EECS 16B Designing Information Systems and Devices II UC Berkeley Spring 2024

Note 6B: Transfer Functions

1 Introduction to Transfer Functions

Informally, *transfer functions* are functions that describe the behavior of some system as a function of the frequency.

Definition 1 (Transfer Function)
Consider the block diagram of a system in Figure 1.

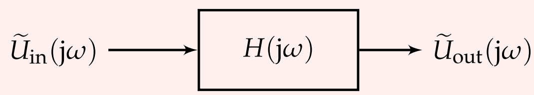

Figure 1: Transfer Function Block Diagram

where $\widetilde{U}_{\text{in}}(\mathrm{j}\omega)$ and $\widetilde{U}_{\text{out}}(\mathrm{j}\omega)$ are the respective inputs and outputs of the system, and the block is the system itself. The transfer function, $H(\mathrm{j}\omega)$, is defined by

$$H(\mathrm{j}\omega) = \frac{\widetilde{U}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{U}_{\text{in}}(\mathrm{j}\omega)} \tag{1}$$

Equivalently, $\widetilde{U}_{\text{out}}(\mathrm{j}\omega) = H(\mathrm{j}\omega)\widetilde{U}_{\text{in}}(\mathrm{j}\omega)$. We use an arbitrary definition of a system above, but we will make this more concrete for the specific case of transfer functions of circuits. When describing the behavior of a transfer function, we typically look at the magnitude and phase of the transfer function, as a function of ω.

2 Transfer Functions of Common Filters

A *filter* is commonly used to block or allow certain ranges of frequencies to pass through as an output, i.e., it allows or restricts certain inputs, based on the frequencies (ω) of the inputs. Generally, filters are written as transfer functions of the form $H(\mathrm{j}\omega) = \frac{p(\omega)}{q(\omega)}$, for $p(\cdot)$ and $q(\cdot)$ being polynomials. From this, we can define the concept of the *order* of a filter.

Definition 2 (Filter Order)
Suppose that a filter's transfer function can be written as a simplified fraction of two polynomials, i.e., $H(\mathrm{j}\omega) = \frac{p(\omega)}{q(\omega)}$. The order of a transfer function is $\max(\deg(p), \deg(q))$, where $\deg(\cdot)$ denotes the degree of the polynomial.

In circuits, we define the "inputs" to our transfer function to be some sort of input voltage phasor, denoted $\widetilde{V}_{\text{in}}$, and the output as some sort of output voltage phasor $\widetilde{V}_{\text{out}}$. We typically encounter two types of first order filters – a low pass filter and a high pass filter.

2.1 Common First Order Filters

Low Pass Filter:

The following transfer function is an example of a low pass filter:

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{2}$$

where ω_c is known as the *cutoff frequency*[1]. This transfer function attenuates the magnitude of outputs where the inputs have frequency $\omega \gg \omega_c$, but does not affect the magnitude for inputs that have frequency $\omega \ll \omega_c$.

Why this is the case?

We can convert the numerator and denominator of eq. (2) into phasor form, namely:

$$1 = (1)\mathrm{e}^{\mathrm{j}0} \tag{3}$$

and

$$1 + \mathrm{j}\frac{\omega}{\omega_c} = \left|1 + \mathrm{j}\frac{\omega}{\omega_c}\right| \mathrm{e}^{\mathrm{j}\measuredangle\left(1+\mathrm{j}\frac{\omega}{\omega_c}\right)} \tag{4}$$

$$= \sqrt{1 + \frac{\omega^2}{\omega_c^2}}\,\mathrm{e}^{\mathrm{j}\,\mathrm{atan2}\left(\frac{\omega}{\omega_c},1\right)} \tag{5}$$

Let $\theta = \mathrm{atan2}\left(\frac{\omega}{\omega_c}, 1\right)$.

So if we were to combine this altogether, we would have

$$H(\mathrm{j}\omega) = \frac{(1)\mathrm{e}^{\mathrm{j}0}}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}\,\mathrm{e}^{\mathrm{j}\theta}} \tag{6}$$

$$= \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}}\mathrm{e}^{-\mathrm{j}\theta} \tag{7}$$

so $|H(\mathrm{j}\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}$. Note that this is a function of ω, which is expected. Now, we can prove the behavior for $\omega \ll \omega_c$ by taking the limit as $\omega \to 0$:

$$\lim_{\omega \to 0} \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}} = 1 \tag{8}$$

so, as $\omega \to 0$, $\left|\widetilde{U}_{\mathrm{out}}(\mathrm{j}\omega)\right| = \left|\widetilde{U}_{\mathrm{in}}(\mathrm{j}\omega)\right|$. Now, we can show the behavior for $\omega \gg \omega_c$ by taking a limit as $\omega \to \infty$:

$$\lim_{\omega \to \infty} \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}} = 0 \tag{9}$$

so, as $\omega \to \infty$, $\left|\widetilde{U}_{\mathrm{out}}(\mathrm{j}\omega)\right| = 0$. Yes but, what does it mean that when $\omega \to 0$, nothing happens (the input is equal to output) and for $\omega \to \infty$ we get zero as the result? Assume we plug as input to this filter a sinusoidal function with a fixed frequency ω, the higher this frequency is, the more attenuated the output will be. If this frequency is high enough, with respect to the cut off frequency, essentially zero signal will pass through the filter and it would effectively not be present at the output. If the frequency is low enough with respect to the cut off frequency, we will essentially recover the exact signal at the output.

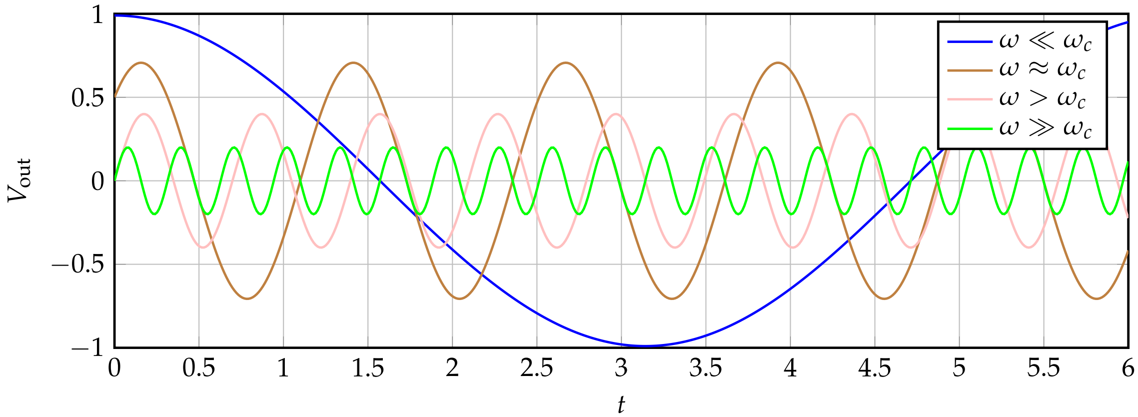

Figure 2: Low pass filter with input a sinusoidal function at different frequencies. The blue and the red sinusoidal functions have the same exact magnitude, because the filter does not attenuate at all the signal. Note that at cutoff we always have some attenuation.

Here is the low pass filter response with input amplitude 1 V:

Example:

We can implement this kind of transfer function in circuits as follows[2]:

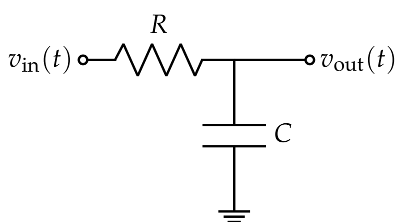

Figure 3: RC Low Pass Circuit

Note that this is not the only way to implement a low pass transfer function in circuit form. Here, the input would be $\widetilde{U}_{in}(j\omega) := \widetilde{V}_{in}(j\omega)$, the phasor for $v_{in}(t)$, and the output would be $\widetilde{U}_{out}(j\omega) := \widetilde{V}_{out}(j\omega)$, the phasor for $v_{out}(t)$. To show that this circuit is an implementation of a low pass transfer function, we can find $\widetilde{V}_{out}$ in terms of $\widetilde{V}_{in}$ and use this to find $H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$. Using the voltage divider formula, we have

$$\widetilde{V}_{out} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \widetilde{V}_{in} = \frac{1}{1 + j\omega RC} \widetilde{V}_{in} \tag{10}$$

so

$$H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{1}{1 + j\omega RC} \tag{11}$$

We identify the cutoff frequency as $\omega_c = \frac{1}{RC}$, and we exactly recover the form of a low pass transfer function.

We can also implement a low pass transfer function using inductors, as follows:

[1]The idea of a cutoff frequency will become more concrete when we plot transfer functions.

[2]For all the transfer function implementations described in this note, the circuits themselves are not unique. That is, it is possible to emulate the same transfer function behavior with different circuit components.

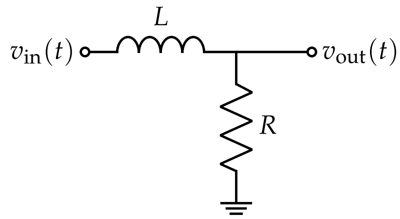

Figure 4: LR Low Pass Circuit

Concept Check: Show that the circuit in Figure 4 implements a low pass transfer function.

High Pass Filter:

The following transfer function is an example of a high pass filter:

$$H(\mathrm{j}\omega) = \frac{\mathrm{j}\frac{\omega}{\omega_c}}{1+\mathrm{j}\frac{\omega}{\omega_c}} \tag{12}$$

with ω_c being the cutoff frequency. This transfer function attenuates the magnitude of outputs where the inputs have frequency $\omega \ll \omega_c$, and not affect the magnitude for inputs that have frequency $\omega \gg \omega_c$.

Why this is the case?

Similar to before, we can separately convert the numerator and denominator to phasors and take limits. For the numerator,

$$\mathrm{j}\frac{\omega}{\omega_c} = \left|\mathrm{j}\frac{\omega}{\omega_c}\right| \mathrm{e}^{\mathrm{j}\measuredangle \mathrm{j}\frac{\omega}{\omega_c}} \tag{13}$$

$$= \frac{\omega}{\omega_c}\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} \tag{14}$$

and for the denominator, the phasor is the same as before, i.e.,

$$1+\mathrm{j}\frac{\omega}{\omega_c} = \left|1+\mathrm{j}\frac{\omega}{\omega_c}\right| \mathrm{e}^{\mathrm{j}\measuredangle(1+\mathrm{j}\frac{\omega}{\omega_c})} \tag{15}$$

$$= \sqrt{1+\frac{\omega^2}{\omega_c^2}}\,\mathrm{e}^{\mathrm{j}\,\mathrm{atan2}\left(\frac{\omega}{\omega_c},1\right)} \tag{16}$$

Let $\theta = \mathrm{atan2}\left(\frac{\omega}{\omega_c}, 1\right)$ as before.

Hence, the resulting phasor expression for the transfer function is

$$H(\mathrm{j}\omega) = \frac{\frac{\omega}{\omega_c}}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{2}-\theta\right)} \tag{17}$$

Taking limits on the magnitude,

$$\lim_{\omega\to 0} \frac{\frac{\omega}{\omega_c}}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}} = 0 \tag{18}$$

and

$$\lim_{\omega\to\infty} \frac{\frac{\omega}{\omega_c}}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}} = \lim_{\omega\to\infty} \frac{1}{\sqrt{\frac{\omega_c^2}{\omega^2}+1}} = 1 \tag{19}$$

which agrees with the qualitative behavior described above.

For the high pass filter we have the opposite result with respect to the low pass filter. Also in this case, what does it mean that when $\omega \to \infty$ nothing happens (the input is equal to output) and for $\omega \to 0$, we get zero as the result? Assume again, we plug as input to this filter a sinusoidal function with a fixed frequency ω; the higher this frequency is, the less it will be attenuated at the output. If this frequency is high enough, with respect to the cut off frequency, essentially all the signal will pass through the filter. Instead if the frequency is low enough with respect to the cut off frequency, we would essentially measure zero at the output.

Here is the high pass filter output with input amplitude 1 V.

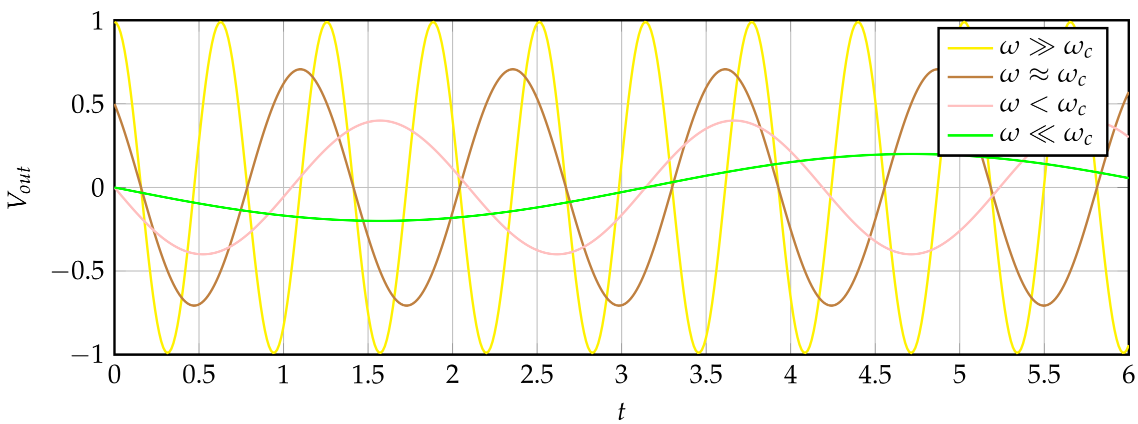

Figure 5: High pass filter with input a sinusoidal function at different frequencies. Note that at cutoff we always have some attenuation.

Example:
We can implement this kind of transfer function in circuits as follows:

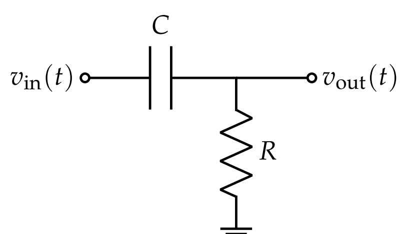

Figure 6: RC High Pass Circuit

We can show that this implements a high pass transfer function by computing the transfer function $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ for this circuit as before. Using the voltage divider formula,

$$\widetilde{V}_{\text{out}} = \frac{R}{R + \frac{1}{\mathrm{j}\omega C}} \widetilde{V}_{\text{in}} = \frac{\mathrm{j}\omega RC}{1 + \mathrm{j}\omega RC} \widetilde{V}_{\text{in}} \tag{20}$$

which matches the high pass transfer function definition if we identify the cutoff frequency as $\omega_c = \frac{1}{RC}$. Another way to implement a high pass transfer function is

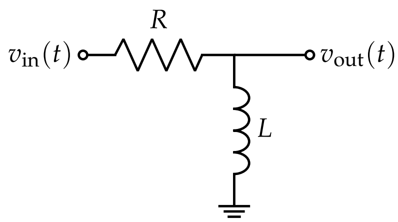

Figure 7: LR High Pass Circuit

Concept Check: Show that the above circuit implements a high pass transfer function. Now we can proceed to discuss filters with more complex behavior by focusing on second order filters.

2.2 Second Order Filters

Based on the transfer functions discussed in the previous subsection, we can define some second order filters to be a product of the filters discussed previously.

Definition 3 (Second Order Low/High Pass)
A second order low/high pass filter is constructed by squaring the transfer function of a first low/high pass filter, i.e.

$$H_{\text{Second Order LP}} = (H_{\text{LP}})^2 \tag{21}$$
$$H_{\text{Second Order HP}} = (H_{\text{HP}})^2 \tag{22}$$

Example:
In practice, we combine filters by connecting them with a unity gain op amp, as shown in Figure 8. The reason for this is that it prevents a loading effect, which would otherwise be present without the unity gain op amp.

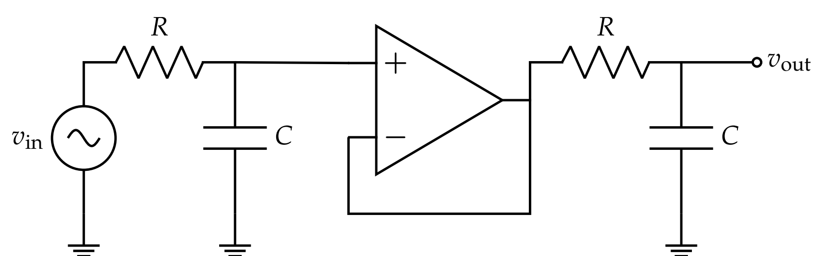

Figure 8: Second Order Low Pass Filter

We can also define a *band pass* filter.

Band Pass Filter:

A band pass filter does not attenuate the magnitude of inputs with frequencies inside of a certain interval, say $\omega \in [a, b]$, and it attenuates frequencies outside this interval.

Example:
We could implement this in a circuit by combining a low pass filter and high pass filter with a unity gain

op amp (note, this is not the only way to create a band pass filter). Mathematically, we can write this as

$$H_{\text{BP}}(\text{j}\omega) = H_{\text{LP}}(\text{j}\omega) \cdot H_{\text{HP}}(\text{j}\omega) \tag{23}$$

The low pass filter needs to have a cutoff frequency higher than that of the high pass filter. Here, as $\omega \to 0$ and $\omega \to \infty$, $|H(\text{j}\omega)| \to 0$ (by virtue of limit multiplication rules) with less attenuation for frequencies somewhere in between. Following the terminology used in the definition above, we would generally set the cutoff frequency of the high pass filter to $\omega_{c,\text{HP}} = a$ and the cutoff frequency of the low pass filter to $\omega_{c,\text{LP}} = b$. A band pass filter implemented in this manner might look like the circuit below. Note that we would want $\frac{1}{R_1C_1} > \frac{1}{R_2C_2}$.

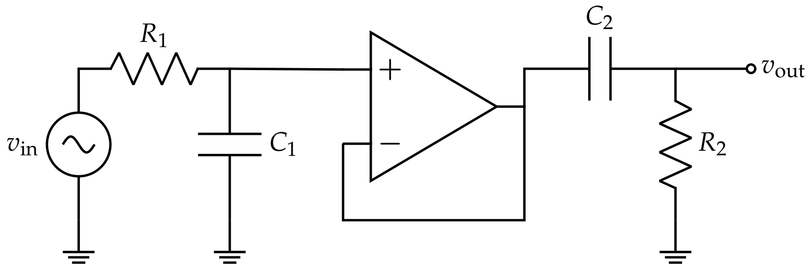

Figure 9: Band Pass Filter

Similarly, we can create a filter that has the opposite effect of a band pass filter, i.e. a notch filter.

Notch Filter:

A notch filter is the opposite of a band pass filter, in that it attenuates the magnitude of inputs with frequencies inside of a certain interval, say $\omega \in [a, b]$, and it does not attenuate frequencies outside this interval. It is called a notch because of the sharp spike that occurs due to LC resonance effects.

Example:

We can implement a notch filter with an LC tank type circuit, as shown in Figure 10.

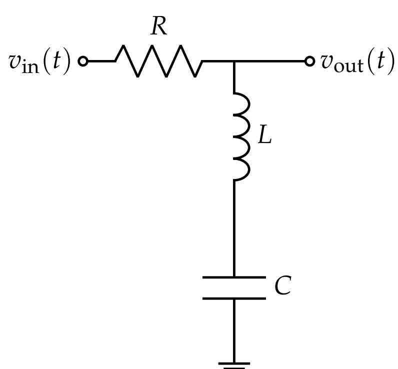

Figure 10: RC Low Pass Circuit

The transfer function, which we derive using voltage dividers, is

$$H(\text{j}\omega) = \frac{\text{j}\omega L + \frac{1}{\text{j}\omega C}}{R + \text{j}\omega L + \frac{1}{\text{j}\omega C}} \tag{24}$$

$$= \frac{1 - \omega^2 LC}{\mathrm{j}\omega RC - \omega^2 LC + 1} \tag{25}$$

$$= \frac{1}{1 + \mathrm{j}\frac{\omega RC}{1-\omega^2 LC}} \tag{26}$$

To find $|H(\mathrm{j}\omega)|$, we can find the magnitude of the top and bottom separately, i.e.,

$$|1| = 1 \tag{27}$$

and

$$\left|1 + \mathrm{j}\frac{\omega RC}{1 - \omega^2 LC}\right| = \sqrt{\left(1 + \mathrm{j}\frac{\omega RC}{1 - \omega^2 LC}\right)\left(1 - \mathrm{j}\frac{\omega RC}{1 - \omega^2 LC}\right)} \tag{28}$$

$$= \sqrt{1 + \left(\frac{\omega RC}{1 - \omega^2 LC}\right)^2} \tag{29}$$

and divide the two to obtain

$$|H(\mathrm{j}\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega RC}{1-\omega^2 LC}\right)^2}} \tag{30}$$

To see what frequency is most attenuated, we can see what frequency minimizes $|H(\mathrm{j}\omega)|$. To do this, we can maximize the denominator, or equivalently maximize $\frac{\omega RC}{1-\omega^2 LC}$. Notice that this term goes to ∞ when $1 - \omega^2 LC = 0 \iff \omega = \frac{1}{\sqrt{LC}}$. Hence, $|H(\mathrm{j}\omega)| = 0$ at $\omega = \frac{1}{\sqrt{LC}}$. **Concept Check:** Show that the transfer function satisfies the remaining requirements of a notch filter by taking limits as $\omega \to 0$ and $\omega \to \infty$.

Key Idea 4 (Choosing Cutoff Frequencies)
When designing transfer functions, you can choose various values for resistance, capacitance, and inductance based on the desired specifications of the system. As we have seen before, the cutoff frequencies in all of these circuit filters are functions of resistance, capacitance, and/or inductance. Hence, it is important to carefully choose values for these based on design requirements and cost constraints.

Note: This section only contains examples of some common transfer functions. It is by no means an exhaustive list of all possible transfer functions, or even an exhaustive list of transfer functions we will cover in the class.

3 Extension Beyond Sinusoidal Functions

3.1 Fourier Transforms

After reading this note, you may think: can we only use phasors to handle sinusoidal inputs? Do we have to use differential equations for other types of inputs? It actually turns out that we can extend the concept of phasors to **all** possible inputs by using **Fourier transforms**, which allow us to convert any function to the frequency/phasor domain, which means we can describe any function as the linear combination of complex exponentials of different frequencies! Then, using superposition, we can essentially apply phasors for each frequency component of the input function and use the inverse Fourier transform to return to the time domain.

What this essentially means is that the frequency/phasor domain does not have to be restricted to just sinusoidal inputs; it can be applied in essentially all scenarios. This ability to change calculus problems into algebra problems is such a powerful tool that circuits are almost always analyzed in the frequency/phasor domain due to the simplicity.

3.2 Laplace Transforms

Yet another question you may have is: what about transient responses? Phasor analysis assumes steady state, correct? This is true, so far our phasor analysis has assumed steady state, which means we do not have information on how the system reached steady state. So, we have to use differential equations for that too? Actually, it turns out we can find the transient response in the frequency/phasor domain as well, with an adjustment to using **Laplace transforms** instead of Fourier transforms. This allows us to account for real exponentials as well, which we see as the behavior of our systems when they move towards the steady state value.

The concepts of Fourier and Laplace transforms are out of scope of this course, but you will have a chance to learn about them in future classes such as EE 120. What you can take away from this section is that the frequency/phasor domain is a very powerful tool that is also versatile and can be used in essentially all situations and you will see it essentially everywhere if you take more courses related to circuits.

4 Check your Understanding

Read through these simple questions to check your basic understanding of the notes.

- What are transfer functions? What is the input variable into the transfer function and what type of value (real or complex) is the output?
- How could we use a transfer function to calculate the output for a specific input?
- What are the common filter types and how do we implement them?
- How could we cascade filters (connect them in sequence) so that the overall filter contains the behavior of both filters combined (such as to create a band pass filter)?

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