinear dynamics.
3. Machine learning and optimization also have highly nonlinear systems.

More generally, a nonlinear model takes the generic form of a difference equation or a differential equation, which we formally define here.

Model 1 (Discrete-Time Time-Invariant Difference Equation Model)
The model is of the form

$$\vec{x}[i+1] = \vec{f}(\vec{x}[i]) \tag{5}$$
$$\vec{x}[0] = \vec{x}_0 \tag{6}$$

for $\vec{x}\colon \mathbb{N} \to \mathbb{R}^n$ the state vector as a function of the timestep and $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ a function.

Model 2 (Continuous-Time Time-Invariant Differential Equation Model)
The model is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t)) \tag{7}$$
$$\vec{x}(0) = \vec{x}_0 \tag{8}$$

for $\vec{x}\colon \mathbb{R}_+ \to \mathbb{R}^n$ the state vector as a function of the timestep and $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ a function.

We will learn how to approximate these models **locally** by linear models, in a process called *linearization* of the function $\vec{f}$.

Key Idea 3 (Local Linearization)
Linearization of a function is the technique of approximating it by a linear function "locally" (i.e., in a small region around some point).

2 Linearization

2.1 Linear Approximations

The key to linearization is the *first derivative* concept. Recall the familiar limit definition of the derivative for a function $f\colon \mathbb{R} \to \mathbb{R}$, i.e., f is differentiable at $x^\star$ with derivative $f'(x^\star)$ if and only if the following limit exists and equality holds:

$$f'(x^\star) = \lim_{x \to x^\star} \frac{f(x) - f(x^\star)}{x - x^\star}. \tag{9}$$

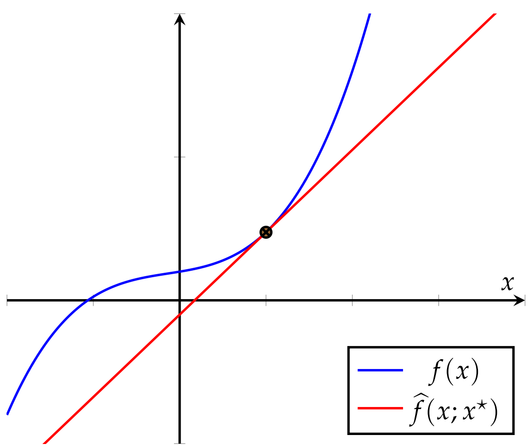
Rearranging, we say that f is differentiable at $x^\star$ with derivative $f'(x^\star)$, if and only if the following limit exists and equality holds:

$$0 = \lim_{x \to x^\star} \frac{f(x) - [f(x^\star) + f'(x^\star) \cdot (x - x^\star)]}{x - x^\star} \tag{10}$$

The bracketed quantity

$$\widehat{f}(x; x^\star) := f(x^\star) + f'(x^\star) \cdot (x - x^\star) \tag{11}$$

is exactly the linearization, i.e., linear approximation of f around $x^\star$. The derivative eq. (10) says that for x very close to $x^\star$, the linear approximation $\widehat{f}(x; x^\star)$ is almost exactly $f(x)$.



This definition of the first derivative – i.e., an intercept $f(x^\star)$ plus a linear function of $x - x^\star$ representing the tangent plane – holds also for vector functions.

Definition 4 (First Derivative, Jacobian)
Let $\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^q$ be a function. We say that $\vec{f}$ is *differentiable* at $\vec{x}^\star \in \mathbb{R}^p$ with derivative $J\vec{f}(\vec{x}^\star) \in \mathbb{R}^{q\times p}$ if and only if the following limit exists and equality holds:

$$0 = \lim_{\vec{x}\to\vec{x}^\star} \frac{\left\| \vec{f}(\vec{x}) - \left[\vec{f}(\vec{x}^\star) + J\vec{f}(\vec{x}^\star)\cdot(\vec{x}-\vec{x}^\star) \right] \right\|}{\|\vec{x}-\vec{x}^\star\|} \tag{12}$$

We call $J\vec{f}(\vec{x}^\star)$ the *Jacobian* or *derivative* of $\vec{f}$ at $\vec{x}^\star$.
More formally, we define the Jacobian $J\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^{q\times p}$ as the matrix-valued function which takes in points $\vec{x}^\star \in \mathbb{R}^p$ and outputs derivative matrices $J\vec{f}(\vec{x}^\star) \in \mathbb{R}^{q\times p}$.
We say that $\vec{f}$ is *differentiable* if it is differentiable at every point $\vec{x}^\star \in \mathbb{R}^p$.

The bracketed quantity

$$\hat{\vec{f}}(\vec{x};\vec{x}^\star) := \vec{f}(\vec{x}^\star) + J\vec{f}(\vec{x}^\star)\cdot(\vec{x}-\vec{x}^\star) \tag{13}$$

is exactly the best linear approximation of $\vec{f}$ around $\vec{x}^\star$. The derivative eq. (12) says that for $\vec{x}$ very close to $\vec{x}^\star$, the linear approximation $\hat{\vec{f}}(\vec{x};\vec{x}^\star)$ is almost exactly $\vec{f}(\vec{x})$.

This motivates the following definition of the best linear approximation, i.e., the linearization.

Definition 5 (Linearization)
Suppose $\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^q$ is differentiable. Then the *linearization* of $\vec{f}$ around $\vec{x}^\star \in \mathbb{R}^p$ is the function $\hat{\vec{f}}(\cdot;\vec{x}^\star)\colon \mathbb{R}^p \to \mathbb{R}^q$ given by

$$\hat{\vec{f}}(\vec{x};\vec{x}^\star) := \vec{f}(\vec{x}^\star) + J\vec{f}(\vec{x}^\star)\cdot(\vec{x}-\vec{x}^\star). \tag{14}$$

2.2 Computing the Derivative

Now, we introduce a much more mechanical and easier way to compute the derivative.

Definition 6 (Partial Derivative)
Let $f\colon \mathbb{R}^p \to \mathbb{R}$ be differentiable. The *partial derivative of f with respect to x_i* is the function $\frac{\partial f}{\partial x_i}\colon \mathbb{R}^p \to \mathbb{R}$ defined by

$$\frac{\partial f}{\partial x_i}(\vec{x}) := \lim_{h\to 0} \frac{f(x_1,\dots,x_i+h,\dots,x_p) - f(x_1,\dots,x_p)}{h}. \tag{15}$$

This definition provides a way to compute the partial derivative.

Key Idea 7 (Computing Partial Derivatives)
To compute a partial derivative $\frac{\partial f}{\partial x_i}$:

- Write out f explicitly in terms of $x_1,\dots,x_p$.
- Pretend all variables x_j are actually constants, *except* the variable x_i.
- Take the single-variable derivative of f in x_i.

The result will be the function $\frac{\partial f}{\partial x_i}$.

We can now give an explicit formula for the Jacobian.

Theorem 8 (Jacobian in terms of Partial Derivatives)
Let $\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^q$ be differentiable at $\vec{x}$. Then the Jacobian at $\vec{x}$, $J\vec{f}(\vec{x})$, is given by

$$J\vec{f}(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_1}{\partial x_p}(\vec{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_q}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_q}{\partial x_p}(\vec{x}) \end{bmatrix} \tag{16}$$

Just like a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ has exactly one derivative at a given point, a differentiable function $\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^q$ has exactly one derivative at a given point. And so we can get an *unique best linearization* using the Jacobian.

The main idea of the proof be obtained by only considering polynomial functions $f\colon \mathbb{R}^p \to \mathbb{R}$. For instance, consider the polynomial $f(x_1, x_2) := (ax_1 + bx_2)^2$. By the chain rule, we have:

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = 2a(ax_1 + bx_2) \qquad \frac{\partial f}{\partial x_2}(x_1, x_2) = 2b(ax_1 + bx_2). \tag{17}$$

Fix a point $\vec{x}^\star \in \mathbb{R}^2$ and a vector $\vec{x}$ which is close to $\vec{x}^\star$. Then

$$\begin{aligned}
f(\vec{x}) &= (ax_1 + ax_2)^2 && (18)\\
&= (ax_1^\star + a(x_1 - x_1^\star) + bx_2^\star + b(x_2 - x_2^\star))^2 && (19)\\
&= a^2 x_1^{\star 2} + 2abx_1^\star x_2^\star + b^2 x_2^{\star 2} && (20)\\
&\quad + 2a(ax_1^\star + bx_2^\star)(x_1 - x_1^\star) + 2b(ax_1^\star + bx_2^\star)(x_2 - x_2^\star) && (21)\\
&\quad + a^2(x_1 - x_1^\star)^2 + 2ab(x_1 - x_1^\star)(x_2 - x_2^\star) + b^2(x_2 - x_2^\star)^2 && (22)\\
&= f(x_1^\star, x_2^\star) + \frac{\partial f}{\partial x_1}(x_1^\star, x_2^\star) \cdot (x_1 - x_1^\star) + \frac{\partial f}{\partial x_2}(x_1^\star, x_2^\star) \cdot (x_2 - x_2^\star) && (23)\\
&\quad + a^2(x_1 - x_1^\star)^2 + 2ab(x_1 - x_1^\star)(x_2 - x_2^\star) + b^2(x_2 - x_2^\star)^2. && (24)
\end{aligned}$$

Now since $\vec{x} - \vec{x}^\star$ is small, the quantities $x_i - x_i^\star$ are small. Thus the quantities $(x_i - x_i^\star)^2$ and $(x_i - x_i^\star)(x_j - x_j^\star)$ are even smaller, so the whole last line is negligible and it is reasonable to consider the linear approximation

$$f(\vec{x}) \approx \widehat{f}(\vec{x}; \vec{x}^\star) := f(x_1^\star, x_2^\star) + \frac{\partial f}{\partial x_1}(x_1^\star, x_2^\star) \cdot (x_1 - x_1^\star) + \frac{\partial f}{\partial x_2}(x_1^\star, x_2^\star) \cdot (x_2 - x_2^\star). \tag{25}$$

Now this quantity can be expressed using the Jacobian, whose formula was explicitly given in Theorem 8.

$$\widehat{f}(\vec{x}; \vec{x}^\star) = f(x_1^\star, x_2^\star) + \begin{bmatrix} \frac{\partial f}{\partial x_1}(x_1^\star, x_2^\star) & \frac{\partial f}{\partial x_2}(x_1^\star, x_2^\star) \end{bmatrix} \begin{bmatrix} x_1 - x_1^\star \\ x_2 - x_2^\star \end{bmatrix} = f(\vec{x}^\star) + Jf(x_1^\star, x_2^\star) \cdot (\vec{x} - \vec{x}^\star) \tag{26}$$

which is exactly the linearization formula given in Definition 5.

3 Linearizing Models

Recall that we originally wanted to linearize state update functions $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$, that took in a state $\vec{x}$, and output either a derivative $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)$ or a new state $\vec{x}[i+1]$. To do this linearization, we introduce some new notation for the Jacobian.

Notation 9

Let $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ be differentiable. We define the Jacobian with respect to $\vec{x}$, i.e., $J_{\vec{x}}\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ as being given by

$$J_{\vec{x}}\vec{f}(\vec{x}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{x}) \end{bmatrix} \tag{27}$$

$$\tag{28}$$

Under this notation, we have the following linearization of the state update function $\vec{f}$.

Key Idea 10 (Linearization of State Update Function)

Let $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. The linearization of $\vec{f}$ around $\vec{x}^\star$ is given by

$$\widehat{\vec{f}}(\vec{x};\vec{x}^\star) := \vec{f}(\vec{x}^\star) + J_{\vec{x}}\vec{f}(\vec{x}^\star)\cdot(\vec{x}-\vec{x}^\star) \tag{29}$$

3.1 Linearization of Discrete-Time Time-Invariant Difference Equation Model

Recall that in the Discrete-Time Time-Invariant Difference Equation Model, the update rule is

$$\vec{x}[i+1] = \vec{f}(\vec{x}[i]). \tag{30}$$

One may linearize $\vec{f}$, but in order to do this, we need to pick a point $\vec{x}^\star$ to linearize around. We would like the system dynamics to be well-behaved around $\vec{x}^\star$). This leads to the concept of *equilibrium point*.

Definition 11 (Equilibrium Point in Discrete-Time Time-Invariant Difference Equation Model)

In the Discrete-Time Time-Invariant Difference Equation Model, the point $\vec{x}^\star$ is an *equilibrium point* if and only if

$$\vec{f}(\vec{x}^\star) = \vec{x}^\star. \tag{31}$$

In certain contexts this is also called an *operating point*.

Intuitively, $\vec{x}^\star$ can be interpreted as a stationary point of the system dynamics, in the following sense: *if the state starts at $\vec{x}^\star$ the state never leaves $\vec{x}^\star$.*

Linearizing the right-hand side around some equilibrium point $\vec{x}^\star$ which is close to $\vec{x}[i]$, we have

$$\begin{aligned} \vec{x}[i+1] &= \vec{f}(\vec{x}[i]) && (32)\\ &\approx \widehat{\vec{f}}(\vec{x}[i];\vec{x}^\star) && (33)\\ &= \vec{f}(\vec{x}^\star) + J_{\vec{x}}\vec{f}(\vec{x}^\star)\cdot(\vec{x}[i]-\vec{x}^\star) && (34)\\ &= \vec{x}^\star + J_{\vec{x}}\vec{f}(\vec{x}^\star)\cdot(\vec{x}[i]-\vec{x}^\star) && (35)\\ & && (36) \end{aligned}$$

If we define the *deviation from the equilibrium point*

$$\delta\vec{x}[i] := \vec{x}[i] - \vec{x}^\star \tag{37}$$

then the linearized equation becomes

$$\delta\vec{x}[i+1] = J_{\vec{x}}\vec{f}(\vec{x}^{\star}) \cdot \delta\vec{x}[i] \tag{38}$$

is a linear system in $\delta\vec{x}[i]$. We can thus analyze its stability, controllability, and so on, using the tools we already developed. We can even do feedback control! Keep in mind that in this system, driving $\delta\vec{x}[i]$ to $\vec{0}_n$ (for instance via feedback stabilization) is equivalent to driving $\vec{x}[i]$ to $\vec{x}^{\star}$. So, one thing we can do is to have $\vec{x}^{\star}$ be the state we want to drive the model to, and then use feedback control to get there.

When we're done, we can go back to the "real" state by setting $\vec{x}[i] = \delta\vec{x}[i] + \vec{x}^{\star}$.

We summarize our findings below.

Proposition 12 (Linearizing the Discrete-Time Time-Invariant Difference Equation Model)
Suppose $(\vec{x}^{\star})$ is an equilibrium point of the Discrete-Time Time-Invariant Difference Equation Model. Define

$$\delta\vec{x}[i] := \vec{x}[i] - \vec{x}^{\star} \tag{39}$$

Then a linearization of Discrete-Time Time-Invariant Difference Equation Model results in the linear model

$$\delta\vec{x}[i+1] = J_{\vec{x}}\vec{f}(\vec{x}^{\star}) \cdot \delta\vec{x}[i] \tag{40}$$

for $\delta\vec{x}[i]$ very small (i.e., $\vec{x}[i] \approx \vec{x}^{\star}$).

Warning 13
The linearization is *only* valid when the state $\vec{x}[i]$ is contained in a small neighborhood of the equilibrium point $\vec{x}^{\star}$.

3.2 Linearization of Continuous-Time Time-Invariant Differential Equation Model

The analysis of the continuous-time goes similarly to the discrete time, with a couple of crucial differences.

Recall that in the Continuous-Time Time-Invariant Differential Equation Model, the update rule is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t)). \tag{41}$$

Again, we need to define *equilibrium point*.

Definition 14 (Equilibrium Point in Continuous-Time Time-Invariant Differential Equation Model)
In the Continuous-Time Time-Invariant Differential Equation Model, the point $\vec{x}^{\star}$ is an *equilibrium point* if and only if

$$\vec{f}(\vec{x}^{\star}) = \vec{0}_n. \tag{42}$$

In certain contexts this is also called an *operating point*.

Intuitively, $\vec{x}^{\star}$ can be interpreted as a stationary point of the system dynamics, in the following sense: *if the state starts at $\vec{x}^{\star}$ the state never leaves $\vec{x}^{\star}$.*

Linearizing the right-hand side around some equilibrium point $\vec{x}^{\star}$ which is close to $\vec{x}(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t)) \tag{43}$$

$$\approx \widehat{\vec{f}}(\vec{x}(t); \vec{x}^{\star}) \tag{44}$$
$$= \vec{f}(\vec{x}^{\star}) + J_{\vec{x}}\vec{f}(\vec{x}^{\star}) \cdot (\vec{x}(t) - \vec{x}^{\star}) \tag{45}$$
$$= J_{\vec{x}}\vec{f}(\vec{x}^{\star}) \cdot (\vec{x}(t) - \vec{x}^{\star}) \tag{46}$$

If we define the *deviation from the equilibrium point*

$$\delta\vec{x}(t) := \vec{x}(t) - \vec{x}^{\star} \tag{47}$$

then the linearized equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}(t) \approx J_{\vec{x}}\vec{f}(\vec{x}^{\star}) \cdot \delta\vec{x}(t) \tag{48}$$

is a linear system in $\delta\vec{x}(t)$. We can thus analyze its stability, controllability, and so on, using the tools we already developed. We can even do feedback control! Keep in mind that in this system, driving $\delta\vec{x}(t)$ to $\vec{0}_n$ (for instance via feedback stabilization) is equivalent to driving $\vec{x}(t)$ to $\vec{x}^{\star}$. So, one thing we can do is to have $\vec{x}^{\star}$ be the state we want to drive the model to, and then use feedback control to get there.

When we're done, we can go back to the "real" state by setting $\vec{x}(t) = \delta\vec{x}(t) + \vec{x}^{\star}$.

We summarize our findings below.

Proposition 15 (Linearizing the Continuous-Time Time-Invariant Differential Equation Model)
Suppose $\vec{x}^{\star}$ is an equilibrium point of the Continuous-Time Time-Invariant Differential Equation Model. Define

$$\delta\vec{x}(t) := \vec{x}(t) - \vec{x}^{\star} \tag{49}$$

Then a linearization of Continuous-Time Time-Invariant Differential Equation Model results in the linear model

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}(t) = J_{\vec{x}}\vec{f}(\vec{x}^{\star}) \cdot \delta\vec{x}(t) \tag{50}$$

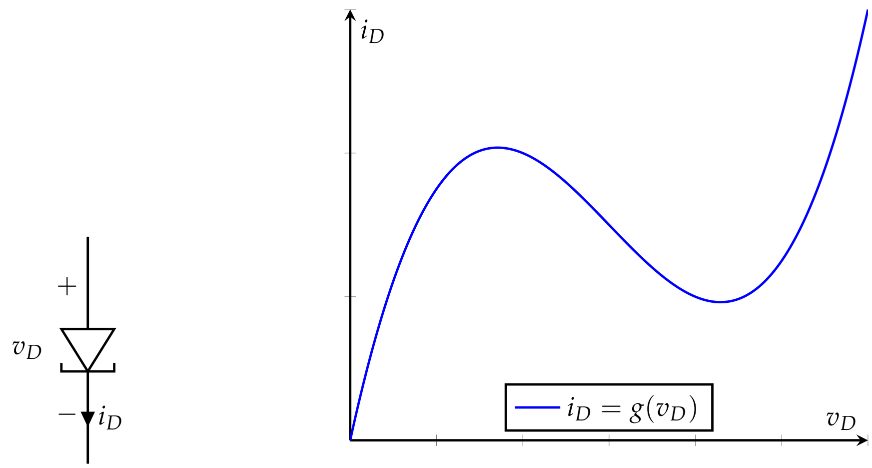
for $\delta\vec{x}(t)$ very small (i.e., $\vec{x}(t) \approx \vec{x}^{\star}$).

Warning 16
The linearization is *only* valid when the state $\vec{x}(t)$ is contained in a small neighborhood of the equilibrium point $\vec{x}^{\star}$.

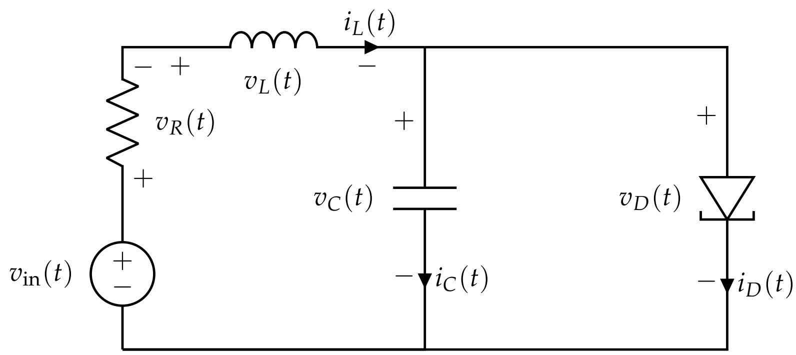
4 Examples

4.1 Circuit Example - Tunnel Diode

A tunnel diode is characterized by an I-V relationship where, for a certain voltage range, the current decreases with increasing voltage. (This is due to a quantum mechanical effect called *tunneling*).



Now consider the circuit below:



Using KVL and KCL, and the fact that $i_D(t) = g(v_D(t))$, we get the state model

$$\frac{\mathrm{d}}{\mathrm{d}t} v_C(t) = -\frac{1}{C} g(v_C(t)) + \frac{1}{C} i_L(t) \tag{51}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} i_L(t) = -\frac{1}{L} v_C(t) + \frac{R}{L} i_L(t) - \frac{1}{L} v_{\text{in}}(t). \tag{52}$$

Thus $\vec{f}$ is given by

$$\vec{f}(\underbrace{v_C, i_L}_{=\vec{x}}, \underbrace{v_{\text{in}}}_{=u}) = \begin{bmatrix} -\frac{1}{C} g(v_C) + \frac{1}{C} i_L \\ -\frac{1}{L} v_C + \frac{R}{L} i_L - \frac{1}{L} v_{\text{in}} \end{bmatrix}. \tag{53}$$

To find an equilibrium point, we set $\vec{f}(v_C^\star, i_L^\star, v_{\text{in}}^\star)$ to 0 (since this is an instance of Continuous-Time Time-Invariant Differential Equation Model) and solve for $v_C^\star$, $i_L^\star$, and $v_{\text{in}}^\star$. Indeed, we have the system of equations

$$0 = -\frac{1}{C} g(v_C(t)) + \frac{1}{C} i_L(t) \tag{54}$$

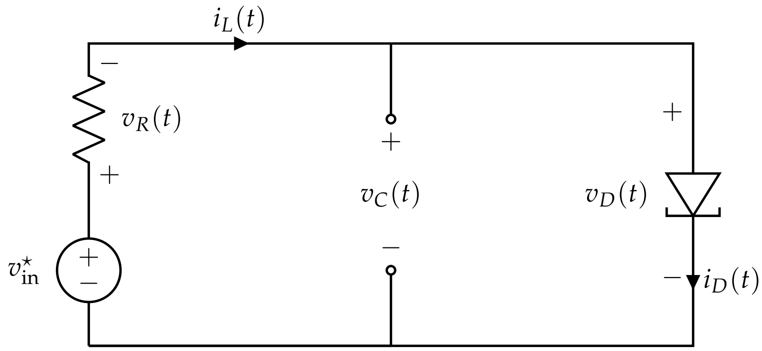
$$0 = -\frac{1}{L} v_C(t) + \frac{R}{L} i_L(t) - \frac{1}{L} v_{\text{in}}(t). \tag{55}$$

Solving, we get that the equilibrium point $(v_C^\star, i_L^\star, v_{\text{in}}^\star)$ is any triple which satisfies the equations

$$i_L^\star = g(v_C^\star) \tag{56}$$

$$i_L^\star = \frac{v_C^\star + v_{\text{in}}^\star}{R}. \tag{57}$$

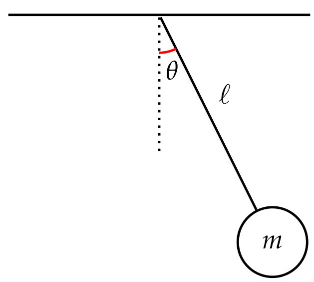
To get further insight into equilibrium states of circuits, note that to solve for the equilibrium we set $\frac{\mathrm{d}}{\mathrm{d}t}v_C(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}i_L(t)$ to 0. Since $i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}$ and $v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$, we thus have that at equilibrium, $i_C(t) = 0$ and $v_L(t) = 0$. Thus at equilibrium, the capacitor acts like an open circuit and the inductor like a short circuit. Redrawing the circuit, this is the picture at equilibrium:



We can linearize this new, simplified system to analyze small perturbations to v_C, i_L, v_{in} from the equilibrium states $v^\star_C, i^\star_L, v^\star_{\text{in}}$.

4.2 Mechanics Example

Consider the following pendulum with mass m:



From physics we know that the equation of motion of this pendulum is governed by the differential equation

$$m\ell\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = -k\ell\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} - mg\sin(\theta(t)) \tag{58}$$

where k is some air resistance coefficient. We define the state space variables

$$x_1(t) := \theta(t) \qquad x_2(t) := \frac{\mathrm{d}}{\mathrm{d}t}\theta(t). \tag{59}$$

This gives the nonlninear system

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = x_2(t) \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = -\frac{g}{\ell}\sin(x_1(t)) - \frac{k}{m}x_2(t) \tag{61}$$

where g is the gravitational acceleration. Thus $\vec{f}$ is given by

$$\vec{f}(x_1, x_2) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - \frac{k}{m}x_2 \end{bmatrix}. \tag{62}$$

There are two distinct equilibrium points, which we get from setting $\vec{f}(x_1^\star, x_2^\star)$ to 0 and solving:

$$(x_1^{\text{down}}, x_2^{\text{down}}) = (0, 0) \qquad \text{and} \qquad (x_1^{\text{up}}, x_2^{\text{up}}) = (\pi, 0) \tag{63}$$

corresponding to the pendulum hanging completely downwards and staying completely upwards, respectively. The Jacobian of $\vec{f}$ is given by

$$J\vec{f}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(x_1) & -\frac{k}{m} \end{bmatrix}. \tag{64}$$

By evaluating the Jacobian at the equilibria, we get that

$$J\vec{f}(x_1^{\text{down}}, x_2^{\text{down}}) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \qquad J\vec{f}(x_1^{\text{up}}, x_2^{\text{up}}) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}. \tag{65}$$

One can show that the eigenvalues of $J\vec{f}(x_1^{\text{down}}, x_2^{\text{down}})$ each have negative real part, so the linearized model at $(x_1^{\text{down}}, x_2^{\text{down}})$ is stable. On the other hand, there is an eigenvalue of $J\vec{f}(x_1^{\text{up}}, x_2^{\text{up}})$ with positive real part, so the linearized model at $(x_1^{\text{up}}, x_2^{\text{up}})$ is unstable. This corresponds with our physical intuition; if we shake a pendulum which is somehow standing straight up, it will immediately fall over and hang downwards, while if we shake a pendulum which is hanging downwards, it will move around a little but will eventually return to hanging downwards.

5 Final Comments

In this note, we learned how to linearize functions, and specifically how to linearize nonlinear control models. *Linearization is what makes linear control useful*, since most physical systems are nonlinear and thus the linear control model would not apply everywhere. Thus, linearization unlocks some rudimentary nonlinear control, allowing us to use linear control methods on nonlinear models. This neatly closes the loop on the controls picture we have developed.

Contributors:

- Druv Pai.
- Rahul Arya.
- Anant Sahai.
- Ashwin Vangipuram.

