Note 11: Stability and Feedback Control

1 Overview and Motivation

An LTI (Linear Time-Invariant) system is a framework used in signal processing and control theory, characterized by two main properties: linearity and time-invariance. Linearity means that the system's response to a linear combination of inputs is the same linear combination of the responses to each input (adhering to the principles of superposition and homogeneity). Time-invariance implies that the system's behavior and characteristics do not change over time. If a system's output response to a specific input is the same regardless of when the input is applied, the system is considered time-invariant. Together, these properties allow for simplified analysis and understanding of complex systems. The linear models we have analyzed in the previous notes are LTI. So, given an LTI model, we know how to determine its behavior at a given time as a function of its initial condition and all inputs. Now, we would like to know how to determine its asymptotics behavior, and get a qualitative idea of what happens to the system in the long-term.

We will begin by introducing the concept of *stability*.

Key Idea 1 (Stability)
A model is *stable* if, given sufficiently "nice" inputs, the model won't have "bad" behavior.

Then, we will learn how to tune the long-term system behavior to where we want, by supplying the right inputs. Such inputs will be functions of the current state, which motivates the idea of *feedback control*.

Key Idea 2 (Feedback Control)
Feedback control is when we choose the control input at each time as a function of the current state.

2 Stability

2.1 State Space Stability or Internal Stability

Model 3 (Discrete-Time LTI Difference Equation Model - no Input)
For checking the internal stability properities we recall the discrete-time model we used. Since this is a property of the system itself, we don't consider the input.

$$\vec{x}[i+1] = A\vec{x}[i] \tag{1}$$
$$\vec{x}[0] = \vec{x}_0 \tag{2}$$

for $\vec{x}\colon \mathbb{N} \to \mathbb{R}^n$ the state vector as a function of timestep, and $A \in \mathbb{R}^{n\times n}$ matrix.

Theorem 4 (Internal Stability in Discrete-Time LTI Difference Equation Model - no Input)
Suppose we are in Discrete-Time LTI Difference Equation Model - no Input where $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$.

(i) If $|\lambda_i| < 1$ for all i, then the model is called *internally stable*.

Notice, that if there is at least one $|\lambda_j| > 1$, then the model is unstable.

The eigenvalues of A may be real or complex. The $|\cdot|$ refers to the complex magnitude function, i.e., $|x + jy| = \sqrt{x^2 + y^2}$. This reduces to absolute value function when λ is real.

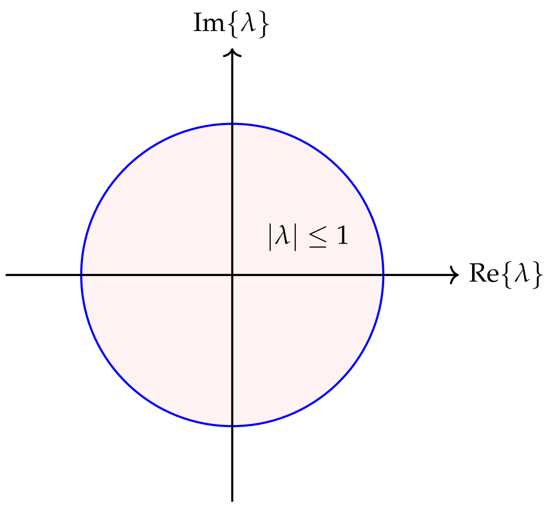

Figure 1: Here, we show the "discrete-time stability region" for the eigenvalues of A. The light red part, $|\lambda| < 1$, is where internal stability is guaranteed, if all eigenvalues are in this region; the blue part, $|\lambda| = 1$, is where so-called *marginal stability* may occur; and the white part, $|\lambda| > 1$ is where internal stability is guaranteed *not* to occur.

Model 5 (Continuous-Time LTI Differential Equation Model no Input)
We may also look at the stability of continuous-time models.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) \tag{3}$$

$$\vec{x}(0) = \vec{x}_0 \tag{4}$$

for $\vec{x}\colon \mathbb{R}_+ \to \mathbb{R}^n$ the state vector as a function of time, and $A \in \mathbb{R}^{n \times n}$,.

Theorem 6 (Internal Stability in Continuous-Time LTI Differential Equation Model no Input)
Suppose we are in Continuous-Time LTI Differential Equation Model no Input where $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$.

(i) If $\mathrm{Re}\{\lambda_i\} < 0$ for all i, then the model is internally stable.

Notice that if there is at least one eigenvalue such that $\mathrm{Re}\{\lambda_j\} > 0$, then the model is unstable.

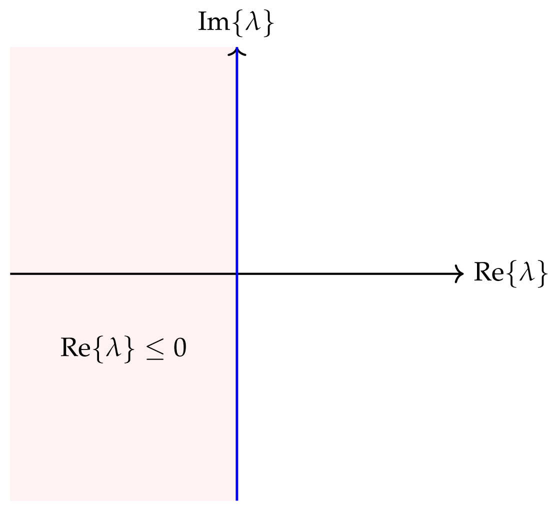

Figure 2: Here, we show the "continuous-time stability region" for the eigenvalues of A. The light red part, $\text{Re}\{\lambda\} < 0$, is where internal stability is guaranteed, if all eigenvalues are in this region; the blue part, $\text{Re}\{\lambda\} = 0$, is where so-called *marginal stability* may occur; and the white part, $\text{Re}\{\lambda\} > 0$, is where internal stability is guaranteed *not* to occur.

2.2 Marginal Stability

In discrete-time, we know whether the system is stable if all eigenvalues of A have magnitude strictly less than 1, and know the system is unstable if any eigenvalues of A has magnitude strictly greater than 1. We *do not* know what happens if all eigenvalues of A have magnitude less than *or equal to* 1, and some eigenvalues have magnitude exactly 1. Correspondingly, in continuous-time, we *do not* know what happens if all eigenvalues of A have real part less than *or equal to* 0, and some eigenvalues have real part exactly 0. Where A is diagonalizable, it turns out that there is a general way to check stability in these regimes. Notice, that the details are out of scope for this class. Recall that for the system to be stable we always make sure **not to include this boundary case**. We have the following theorems characterizing stability in this context. Models which satisfy the following characterizations, but not the internal stability characterizations, are called *marginally stable*.

Theorem 7 (Marginal Stability in Discrete-Time LTI Difference Equation Model - no Input when A is Diagonalizable)

Suppose we are in Discrete-Time LTI Difference Equation Model - no Input where $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. For a homogeneous, discrete-time, linear, time-invariant system to be marginally stable, two criteria must be satisfied:

1. First, the absolute largest value among all the poles (or eigenvalues) of its transfer function must be exactly 1.
2. Second, any pole that does have this maximum magnitude of 1 must be unique—there can't be duplicates.

Theorem 8 (Marginal Stability in Continuous-Time LTI Differential Equation Model no Input when A is Diagonalizable)
Suppose we are in Continuous-Time LTI Differential Equation Model no Input where $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$.
For a homogeneous, continuous, linear, time-invariant system to be considered marginally stable, it needs to meet three conditions:

1. First, the real part of every pole (or eigenvalue) in the system's transfer function must be zero or negative.
2. Second, there must be at least one pole with a real part exactly equal to zero.
3. Lastly, all poles with a zero real part must be unique — no two poles on the imaginary axis can be the same.

2.3 BIBO Stability or External Stability

We have seen one definition of stability. *Internal Stability* where inputs are not considered. Another definition of stability is (*external*) the one that characterize the system based on its inputs - how the system responds to an input.

For the sake of brevity, we are not going to define the Continuous Time-LTI model with inputs, as well as the Discrete one. The reader should be familiar with those models from previous notes. This particular definition of stability is called *bounded-input, bounded-output (BIBO) stability*. We will define it shortly, but first we need to know what *bounded* even means.

Definition 9 (Boundedness)

- A discrete-time function $\vec{z}_d \colon \mathbb{N} \to \mathbb{R}^k$ is *bounded* if there exists some number $R_d \in \mathbb{R}$ such that $\|\vec{z}_d[i]\| \leq R_d$ for all i.
- Similarly, a continuous-time function $\vec{z}_c \colon \mathbb{R}_+ \to \mathbb{R}^k$ is *bounded* if there exists some number $R_c \in \mathbb{R}$ such that $\|\vec{z}_c(t)\| \leq R_c$ for all t.

Note that we need the constants R_d or R_c to be independent of i or t respectively. One can think of them as denoting a radius for the boundary of the region that $\vec{z}_d$ or $\vec{z}_c$ have to stay in for all time.

Definition 10 (BIBO Stability)
A control model is *(BIBO) stable* if and only if, for *every* bounded input function $\vec{u}$, and *every* initial condition $\vec{x}_0$, the resulting state trajectory $\vec{x}$ is bounded. The same definition applies to both continuous-time or discrete-time models.

Generally, stability is *desirable* in our control models, because it means that the model will produce well-behaved state trajectories. A control model is *unstable* if we can find a bounded input and an initial condition which results in an unbounded state trajectory (i.e., one without an upper bound R on the norm that holds for all time) as the time(step) goes to ∞. Generally, instability is undesirable in our control models.

We conclude this section with a warning; that these conditions only hold when A is diagonalizable. The conditions for general A are much more subtle and mathematically sophisticated, and are best left to later study. Later, we present examples of where the theorems break down.

2.4 Stability Sanity-Checking

Right now, it may not be clear why our conditions for discrete-time and continuous-time stability are different. For intuition about this, it may help to review the scalar cases.

In the scalar case, the Discrete-Time LTI Difference Equation Model - no Input has state trajectory

$$x[i] = a^i x_0 + \sum_{k=0}^{i-1} a^{i-1-k} b u[k]. \tag{5}$$

Looking at only the first term, we can see whether its magnitude goes to ∞ ("blows up"). If the first term "blows up", we know that the system is unstable, because we can feed in $u[k] = 0$ for all k, and let the state "blow up". If the first term does not "blow up", then we would need to show that the second term does not "blow up" either.

For now, it is important to get an intuitive idea of what is happening with the first term, and in particular the behavior of z^i for a complex number z. More formally, suppose $z := re^{j\omega} \in \mathbb{C}$ is a complex number. Then

$$z^i = r^i e^{j\cdot\omega i} = r^i \cos(\omega i) + j r^i \sin(\omega i). \tag{6}$$

The idea is that r controls the rate of growth of $|z^t|$, and ω controls any oscillatory behavior.

- When $|z| < 1$, the envelope $r^i \to 0$, so z^i decays to 0, although if $\omega \neq 0$ it also has oscillatory behavior due to the sine/cosine.
- When $|z| = 1$, the envelope $r^i = 1$, so if $\omega \neq 0$ then $z^i = \cos(\omega i) + j\sin(\omega i)$ oscillates around the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$.
- When $|z| > 1$, the envelope $r^i \to \infty$, so z^i blows up to ∞, although if $\omega \neq 0$ it also has oscillatory behavior due to the sine/cosine.

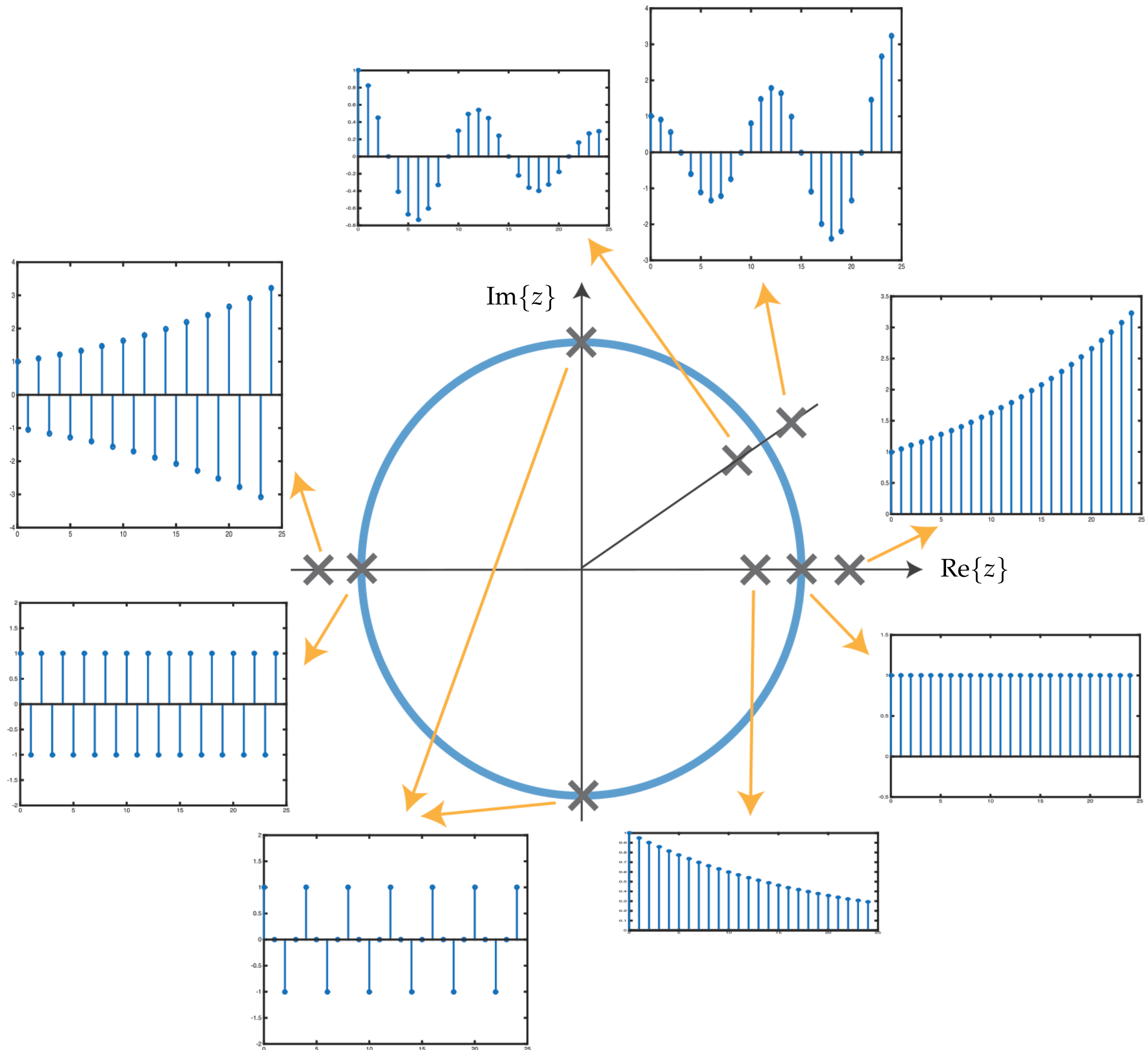

Figure 3: The real part of z^t for various values of z in the complex plane. It grows unbounded when $|z| > 1$, decays to zero when $|z| < 1$, and has constant amplitude when z is on the unit circle ($|z| = 1$).

So when $|z| > 1$ then the first term "blows up", and we can see that this is an indicator of instability.

We can also consider the continuous-time and do a similar analysis. In the scalar case, the Continuous-Time LTI Differential Equation Model no Input has state trajectory

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau\,. \tag{7}$$

Looking again at only the first term, we can see whether it "blows up". If the first term "blows up", we know that the system is unstable, because we can feed in $u(\tau) = 0$ for all τ, and let the state "blow up". If the first term does not "blow up", then we would need to show that the second term does not "blow up" either. For now, it is important to get an idea of what is happening with the first term, and in particular the behavior of e^{st} for a complex number s. More formally, suppose $s := \alpha + j\omega \in \mathbb{C}$ is a complex number. Then

$$e^{st} = e^{(\alpha+j\omega)t} = e^{\alpha t}e^{j\omega t} = e^{\alpha t}\cos(\omega t) + je^{\alpha t}\sin(\omega t). \tag{8}$$

The idea is that α controls the rate of growth of $|e^{st}|$, and ω controls any oscillatory behavior.

- When $\text{Re}\{s\} < 0$, the envelope $e^{\alpha t} \to 0$, so e^{st} decays to 0, although if $\omega \neq 0$ then it also has oscillatory behavior due to the sine/cosine.
- When $\text{Re}\{s\} = 0$, the envelope $e^{\alpha t} = 1$, so if $\omega \neq 0$ then $e^{st} = \cos(\omega t) + j\sin(\omega t)$ oscillates around the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$.
- When $\text{Re}\{s\} > 0$, the envelope $e^{\alpha t} \to \infty$, so e^{st} blows up to ∞, although if $\omega \neq 0$ it also has oscillatory behavior due to the sine/cosine.

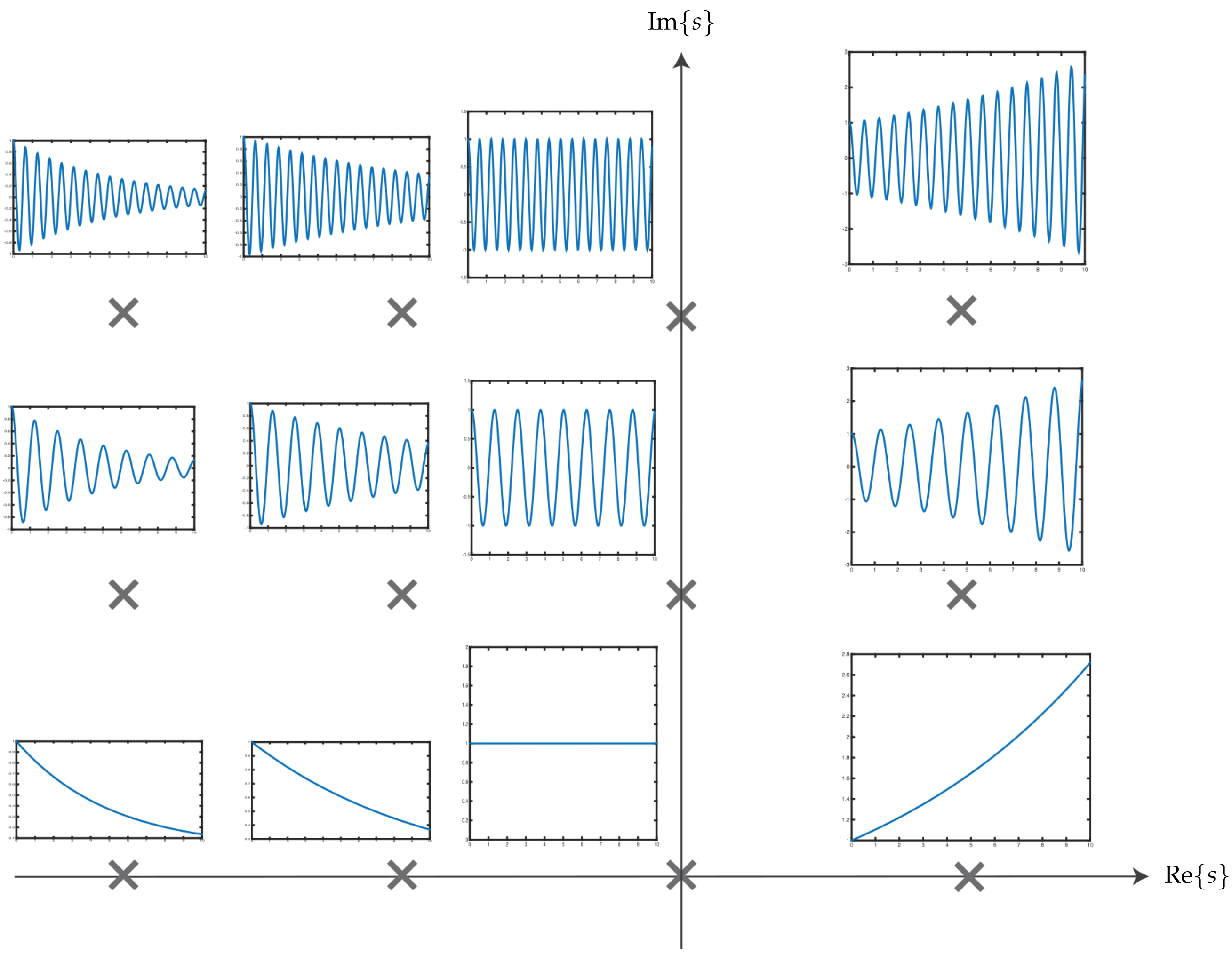

Figure 4: The real part of e^{st} for various values of s in the complex plane. Note that e^{st} is oscillatory when s has an imaginary component. It grows unboundedly when $\text{Re}\{s\} > 0$, decays to 0 when $\text{Re}\{s\} < 0$, and has constant amplitude when $\text{Re}\{s\} = 0$.

When $\text{Re}\{s\} > 0$ then the first term "blows up", and this is an indicator of instability.

2.5 Examples

The discrete-time model

$$x[i+1] = 3x[i] + 2u[i] \tag{9}$$

has the solution

$$x[i] = 3^i x[0] + 2\sum_{k=0}^{i-1} 3^{i-1-k} u[k] \tag{10}$$

and even if $u[k] = 0$ for all k, we still have $x[i] = 3^i x[0]$ which goes off to $\pm\infty$ as long as $x[0] \neq 0$. So this system is unstable.

If instead the model is

$$x[i+1] = \frac{1}{3}x[i] + 2u[i] \tag{11}$$

then the solution is

$$x[i] = \frac{x[0]}{3^i} + 2\sum_{k=0}^{i-1} \frac{u[k]}{3^{i-1-k}}. \tag{12}$$

It can be shown that no matter what $x[0]$ is, and with bounded u, that x is bounded, and therefore stable.

One can come up with continuous-time examples as well. For example, if we have the model

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 3x(t) + 2u(t) \tag{13}$$

then the solution is

$$x(t) = \mathrm{e}^{3t}x(0) + 2\int_0^t \mathrm{e}^{3(t-\tau)}u(\tau)\,\mathrm{d}\tau\,. \tag{14}$$

Even if $u(\tau) = 0$ for all τ, we still have $x(t) = \mathrm{e}^{3t}x(0)$ which goes off to $\pm\infty$ as long as $x(0) \neq 0$. So this system is unstable.

We could instead consider the model

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -3x(t) + 2u(t) \tag{15}$$

which has solution

$$x(t) = \mathrm{e}^{-3t}x(0) + 2\int_0^t \mathrm{e}^{-3(t-\tau)}u(\tau)\,\mathrm{d}\tau\,. \tag{16}$$

It can be shown that no matter what $x(0)$ is, and with bounded u, that x is bounded, and therefore stable.

3 Feedback Control

We usually want a given LTI model to be stable. Sometimes, our system identification process, or nature itself, gives us an unstable model. In this case we use *state feedback control*.

Definition 11 (Feedback Control)
Feedback control is when the input at a given time is a function of the state at that time:

$$\vec{u}[i] = \vec{f}(\vec{x}[i]) \qquad \text{or} \qquad \vec{u}(t) = \vec{f}(\vec{x}(t)) \tag{17}$$

for some function $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^m$.

Feedback control is also called *closed loop* control, because the input is a function of the state, which itself is a linear function of the previous input, and so on.

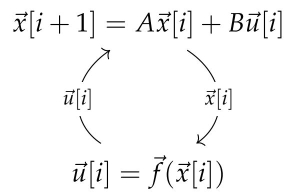

Figure 5: Closed-loop system in discrete-time.

This is in opposition to *open loop* control, which is when the inputs are not a function of the state.

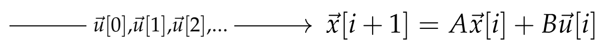

Figure 6: Open-loop system in discrete-time.

3.1 Discrete-Time Feedback Control

Theorem 12 (Discrete-Time Feedback Control)
Suppose in the Discrete-Time LTI Difference Equation Model - no Input we apply the closed-loop control:

$$\vec{u}[i] := \vec{u}_{\mathrm{CL}}[i] := F\vec{x}[i] + \vec{u}_{\mathrm{OL}}[i] \tag{18}$$

for some matrix $F \in \mathbb{R}^{m \times n}$ and some open-loop sequence of inputs $\vec{u}_{\mathrm{OL}} \colon \mathbb{N} \to \mathbb{R}^m$. Then the model becomes

$$\vec{x}[i+1] = A_{\mathrm{CL}}\vec{x}[i] + B\vec{u}_{\mathrm{OL}}[i] \tag{19}$$

$$\vec{x}[0] = \vec{x}_0 \tag{20}$$

where $A_{\mathrm{CL}} := A + BF$.

Proof.

$$\begin{aligned}
\vec{x}[i+1] &= A\vec{x}[i] + B\vec{u}[i] && (21)\\
&= A\vec{x}[i] + B\vec{u}_{\mathrm{CL}}[i] && (22)\\
&= A\vec{x}[i] + B(F\vec{x}[i] + \vec{u}_{\mathrm{OL}}[i]) && (23)\\
&= A\vec{x}[i] + BF\vec{x}[i] + B\vec{u}_{\mathrm{OL}}[i] && (24)\\
&= (A + BF)\vec{x}[i] + B\vec{u}_{\mathrm{OL}}[i] && (25)\\
&= A_{\mathrm{CL}}\vec{x}[i] + B\vec{u}_{\mathrm{OL}}[i]. && (26)
\end{aligned}$$

□

NOTE: We usually take $\vec{u}_{\mathrm{OL}}[i] = 0$ for all i, so that $\vec{u}[i] = F\vec{x}[i]$ for all i. There are some cases, however, where we would like the open-loop input to be nonzero. By Definition 10, *if all eigenvalues λ of A_{CL} have $|\lambda| < 1$, this system is stable.*

3.2 Continuous-Time Feedback Control

The analysis for the continuous-time feedback control is not much different from the discrete-time feedback control.

Theorem 13 (Continuous-Time Feedback Control)
Suppose in the Continuous-Time LTI Differential Equation Model no Input we apply the closed-loop control:

$$\vec{u}(t) := \vec{u}_{\mathrm{CL}}(t) := F\vec{x}(t) + \vec{u}_{\mathrm{OL}}(t) \tag{27}$$

for some matrix $F \in \mathbb{R}^{m\times n}$ and some open-loop input function $\vec{u}_{\mathrm{OL}} \colon \mathbb{R}_+ \to \mathbb{R}^m$. Then the model becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A_{\mathrm{CL}}\vec{x}(t) + B\vec{u}_{\mathrm{OL}}(t) \tag{28}$$

$$\vec{x}(0) = \vec{x}_0 \tag{29}$$

where $A_{\mathrm{CL}} := A + BF$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{30}$$

$$= A\vec{x}(t) + B\vec{u}_{\mathrm{CL}}(t) \tag{31}$$

$$= A\vec{x}(t) + B(F\vec{x}(t) + \vec{u}_{\mathrm{OL}}(t)) \tag{32}$$

$$= A\vec{x}(t) + BF\vec{x}(t) + B\vec{u}_{\mathrm{OL}}(t) \tag{33}$$

$$= (A + BF)\vec{x}(t) + B\vec{u}_{\mathrm{OL}}(t) \tag{34}$$

$$= A_{\mathrm{CL}}\vec{x}(t) + B\vec{u}_{\mathrm{OL}}(t). \tag{35}$$

□

NOTE: Again, we usually take $\vec{u}_{\mathrm{OL}}(t) = 0$ for all t. By Definition 10, *if all eigenvalues λ of A_{CL} have $\mathrm{Re}\{\lambda\} < 0$, then the system is stable.*

3.3 Stabilizing a Model

Since in both cases the A matrix for the closed-loop system is $A_{\mathrm{CL}} = A + BF$, we are able to use the following procedure to see if a model is stabilizable by linear closed-loop control.

An example of this algorithm will be included in the next subsection.

3.4 Example

We do a discrete-time feedback control example; the continuous-time case works similarly, with the caveat that the conditions for stability as described by Definition 10 are different from those described by Defini-

Algorithm 14 Stabilizing Models

Input: Original $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ matrices

Output: Whether or not closed-loop control can be used to make the model stable

Write symbolically $F := \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}$

Compute eigenvalues of $A_{\text{CL}} := A + BF$ in terms of $f_{11}, \ldots, f_{mn}$.

if there is a way to set $f_{11}, \ldots, f_{mn}$ so that system is stable **then**

  return STABILIZABLE, and such a $(f_{11}, \ldots, f_{mn})$

else

  return NOT STABILIZABLE

end if

tion 10, so it is important to be careful in determining which one applies.

Imagine our original model is:

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]. \tag{36}$$

If we apply the feedback

$$u[i] = F\vec{x}[i] \qquad \text{where} \qquad F := \begin{bmatrix} f_1 & f_2 \end{bmatrix} \tag{37}$$

then we get

$$A_{\text{CL}} := A + BF = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 + f_1 & 3 + f_2 \end{bmatrix}. \tag{38}$$

The eigenvalues of this are determined by the characteristic polynomial

$$p_{A_{\text{CL}}}(\lambda) := \det(A_{\text{CL}} - \lambda I_2) \tag{39}$$

$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ 2 + f_1 & 3 + f_2 - \lambda \end{bmatrix}\right) \tag{40}$$

$$= (-\lambda)(3 + f_2 - \lambda) - 1 \cdot (2 + f_1) \tag{41}$$

$$= \lambda^2 - (3 + f_2)\lambda - (2 + f_1). \tag{42}$$

We can find the roots, and thus the eigenvalues using the following method. Imagine that $p_{A_{\text{CL}}}$ has two roots λ_1 and λ_2. Since the leading coefficient of $p_{A_{\text{CL}}}(\lambda)$ is 1, we know that it has the form

$$p_{A_{\text{CL}}}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \tag{43}$$

$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. \tag{44}$$

Thus, matching coefficients of λ, we have the system of equations

$$3 + f_2 = \lambda_1 + \lambda_2 \tag{45}$$

$$-2 - f_1 = \lambda_1\lambda_2. \tag{46}$$

Thus

$$f_1 = -2 - \lambda_1\lambda_2 \tag{47}$$

$$f_2 = \lambda_1 + \lambda_2 - 3. \tag{48}$$

If we want to set particular eigenvalues λ_1 and λ_2 for A_{CL}, we use the above expression to determine the feedback control.

4 Final Comments

Overall, we have discussed the limiting behavior of the LTI models we usually work with. We have also discussed when and how we can change this behavior by applying the correct desired sequence of inputs.

This is the first time that we explicitly discuss the effect inputs may have on a control system. In this note, the analysis is somewhat qualitative and centers around limiting behavior. In future notes, we will be more quantitative about how to use control inputs to reach a particular state in a certain amount of steps and minimal energy consumption.

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