Note 6C: AC Power (OPTIONAL)

With the concept of phasors, we now have a way to work with sinusoidal voltages and currents without complicated differential equations or integrals. Another topic that comes up in the discussion of phasors is how to calculate power, an important quantity in many applications, with phasors.

1 Average and Reactive Power

We start out with the standard definition of power:

$$p(t) = v(t)i(t) \tag{1}$$

Notice that we include the dependence on time; the instantaneous power for a system is the instantaneous product of the relevant voltage and current.

However, this definition is not very efficient to understand the behavior of sinusoidal signals, which are very common. Especially for high frequency sinusoidal functions, the instantaneous power is not very relevant; what is more relevant is the **average power**.

To think about how this translates to phasors, assume that the voltage for our system is $v(t) = V_0 \cos(\omega t + \phi_V)$ and the current is $i(t) = I_0 \cos(\omega t + \phi_I)$ (notice that we account for the phase shift between the voltage and current). The average power is calculated over one period of the sinusoidal function (which is from 0 to $T = \frac{2\pi}{\omega}$, where ω is the frequency of the sinusoid):

$$\begin{aligned} P_{\text{avg}} &= \frac{1}{T}\int_0^T p(t)dt &(2)\\ &= \frac{1}{T}\int_0^T v(t)i(t)dt &(3)\\ &= \frac{1}{T}\int_0^T V_0 I_0 \cos(\omega t + \phi_V)\cos(\omega t + \phi_I)dt &(4)\end{aligned}$$

We will use the identity that $\cos(\omega t + \phi) = \cos(\theta_1)\cos(\theta_2) = \frac{1}{2}\cos(\theta_1 + \theta_2) + \frac{1}{2}\cos(\theta_1 - \theta_2)$:

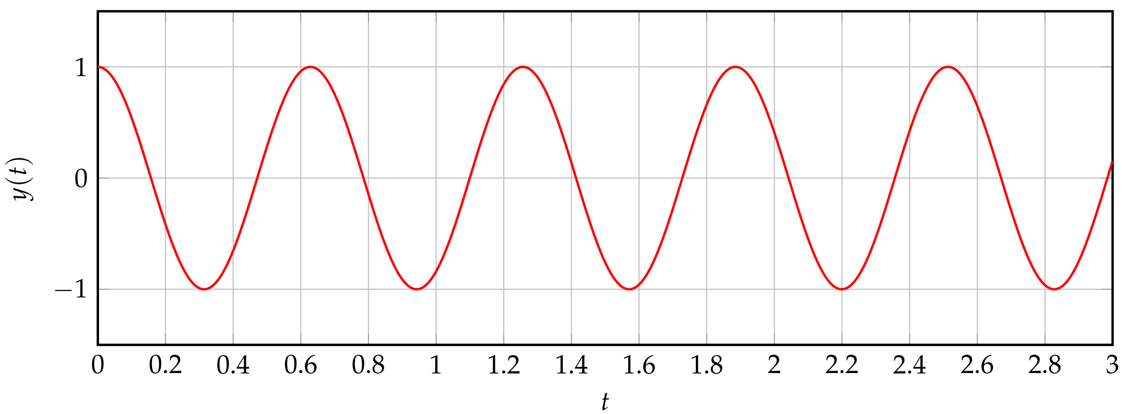
$$P_{\text{avg}} = \frac{1}{2}V_0 I_0 \frac{1}{T}\int_0^T \cos(2\omega t + \phi_V + \phi_I)dt + \frac{1}{2}V_0 I_0 \frac{1}{T}\int_0^T \cos(\phi_V - \phi_I)dt \tag{5}$$

$$\tag{6}$$

Notice that our integral involves the average of $\cos(2\omega t + \phi_V + \phi_I)$:

$$\frac{1}{T}\int_0^T \cos(2\omega t + \phi_V + \phi_I)dt = 0 \tag{7}$$

Here is a plot of $\cos(2\omega t)$ to show this:



From this plot, we can see that $\cos(2\omega t)$ is a sinusoidal function centered at 0 so the average value is 0 (the phase does not affect this).

With this, our average power expression simplifies to a simple integral of a constant:

$$P_{\text{avg}} = \frac{1}{2} V_0 I_0 \frac{1}{T} \int_0^T \cos(\phi_V - \phi_I) dt \tag{8}$$

$$P_{\text{avg}} = \frac{1}{2} V_0 I_0 \cos(\phi_V - \phi_I) \tag{9}$$

Here are some important details to notice:

- The average power is related to the product of V_0 and I_0, the amplitudes of voltage and current. This should make sense since this looks like a direct application of the power formula.
- The average power depends on the phase difference between the voltage and current, which is $\phi_V - \phi_I$. What happens if $\phi_V - \phi_I = \frac{\pi}{2}$? More on that later...
- There is a factor of $\frac{1}{2}$. Why? This is because the sinusoidal functions oscillate and thus do not maintain their full amplitude for the full cycle; the average value of a squared sinusoidal function is $\frac{1}{2}$, which is where this comes from. In fact, in some contexts, you may define some **root-mean-squared** value ($V_{rms} = \frac{V_0}{\sqrt{2}}$ for voltage, $I_{rms} = \frac{I_0}{\sqrt{2}}$ for current) such that the factor of $\frac{1}{2}$ disappears:

$$P_{\text{avg}} = V_{rms} I_{rms} \cos(\phi_V - \phi_I) \tag{10}$$

How could we reach this result with phasors? Note that the phasors associated with $v(t)$ and $i(t)$ as defined earlier are $\widetilde{V} = V_0 e^{j\phi_V}$ and $\widetilde{I} = I_0 e^{j\phi_I}$. We could reach the same result with the following calculation:

$$P_{\text{avg}} = \frac{1}{2} \text{Re}\{\widetilde{V}\widetilde{I}^*\} \tag{11}$$

(We use $\widetilde{I}^*$ here to be the complex conjugate of $\widetilde{I}$.)

You could verify, with Euler's theorem, that this is consistent with the previous expression. You may think: why do we need the complex conjugate? The purpose of this is to extract the phase difference between voltage and current, which we saw from the integral calculation is important to describe average power.

Notice that $\widetilde{V}\widetilde{I}^*$ is a complex number; we take the real part since this matches the calculation we did earlier in the time domain. What is the importance of the imaginary part? We call this **reactive power**:

$$P_{\text{reactive}} = \frac{1}{2} \text{Im}\{\widetilde{V}\widetilde{I}^*\} \tag{12}$$

What does reactive power represent? Reactive power represents a "storage" of energy/power to be used later by the system. Notice that this is not the same as our traditional definition of power; the average power calculated earlier represents a measure of the average "dissipated" or "used" power that cannot be used later by the system.

2 AC Power with Different Elements

Now, let's examine the AC power associated with the three major elements we have discussed: resistors, capacitors, and inductors.

Suppose for each element, there is a voltage with phasor $\widetilde{V}$ across the element.

For resistors, the current through the resistor would be $\widetilde{I}_R = \frac{\widetilde{V}}{Z_R} = \frac{\widetilde{V}}{R}$. Thus, the average power would be:

$$P_{\text{avg}} = \frac{1}{2}\text{Re}\{\widetilde{V}\widetilde{I}_R^*\} = \frac{1}{2}\text{Re}\{\widetilde{V}\frac{\widetilde{V}^*}{R}\} = \frac{1}{2}\text{Re}\{\frac{|\widetilde{V}|^2}{R}\} = \frac{1}{2}\frac{|\widetilde{V}|^2}{R} \tag{13}$$

This is very similar to the power equation we would see with DC sources.

The reactive power would be:

$$P_{\text{reactive}} = \frac{1}{2}\text{Im}\{\widetilde{V}\widetilde{I}_R^*\} = \frac{1}{2}\text{Im}\{\frac{|\widetilde{V}|^2}{R}\} = 0 \tag{14}$$

Notice that resistors have no reactive power; they just "dissipate/use" power.

For capacitors, the current through the capacitor would be $\widetilde{I}_C = \frac{\widetilde{V}}{Z_C} = \frac{\widetilde{V}}{\frac{1}{j\omega C}} = j\omega C\widetilde{V}$. Thus, the average power would be:

$$P_{\text{avg}} = \frac{1}{2}\text{Re}\{\widetilde{V}\widetilde{I}_C^*\} = \frac{1}{2}\text{Re}\{\widetilde{V}(-j\omega C\widetilde{V}^*\} = \frac{1}{2}\text{Re}\{-j\omega C|\widetilde{V}|^2\} = 0 \tag{15}$$

The reactive power would be:

$$P_{\text{reactive}} = \frac{1}{2}\text{Im}\{\widetilde{V}\widetilde{I}_C^*\} = \frac{1}{2}\text{Im}\{-j\omega C|\widetilde{V}|^2\} = -\frac{1}{2}\omega C|\widetilde{V}|^2 \tag{16}$$

For inductors, the current through the inductor would be $\widetilde{I}_L = \frac{\widetilde{V}}{Z_L} = \frac{\widetilde{V}}{j\omega L} = -j\frac{\widetilde{V}}{\omega L}$. Thus, the average power would be:

$$P_{\text{avg}} = \frac{1}{2}\text{Re}\{\widetilde{V}\widetilde{I}_L^*\} = \frac{1}{2}\text{Re}\{\widetilde{V}(j\frac{\widetilde{V}^*}{\omega L}\} = \frac{1}{2}\text{Re}\{j\frac{|\widetilde{V}|^2}{\omega L}\} = 0 \tag{17}$$

The reactive power would be:

$$P_{\text{reactive}} = \frac{1}{2}\text{Im}\{\widetilde{V}\widetilde{I}_L^*\} = \frac{1}{2}\text{Im}\{j\frac{|\widetilde{V}|^2}{\omega L}\} = \frac{1}{2}\frac{|\widetilde{V}|^2}{\omega L} \tag{18}$$

The observations we make here are important. Both capacitors and inductors **do not dissipate power**; this is equivalent to the $\phi_V - \phi_I = \frac{\pi}{2}$ case we saw earlier! They only have reactive power, which is accessible to the system later. Thus, capacitors and inductors are often known as **reactive elements**; elements that store power to be used later (this is most obvious with a capacitor that stores charge that could be used later, but it happens with inductors too). In fact, the imaginary part of impedance is called reactance ($Z = R + jX$, where X is the variable for reactance)! An important example of the implications of this are seen in the LC tank example; earlier in the course, we saw how the LC tank circuit (with no resistance) oscillates continuously, without loss of amplitude. This happened because the capacitor and inductor did not dissipate the power of the system; they just moved power back and forth between each other, represented by the continuous oscillation! This idea that capacitors and inductors do not dissipate power is important to remember.

3 Quality Factor

One aspect of circuits, especially RLC circuits, that we will introduce now is the **quality factor** (Q), which is a value that essentially measures how much loss (power dissipation) a circuit has (a higher quality factor corresponds to less loss). As you may think after reading about AC power, this will be related to having a low amount of associated resistance, since resistance is what dissipates power (reactance only stores power).

The quality factor can be defined in multiple ways. In the context of power, we will define the quality factor as the ratio of the magnitude of reactive power (either of the inductor or capacitor) to average power *at the resonant frequency (ω_0)*:

$$Q = \frac{P_{\text{reactive}}}{P_{\text{avg}}} \tag{19}$$

For now, we will just leave the resonant frequency as ω_0, but later, you will understand what the resonant frequency is for RLC circuits (you may already remember for second-order differential equations).

Suppose we have a standard series RLC circuit:

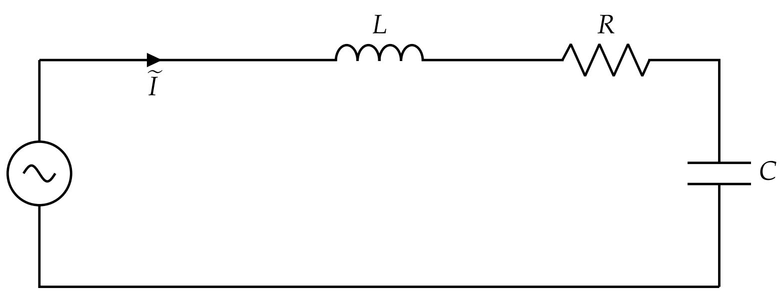


Figure 1: Series RLC Circuit

The current through all of the elements is $\widetilde{I}$. In terms of this current, the reactive power across the inductor at $\omega = \omega_0$ is:

$$P_{\text{reactive}} = \frac{1}{2}\text{Im}\{\widetilde{V}_L\widetilde{I}_L^*\} = \frac{1}{2}\text{Im}\{\widetilde{I}(\text{j}\omega_0 L)\widetilde{I}^*\} = \frac{1}{2}\omega_0 L|\widetilde{I}|^2 \tag{20}$$

The average power must be from the resistor:

$$P_{\text{avg}} = \frac{1}{2}\text{Re}\{\widetilde{V}_R\widetilde{I}_R^*\} = \frac{1}{2}\text{Re}\{\widetilde{I}(R)\widetilde{I}^*\} = \frac{1}{2}R|\widetilde{I}|^2 \tag{21}$$

Thus, the quality factor is

$$Q = \frac{P_{\text{reactive}}}{P_{\text{avg}}} = \frac{\frac{1}{2}\omega_0 L|\widetilde{I}|^2}{\frac{1}{2}R|\widetilde{I}|^2} = \frac{\omega_0 L}{R} \tag{22}$$

This is a useful fact to know about series RLC circuits if you work more with them in the future (such as for RF, or high frequency, applications)! The quality factor will be a little different in different situations (for example, the parallel RLC has a different quality factor), but the concepts are the same; at the resonant frequency, there is some ratio between the reactive power through the inductor/capacitor and the average power disspiated by the resistance.

4 Check your Understanding

Read through these simple questions to check your basic understanding of the notes.

- Why do we care about average power for systems with sinusoidal inputs?
- Where does the factor of $\frac{1}{2}$ come from in the expressions for power?
- How do we interpret the real and imaginary parts of complex power with phasors?
- What elements dissipate power? What elements store power?
- What value would be the ideal quality factor if we want minimal power loss?
- How would you find an expression for quality factor for a parallel RLC circuit?

Contributors:

- Nikhil Jain.