

Overview

In this note, we introduce the concept of Bode Plots, which are piecewise-linear approximations of transfer function plots. These will be useful in plotting a filter's frequency response by hand, and also help us better understand cascaded filters.

Having analyzed our first order filters and gone through a design example in the previous note to show why filter design is important, we will now plot their transfer functions $H(j\omega)$ (or frequency responses). There is immense value in visualizing transfer functions across a wide range of frequencies, Bode Plots are useful for performing filter design by hand quickly for various applications.

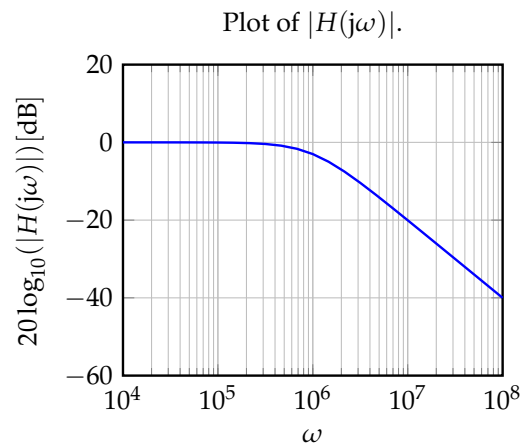
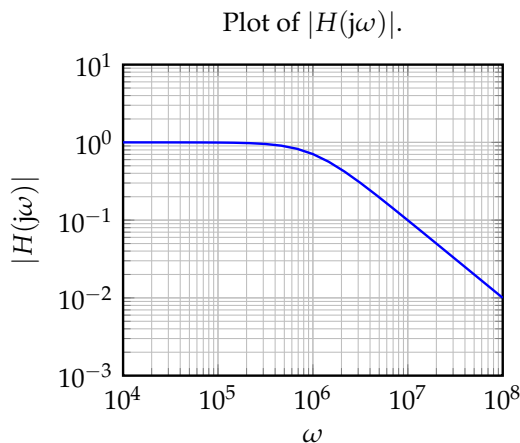
This note will present 2 key ideas, which build on what you've learned about Transfer Functions.

- **Linear Approximations:** Linear approximations applied to a transfer function plot define a Bode Plot, and this is where the key benefit comes in with respect to design.
- **y -axis scaling:** For the magnitude plot, we will plot $H(j\omega)$ in units of decibels, which implies switching from a linear scale to a logarithmic scale.

1 Bode Plots

Instead of plotting $|H(j\omega)|$ directly, we will plot $20 \log_{10}(|H(j\omega)|)$ and label the units as "decibels" (dB). We use *dB* for historical reasons going back to the inventor of the telephone, Alexander Graham Bell. Originally, one "bel" was used to describe a 10x power gain. Bels were deemed not granular enough, so decibels (one tenth of a bel) became the standard.

The constant 20 appears because the units are designed for power gain, not voltage (or current) gain. Because power increases with the square of voltage, and because of the properties of the log function, the constant becomes 20 instead of 10.



1.1 Low-Pass Filter

As a reminder, our low-pass filter has the following form:

$$H_{LP}(j\omega) = \frac{1}{1 + j\omega/\omega_c} \quad (1)$$

1.1.1 Low-Pass Filter: Magnitude Bode Plot

In addition to plotting the magnitude of the frequency response (that is, the exact transfer function magnitude), we would like to develop a piecewise-linear approximation as well.

Here is the expression for $|H_{LP}(j\omega)|_{dB}$:

$$|H_{LP}(j\omega)|_{dB} = 20 \log\left(\frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}}\right) = -20 \log\left(\sqrt{1 + \frac{\omega^2}{\omega_c^2}}\right) \quad (2)$$

To create a linear approximation for the magnitude of the frequency response we will consider the plot for different regions ($\omega \ll \omega_c$, $\omega = \omega_c$, $\omega \gg \omega_c$):

- For $\omega \ll \omega_c$, $|H_{LP}(j\omega)|_{dB} \approx -20 \log(1) = 0$ dB (horizontal line at 0 dB)
- For $\omega \gg \omega_c$, $|H_{LP}(j\omega)|_{dB} \approx -20 \log\left(\sqrt{\frac{\omega^2}{\omega_c^2}}\right) = -20 \log(\omega) + 20 \log(\omega_c)$ (line with slope $-20 \frac{dB}{dec}$ since ω is plotted on log scale)

At this point, we have a plot as shown in fig. 1. Note the use of 2 equivalent y -axes; on the left is the log scale, and on the right is the decibel scale introduced in this note.

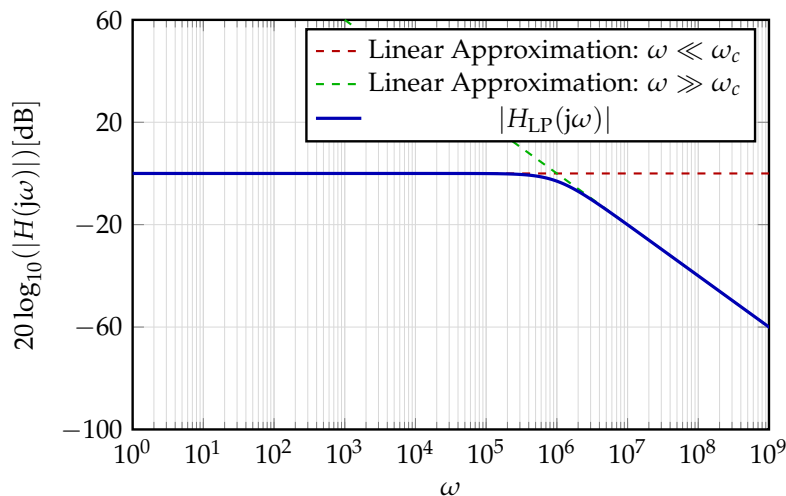


Figure 1: Low-pass filter magnitude plot, with 2 asymptotes drawn (dashed).

Now, once we plot these both, there is a point of conflict in the middle, right around ω_c . In this region, we will join the two models at a point, and pick the corresponding model for a given region (horizontal for $\omega < \omega_c$, sloped for $\omega > \omega_c$.) This means our final Bode Plot for the magnitude of a low-pass filter is as shown in fig. 2.

The linear model works well for most frequencies; it differs most to the actual response around the cutoff frequency. Can we be more precise about this bound on the error? It is a factor of $\frac{1}{\sqrt{2}}$, and this occurs

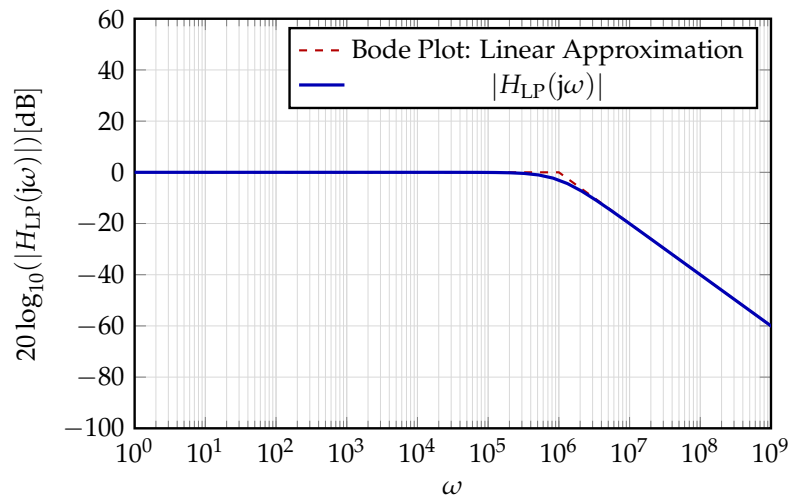


Figure 2: Low-pass filter magnitude plot.

right at the cutoff frequency. In other words, using the Bode Plot approximation will be inaccurate by at most 30%, and this level of inaccuracy arises at frequencies ω that are "close" to the cutoff frequency ω_c . It is a common rule of thumb to say that Bode Plot is correct for frequencies at least a factor of 10 away from cutoff; for anything within $0.1\omega_c$ to $10\omega_c$, it's safest to evaluate the transfer function exactly when possible.) The farther away we are from ω_c , the better the approximation will be, as seen in the plot.

Note that after the cutoff frequency the magnitude drop of $20\text{dB}/\text{dec}$, this means that every decade, the magnitude decrease of 20dB

1.1.2 Low-Pass Filter: Phase Bode Plot

Now, let's perform the same approximation process for the phase plot of a low-pass filter's transfer function (to arrive at the Bode Plot for the phase of a low-pass filter). In this case, there are 3 regions to examine.

Here is the expression for $\angle H_{\text{LP}}(j\omega)$:

$$\angle H_{\text{LP}}(j\omega) = \arctan\left(\frac{0}{1}\right) - \arctan\left(\frac{\omega}{\omega_c}\right) = -\arctan\left(\frac{\omega}{\omega_c}\right) \quad (3)$$

The analysis for each region is similar to the magnitude plot:

- For $\omega \ll \omega_c$, $\angle H_{\text{LP}}(j\omega) \approx -\arctan(0)$ (horizontal line at 0°)
- For $\omega \gg \omega_c$, $\angle H_{\text{LP}}(j\omega) \approx -\arctan(\infty) = -90^\circ$ (horizontal line at -90°)

Finally, there is the middle transition region during which we curve from 0° at low frequencies down to -90° at high frequencies. For this region too, we will use a line.

However, there's a point of subtlety here; how do we choose the *slope* of this line? The piecewise-linear approximation in this linear region should probably go through -45° at $\omega = \omega_c$, since this is exactly correct based on the true transfer function plot. But, where will this line intersect the other two horizontal lines? For design simplicity, we will choose to model the region between $10\omega_c$ and $\frac{\omega_c}{10}$ with this line. This is consistent with our prior approximation.

At this point, we have a plot as shown in fig. 3, with the 3 asymptotes drawn. The y -axis is already linear in degrees¹, so we don't have a decibel scale or anything.

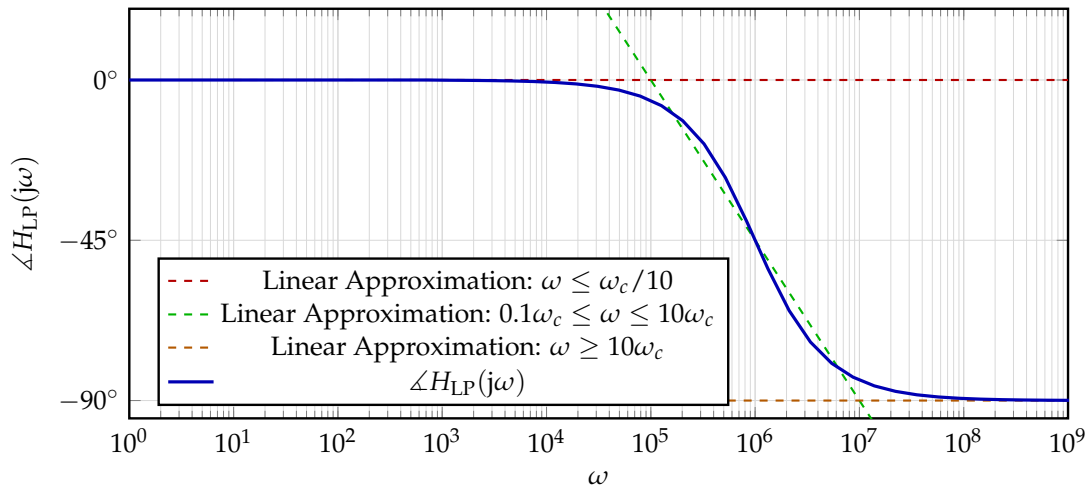


Figure 3: Low-pass filter phase plot, with 3 asymptotes drawn (dashed).

Once we join the asymptotes in their corresponding regimes, as discussed above, we arrive at fig. 4 for the Bode Plot of a low-pass filter's phase.

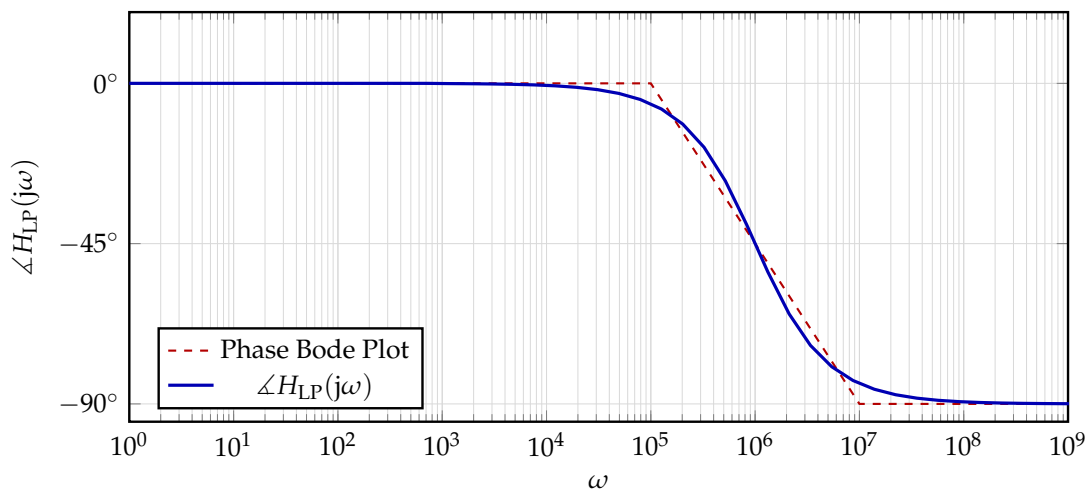


Figure 4: Low-pass filter phase plot.

1.2 High-Pass Filter

We can similarly analyze our generalized high-pass filter model:

$$H_{\text{HP}}(j\omega) = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c} \quad (4)$$

All of the same reasoning as for the low-pass filter holds here but in "reverse," so we will directly draw the Bode Plots for the high pass filter's magnitude (fig. 5) and phase (fig. 6) assuming $\omega_c = 10^6$.

¹Such plots could easily be formulated in terms of radians too.

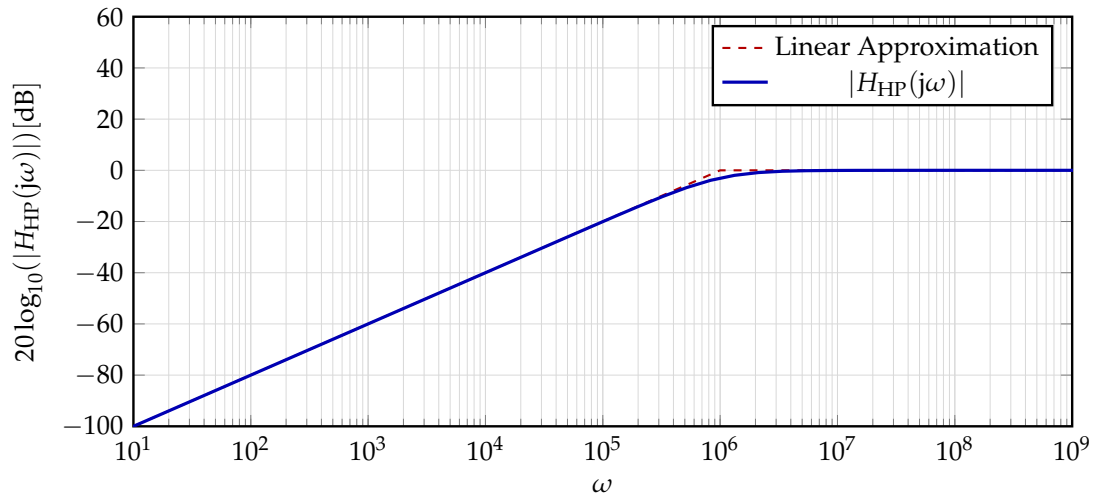


Figure 5: High-pass filter magnitude plot.

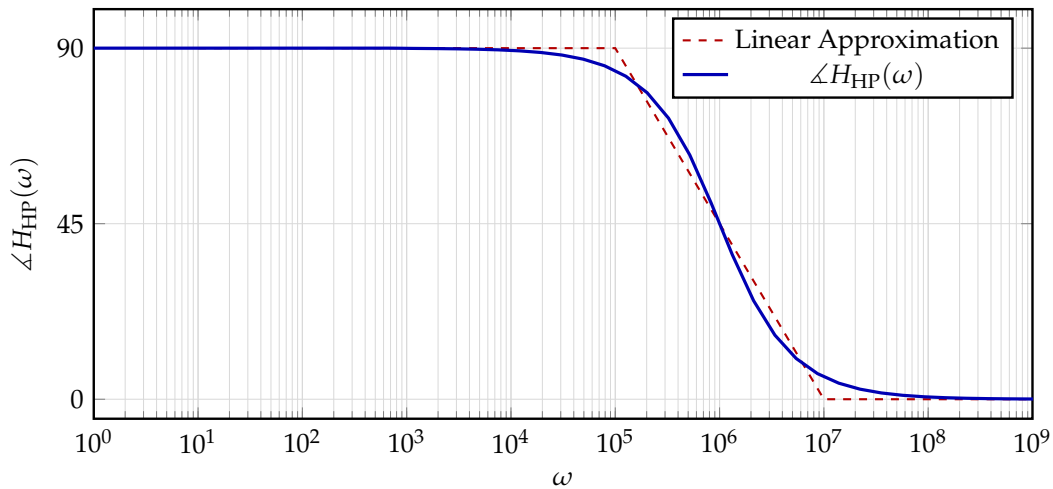


Figure 6: High-pass filter phase plot.

2 Plotting Linear Approximations of General Transfer Functions

2.1 Composing Transfer Functions

For two transfer functions $H_1(j\omega)$ and $H_2(j\omega)$, if $H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$,

$$|H(j\omega)|_{\text{dB}} = 20 \log |H(j\omega)| = 20 \log |H_1(j\omega) \cdot H_2(j\omega)| = 20 \log |H_1(j\omega)| + 20 \log |H_2(j\omega)| \quad (5)$$

$$\angle H(j\omega) = \angle(H_1(j\omega) \cdot H_2(j\omega)) = \angle H_1(j\omega) + \angle H_2(j\omega) \quad (6)$$

As a consequence, when plotting $|H(j\omega)|$ on a decibel scale plot, the Bode Plot, we can simply plot $|H_1(j\omega)|$ and $|H_2(j\omega)|$ and **add them**.

3 Rational Transfer Functions

When we write the transfer function of an arbitrary circuit involving linear circuit elements, it can always be rearranged to take the following form, called a "rational transfer function":

$$H(s) = K \cdot \frac{N(s)}{D(s)} \quad (7)$$

where the numerator $N(s)$ is a polynomial and so is the denominator $D(s)$.

Note, for formality, we will use the variable $s = j\omega$ (this is called Laplace notation; the substitution $s = j\omega$ is called Fourier notation).

We like to factor the numerator and denominator so that they become easier to work with and plot:

$$H(s) = K \cdot \frac{N(s)}{D(s)} = K \frac{s^{N_{z0}} \left(1 \pm \frac{s}{\omega_{z1}}\right) \left(1 \pm \frac{s}{\omega_{z2}}\right) \cdots \left(1 \pm \frac{s}{\omega_{zn}}\right)}{s^{N_{p0}} \left(1 \pm \frac{s}{\omega_{p1}}\right) \left(1 \pm \frac{s}{\omega_{p2}}\right) \cdots \left(1 \pm \frac{s}{\omega_{pm}}\right)} \quad (8)$$

The above is a consequence of the Fundamental Theorem of Algebra which asserts that all polynomials with complex coefficients can be factored into monomials — i.e. all complex polynomials of degree d have exactly d roots, if one counts repetitions. This is a theorem that you have probably seen, but is usually only proved in upper-division mathematics courses.

The above factorization is interesting because it says that no matter what, we can think of a transfer function as though it were a composition of elementary filters connected by unity-gain buffers. Mathematically, because polynomials might in principle have no constant terms, we need to deal with two more new objects that don't correspond to the simple low-pass and high-pass filters that we have seen so far.

To summarize the components, each transfer function is the product of constant gain K , one or more "origin poles" (s in the denominator) or "origin zeros" (s in the numerator) — these are the two new things — and one or more "poles" ($\left(1 \pm \frac{s}{\omega_{pi}}\right)$ in the denominator) or "zeros" ($\left(1 \pm \frac{s}{\omega_{zi}}\right)$ in the numerator).

Here, we define the constants $\mp\omega_z$ as *zeros* and $\mp\omega_p$ as *poles*. The zeros are the roots of $N(s)$ (the values of s for which $N(s) = 0$) while poles are the roots of $D(s)$ (the values of s for which $D(s) = 0$).

We will now substitute back $j\omega$; the values of ω_z will be *zero frequencies* and ω_p will be *pole frequencies* in this notation and will be extremely relevant when we start to draw Bode plots with poles and zeros:

$$H(j\omega) = K \cdot \frac{N(j\omega)}{D(j\omega)} = K \frac{(j\omega)^{N_{z0}} \left(1 \pm j\frac{\omega}{\omega_{z1}}\right) \left(1 \pm j\frac{\omega}{\omega_{z2}}\right) \cdots \left(1 \pm j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 \pm j\frac{\omega}{\omega_{p1}}\right) \left(1 \pm j\frac{\omega}{\omega_{p2}}\right) \cdots \left(1 \pm j\frac{\omega}{\omega_{pm}}\right)} \quad (9)$$

Using our rules for composing Bode plots, we know how to decompose our so-called general rational transfer function in terms of its magnitude and phase:

$$|H(j\omega)| = |K| \cdot \frac{|j\omega|^{N_{z0}} \cdot \left|1 \pm j\frac{\omega}{\omega_{z1}}\right| \cdots \left|1 \pm j\frac{\omega}{\omega_{zn}}\right|}{|j\omega|^{N_{p0}} \cdot \left|1 \pm j\frac{\omega}{\omega_{p1}}\right| \cdots \left|1 \pm j\frac{\omega}{\omega_{pm}}\right|} \quad (10)$$

$$= |K| \cdot |\omega|^{N_{z0} - N_{p0}} \cdot \frac{\sqrt{1 + \frac{\omega^2}{\omega_{z1}^2}} \cdots \sqrt{1 + \frac{\omega^2}{\omega_{zn}^2}}}{\sqrt{1 + \frac{\omega^2}{\omega_{p1}^2}} \cdots \sqrt{1 + \frac{\omega^2}{\omega_{pm}^2}}} \quad (11)$$

$$\angle H(j\omega) = \angle(K) + \angle(j\omega)^{N_{z0}} + \sum_{i=1}^n \angle\left(1 \pm j\frac{\omega}{\omega_{zi}}\right) - \angle(j\omega)^{N_{p0}} - \sum_{i=1}^m \angle\left(1 \pm j\frac{\omega}{\omega_{pi}}\right) \quad (12)$$

$$= \angle(K) + (N_{z0} - N_{p0})\angle(j) + \sum_{i=1}^n \pm \arctan\left(\frac{\omega}{\omega_{zi}}\right) - \sum_{i=1}^m \pm \arctan\left(\frac{\omega}{\omega_{pi}}\right). \quad (13)$$

Now, we have simplified as much as we can, generally. We could now convert j into either $e^{j\frac{\pi}{2}}$ or e^{j90° , depending on whether we're using radians or degrees for phase.

IMPORTANT: Note that the \pm does not impact the magnitude of the transfer function, but does impact the phase. In fact, it precisely reverses the phase:

$$\angle\left(1 - j\frac{\omega}{\omega_{zp}}\right) = -\angle\left(1 + j\frac{\omega}{\omega_{zp}}\right) \quad (14)$$

In the following sections, we will mostly deal with the "+" case, but if you have the "-" case, the only difference is for that specific zero or pole factor, you flip the phase effect. This will be discussed more once we derive the Bode plots for poles and zeros.

3.1 Poles, Zeros, and Constants

3.1.1 Simple Pole, Simple Zero

The notion of a **pole** and **zero** frequency is a generalization of the term cutoff frequency.² Let's first look back at a plot of our RC low-pass filter, which has the following form (except we've substituted the more general ω_p for ω_c):

$$H_P(j\omega) = \frac{1}{1 + j\omega/\omega_p} \quad (15)$$

In what follows, pay special attention to the Linear Approximations! When drawing Bode plots, we claim that the plot drops off with a slope of $20\text{dB}/\text{dec}$ after a pole ω_p . Suppose our transfer function has a simple pole at $\omega_p = 10^6$. Then magnitude plot has a familiar shape as in fig. 7 (resembling a low-pass filter's transfer function!).

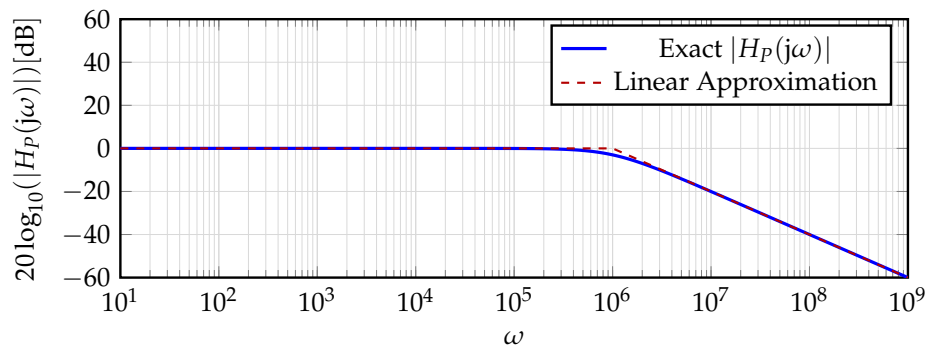


Figure 7: Simple Pole, Magnitude Plot.

We can look at the phase plot in fig. 8 as well:

Now let's take a look at a simple zero.

$$H_Z(j\omega) = 1 + j\omega/\omega_z \quad (16)$$

²How is it more general? As an example, in all our previous plots and transfer functions, our magnitude has always *dropped* before or after the cutoff frequency relative to the passband; for a zero, it will rise.

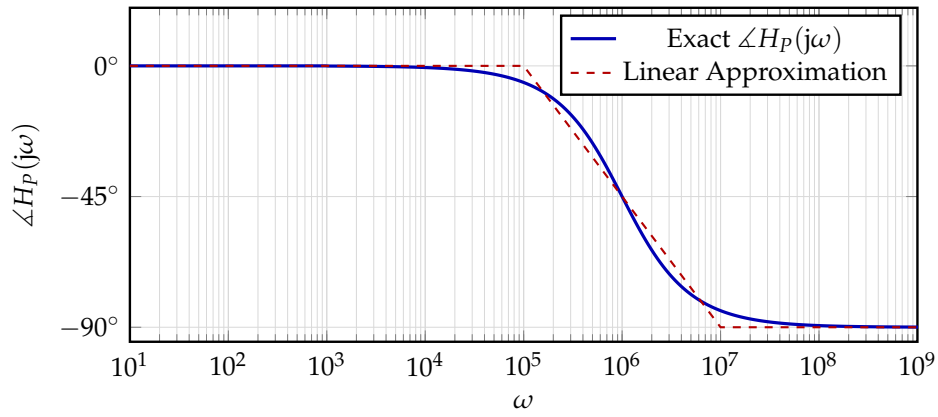


Figure 8: Simple Pole, Phase Plot.

We see that this Magnitude Bode plot in fig. 9 rises with a slope of $20\text{dB}/\text{dec}$ after the zero at ω_z . Suppose $\omega_z = 10^6$ also.

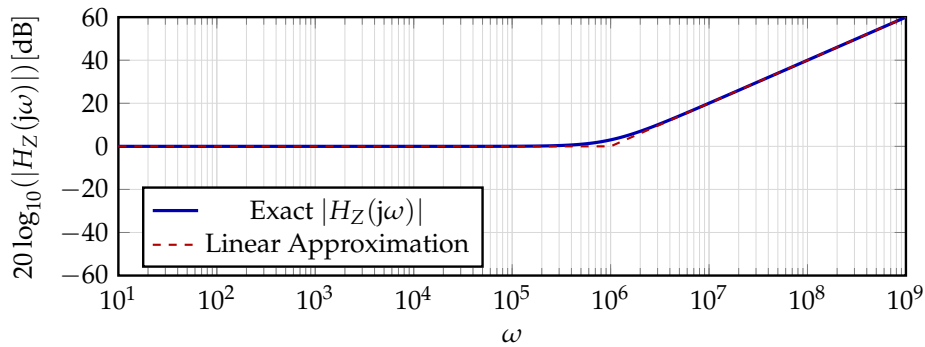


Figure 9: Simple Zero, Magnitude Plot.

The phase plot is in fig. 10.

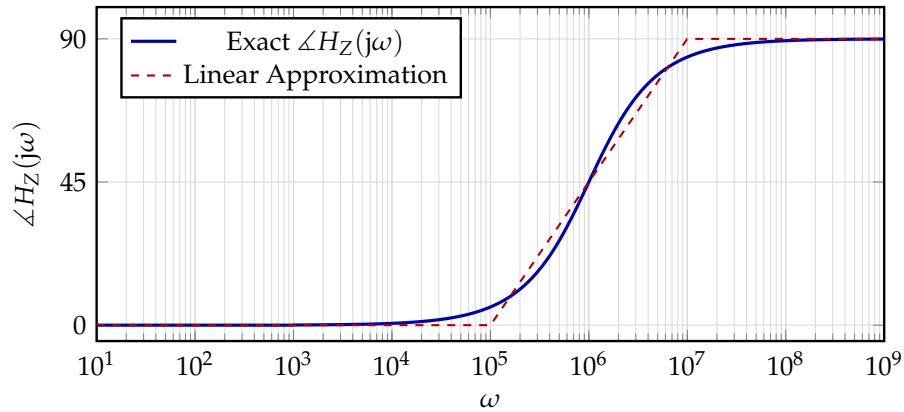


Figure 10: Simple Zero, Phase Plot.

Definition 1 (Bode Plot 101)

Here are some observations to make (for linear approximation purposes):

- Poles: A pole frequency of ω_p causes the slope of $|H(j\omega)|_{\text{dB}}$ to **decrease** by $20 \frac{\text{dB}}{\text{dec}}$ at $\omega = \omega_p$. The phase will *decrease* by $\frac{\pi}{2}$, or 90° from $\omega = \frac{1}{10}\omega_p$ to $10\omega_p$ if it is a "+" factor. For a "-" factor, the phase will instead *increase* by $\frac{\pi}{2}$, or 90° from $\omega = \frac{1}{10}\omega_p$ to $10\omega_p$.
- Zeros: A zero frequency of ω_z causes the slope of $|H(j\omega)|_{\text{dB}}$ to **increase** by $20 \frac{\text{dB}}{\text{dec}}$ at $\omega = \omega_z$. The phase will *increase* by $\frac{\pi}{2}$, or 90° from $\omega = \frac{1}{10}\omega_p$ to $10\omega_p$ if it is a "+" factor. For a "-" factor, the phase will instead *decrease* by $\frac{\pi}{2}$, or 90° from $\omega = \frac{1}{10}\omega_p$ to $10\omega_p$.

3.1.2 Pole/Zero at the Origin

To plot a pole at the origin (as in fig. 11), recall that $H(j\omega) = \frac{1}{j\omega}$ has magnitude $\frac{1}{\omega}$ and phase -90° .³ If our transfer function has a pole at the origin, it will start off with a slope of $-20\text{dB}/\text{dec}$. The phase of a pole at the origin is -90° at all frequencies.

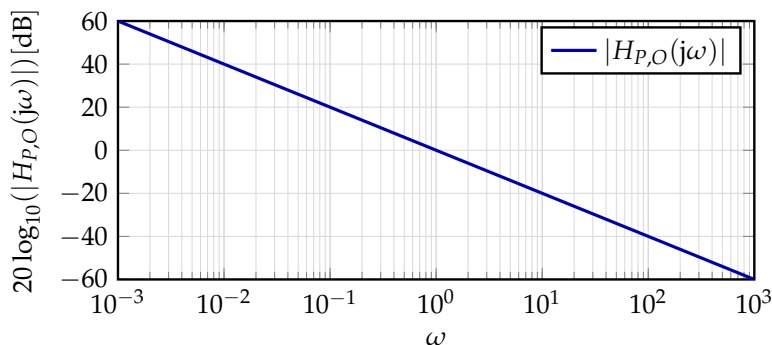


Figure 11: Origin Pole, Magnitude Plot.

We plot a zero at the origin in fig. 12, recall that $H(j\omega) = j\omega$ has magnitude ω and phase 90° . If our transfer function has a zero at the origin, it will start off with a slope of $20\text{dB}/\text{dec}$. The phase of a zero at the origin is 90° at all frequencies.

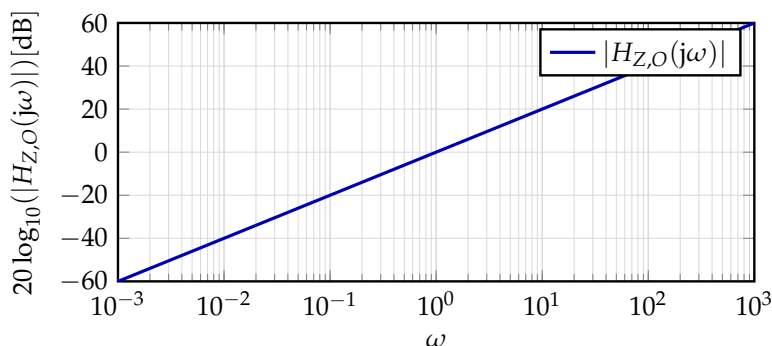


Figure 12: Origin Zero, Magnitude Plot.

³For this subsection and the next, our linear approximations are actually exactly correct.

3.1.3 Constant Terms

Lastly, we show the plot of a constant $K = 100$ in fig. 13. As expected, the plot remains constant. This implies that multiplication by K will shift up the entire bode plot up by K . Note that positive constants have a constant phase of 0° at all frequencies, while negative constants have a constant phase of $180^\circ \equiv -180^\circ$ at all frequencies.

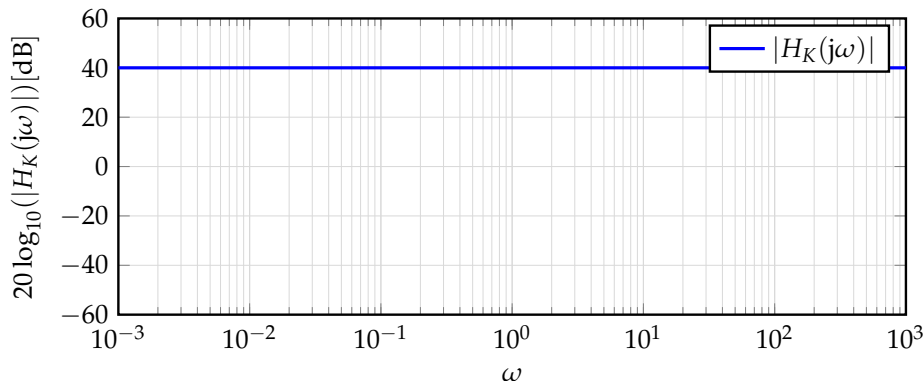


Figure 13: Constant Gain Term, Magnitude Plot.

3.1.4 RLC filter Bode plot

For an RLC circuit, which is composed of a resistor (R), an inductor (L), and a capacitor (C) connected in series or parallel, the Bode plot illustrates how the circuit responds to different frequencies. The key features of the Bode plot for an RLC circuit are determined by the circuit's resonant frequency and its quality factor (Q factor). The *resonant frequency* of an RLC circuit is the frequency at which the circuit naturally oscillates when not driven by an external source. It is defined as:

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (17)$$

This frequency corresponds to the point on the magnitude plot of the Bode diagram where the circuit can exhibit special behavior (either a maximum or minimum), assuming the Q factor is high enough to produce resonance.

The quality factor is a dimensionless parameter that describes how underdamped an oscillator or resonator is, and characterizes a resonator's bandwidth relative to its center frequency. This is the same factor a discussed with AC power; higher Q indicates a lower rate of energy loss relative to the stored energy of the resonator; hence oscillations die out more slowly. Different RLC circuit structures have different Q factors.

For a series RLC circuit:

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} = \frac{1}{R} \sqrt{\frac{L}{C}} \quad (18)$$

The derivation for this is located in the note about AC power.

For a parallel RLC circuit:

$$Q = \frac{R}{\omega_0 L} = \omega_0 C R = R \sqrt{\frac{C}{L}} \quad (19)$$

The derivation for this is similar to the series case.

At resonance, the inductive and capacitive reactances in an RLC circuit cancel each other out, leading to a condition where the circuit's impedance is purely resistive and at its minimum for a series RLC circuit, or maximum for a parallel RLC circuit.

To be more specific, a capacitor and inductor in series will be equivalent to a *short circuit* at resonance, while a capacitor and inductor in parallel will be equivalent to an *open circuit* at resonance, both of which leave just the resistor for an RLC circuit.

This results in maximum current for a series RLC or minimum current for a parallel RLC, which corresponds to the peak in the Bode magnitude plot.

The Q factor affects the Bode plot in two key ways. *Magnitude*: A higher Q factor means a sharper peak at the resonant frequency. If the Q factor is less than 0.5, the system does not resonate, and no peak is observed in the Bode magnitude plot. As the Q factor exceeds 0.5, a peak develops, and it becomes more pronounced as the Q factor increases. *Bandwidth*: The bandwidth (B) of the circuit, defined as the range of frequencies for which the circuit's response is above -3 dB of the peak response, is inversely proportional to the Q factor:

$$B \approx \frac{\omega_0}{Q} \quad (20)$$

A higher Q means a narrower bandwidth.

Here is the magnitude Bode plot for an RLC filter with different Q factors:

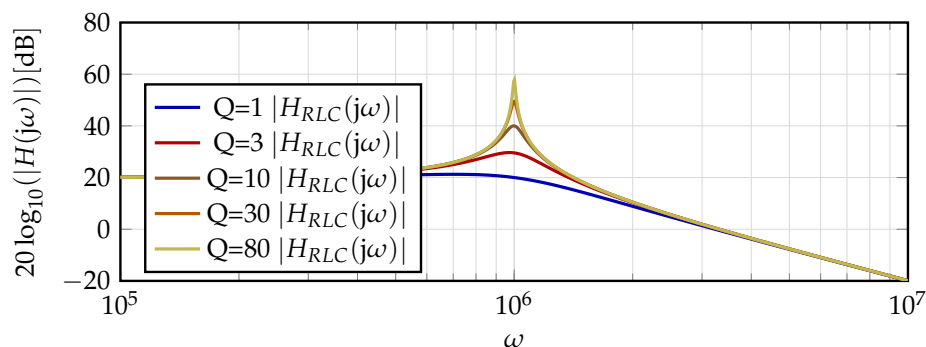


Figure 14: RLC circuit $\omega_0 = 10^6$, Magnitude Plot.

3.2 Bode Plot: Complicated Examples

Transfer Function Example

Now let's take a look at the Bode plot of a new transfer function in eq. (21).

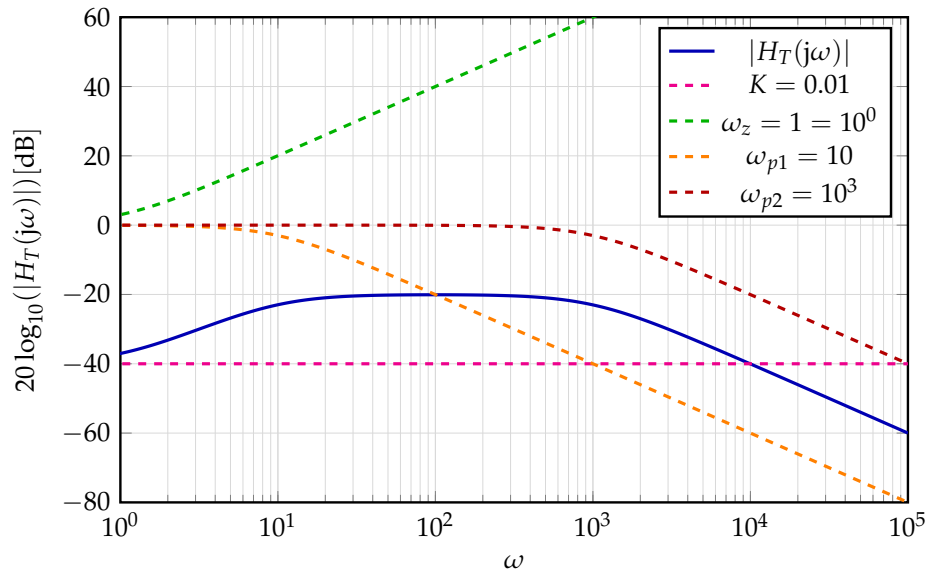
$$H_T(j\omega) = 100 \frac{(1 + j\omega)}{(j\omega)^2 + 1010(j\omega) + 10^4} \quad (21)$$

Our first step is to factor this into its rational transfer function form:

$$H_T(j\omega) = 0.01 \frac{(1 + j\omega)}{\left(1 + \frac{j\omega}{10}\right)\left(1 + \frac{j\omega}{10^3}\right)} \quad (22)$$

With $H_T(j\omega)$ in its rational form, we see that $K = 0.01$, $\omega_z = 1$, $\omega_{p1} = 10$, $\omega_{p2} = 10^3$. In ?? is a magnitude plot of each constituent component (following the building-block rules presented above), and the multiplication

of all of these provides $|H_T(j\omega)|$. The linear approximations are omitted to keep the plot legible, but the approximate result will very closely match the exact one.



To provide an analysis for this Bode plot, we see that the plot starts off at $K = 0.01$. Then at $\omega_z = 1$, it starts rising with slope $20dB$. When it hits the pole at $\omega_{p1} = 10$, the slope of $20dB$ is cancelled out by the $-20dB$ slope that the pole provides. Then the Bode plot stays constant until $\omega_{p2} = 10^3$ at which it drops off with a slope of $20dB$. We've provided Bode plots of the individual terms to give you a sense of how we "add" Bode plots together.

We can also plot the phase in fig. 15, in a very similar way using our building blocks:

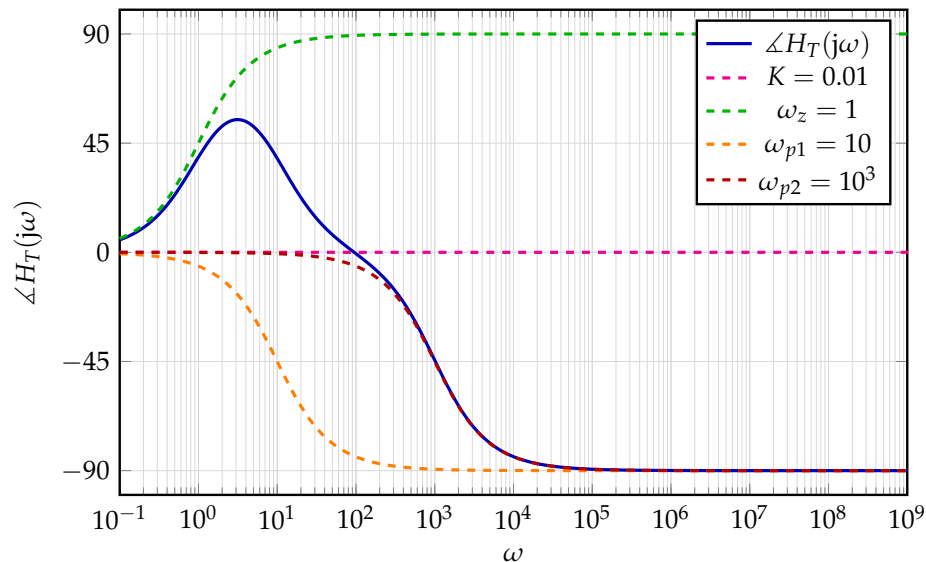


Figure 15: Phase Bode Plot of $H_T(j\omega)$

Zero at the Origin

In our final example, we examine the effects of a zero at the origin. Only the final results are shown; the intermediate building blocks are left for the reader to consider. We are given the transfer function eq. (23) in rational form.

$$H_O(\omega) = 0.1 \frac{(j\omega)(1 + \frac{j\omega}{10^6})}{(1 + \frac{j\omega}{10^2})^2} \quad (23)$$

Our magnitude plot is in fig. 16.

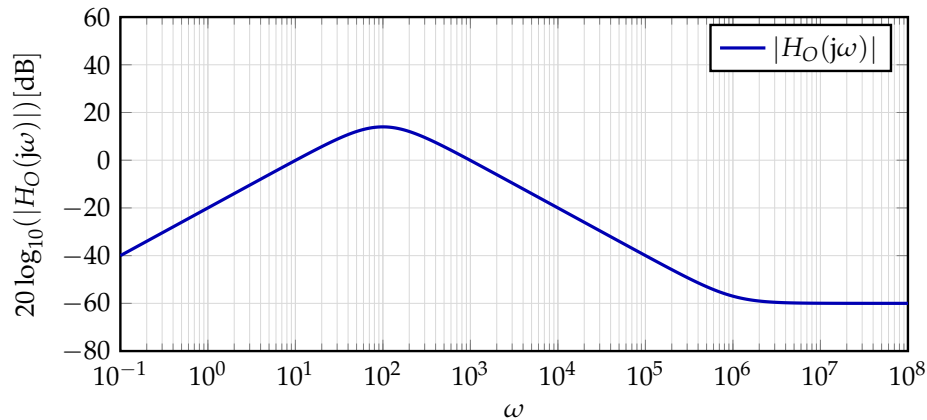


Figure 16: Magnitude Bode Plot of $H_O(j\omega)$

Since there is a zero at the origin, the plot will initially start with a slope of 20. There are no additional zeros or poles before $\omega = 1$, so we can approximate $|H_O(1)| = K = 0.1$. Then the double pole at $\omega_p = 10^2$ provides a slope of -40 that will cancel out the slope of 20 making the overall slope after ω_p equal to -20 . Lastly, there is a zero at $\omega_z = 10^6$ and we see that the addition of a slope of 20 makes $|H_O(\omega)|$ remains constant after ω_z .

And for the phase, we have the plot in fig. 17.

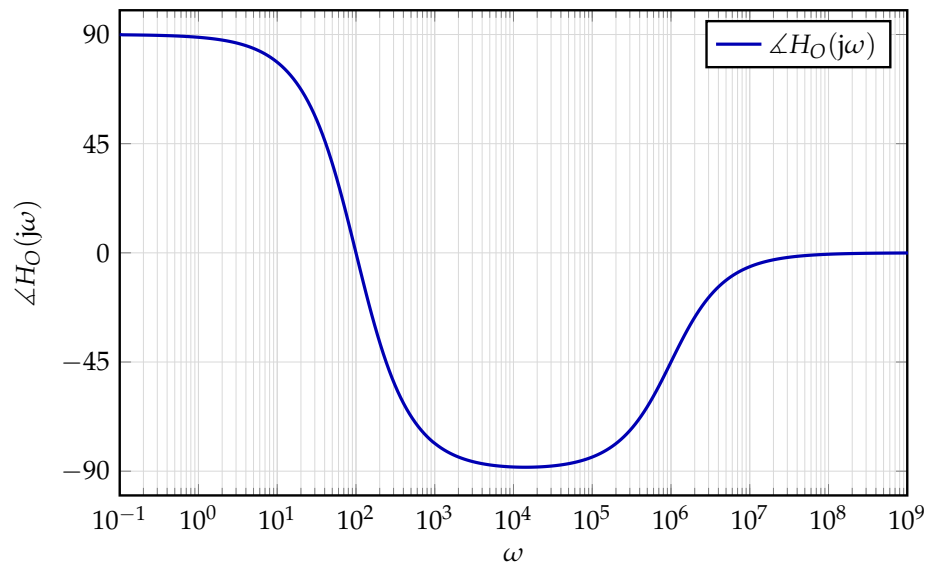


Figure 17: Phase Bode Plot of $H_O(j\omega)$

4 Check your Understanding

Read through these simple questions to check your basic understanding of the notes.

- What is a Bode plot and why it is so widely used?
- What is the Bode plot of a band-pass filter implemented as a combination of a low-pass and high-pass filter? (Hint: How do we combine multiplied Bode plots?)
- What is represented on the axis of a magnitude Bode plot? What about the axis of a phase Bode plot?
- How do we identify the poles and zeros of a transfer function?
- Poles: How does a pole change the slope of the magnitude Bode plot? How does a pole change the phase for the phase Bode plot?
- Zeros: How does a zero change the slope of the magnitude Bode plot? How does a zero change the phase for the phase Bode plot?
- How does the quality factor Q relate to the Bode plot of a RLC circuit?

Contributors:

- Neelesh Ramachandran.
- Rahul Arya.
- Anant Sahai.
- Jaijeet Roychowdhury.
- Taejin Hwang.
- Druv Pai.
- Anish Muthali.
- Matteo Guarrera.
- Nikhil Jain.