Note 6A: Phasors

1 Introduction to Phasors

1.1 Magnitude-Phase Representations of Complex Numbers

We can represent any complex number with a magnitude and phase. That is, for all complex numbers x, $x = Ae^{j\phi}$ for some real values A, ϕ.

Theorem 1 (Magnitude-Phase Representation)
Given a complex number $x = a + jb$, we can equivalently represent it in the form $x = Ae^{j\phi}$ where $A = |x| = \sqrt{x\overline{x}} = \sqrt{a^2 + b^2}$ and $\phi = \text{atan2}(b, a)$.

Proof. We can set the two representations equal and solve for A and ϕ. That is,

$$a + jb = Ae^{j\phi} \tag{1}$$

$$= A\cos(\phi) + jA\sin(\phi) \tag{2}$$

so we have that $a = A\cos(\phi)$ and $b = A\sin(\phi)$. We have that

$$A^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) \tag{3}$$

$$= a^2 + b^2 \tag{4}$$

so $A = \sqrt{a^2 + b^2}$. Next, we have that

$$\frac{b}{a} = \frac{\sin(\phi)}{\cos(\phi)} \tag{5}$$

$$= \tan(\phi) \tag{6}$$

so $\phi = \text{atan2}(b, a)$[1]. □

1.2 Determining Phasors for Sine and Cosine Functions

First, we should note a corollary of Euler's formula.

Theorem 2 (Euler's Theorem)
The following identities hold:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{7}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{8}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{9}$$

[1] We choose to use two argument atan (i.e., atan2) because this preserves the sign of the angles and we will not encounter division by 0 this way.

Now, we can derive a formula for the phasor representation of a sine/cosine. Note that $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$, so it suffices to derive a phasor representation for an arbitrary cosine function.

Key Idea 3 (Cosine Phasors)
Suppose we are given an arbitrary, time-varying cosine function of the form $v(t) = V_0 \cos(\omega t + \phi)$, where V_0 is the amplitude, ω is the frequency, and ϕ is a phase shift. The function $v(t)$'s phasor for the frequency ω is given by $\widetilde{V} = V_0 e^{j\phi}$[a].

[a] We denote the phasor for $v(t)$ as $\widetilde{V}$, dropping the time input, capitalizing, and putting a tilde on top.

There are different ways to interpret this transformation. One common way to do so is to say that we operate in the complex domain, where the cosine function is the real part of its associated complex exponential function as follows:

$$\mathrm{Re}\{V_0 e^{j\phi} e^{j\omega t}\} = \mathrm{Re}\{V_0 \cos(\omega t + \phi) + jV_0 \sin(\omega t + \phi)\} = V_0 \cos(\omega t + \phi) \tag{10}$$

The essential idea here is that we will do computation in the complex domain, where the $e^{j\omega t}$ factors will essentially disappear from the calculations (since in steady state, all the voltages and currents in the system will become complex exponentials with the same frequency as well), so our phasor definition only needs to keep track of the coefficient of the $e^{j\omega t}$ factors. The actual currents and voltages correspond to the real parts of the complex exponentials, which are the sinusoidal functions. In discussions, you will see another valid way to interpret/derive the phasor definition we present here.

Corollary 4 (Sine Phasors)
Suppose we are given an arbitrary, time-varying sine function of the form $v(t) = V_0 \sin(\omega t + \phi)$, where V_0 is the amplitude, ω is the frequency, and ϕ is a phase shift. The function $v(t)$'s phasor for the frequency ω is given by $\widetilde{V} = \frac{V_0 e^{j\phi}}{j}$.

Concept Check: Prove this corollary, using the fact that $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$ and that $e^{-j\frac{\pi}{2}} = -j = \frac{1}{j}$.
Example:
Suppose $v(t) = 10\cos\left(20t + \frac{3\pi}{2}\right)$. To find the phasor for this function, we can begin by pattern matching $V_0 = 10$ and $\phi = \frac{3\pi}{2}$. Applying this to the resulf of Key idea 3, we have $\widetilde{V} = 10e^{j\frac{3\pi}{2}} = -10j$.

2 Computing Impedances in Phasor Domain

We can look at the phasor domain "resistances" of all passive circuit elements we have learned so far. The technical term for these "resistances" is impedance. Formally, we denote this as

$$Z = \frac{\widetilde{V}}{\widetilde{I}} \tag{11}$$

We are leveraging the I-V relationship of each circuit element in phasor domain so that we can derive their phasor domain impedances.

Theorem 5 (Impedance of a Capacitor)
Suppose we applied an input voltage $v_C(t) = V_0 \cos(\omega t + \phi)$ across a capacitor with capacitance C. Its phasor domain impedance is given by $Z_C = \frac{1}{j\omega C}$.

Proof. We can find $i_C(t)$ and then find its phasor domain representation, i.e., $\widetilde{I}_C$. We can apply the equation relating current and voltage across a capacitor, namely

$$\begin{aligned} i_C(t) &= C\frac{\mathrm{d}}{\mathrm{d}t} v_C(t) && (12) \\ &= C\frac{\mathrm{d}}{\mathrm{d}t}(V_0 \cos(\omega t + \phi)) && (13) \\ &= -\omega C V_0 \sin(\omega t + \phi) && (14) \end{aligned}$$

Using Corollary 4, we have that

$$\begin{aligned} \widetilde{I}_C &= \frac{-\omega C V_0 e^{j\phi}}{j} && (15) \\ &= j\omega C V_0 e^{j\phi} && (16) \end{aligned}$$

and by Key idea 3, we have that

$$\widetilde{V}_C = V_0 e^{j\phi} \tag{17}$$

Hence,

$$Z_C = \frac{\widetilde{V}_C}{\widetilde{I}_C} = \frac{1}{j\omega C} \tag{18}$$

□

Theorem 6 (Impedance of a Resistor)
Suppose we applied an input voltage $v_R(t) = V_0 \cos(\omega t + \phi)$ across a resistor with resistance R. Its phasor domain impedance is given by $Z_R = R$.

Proof. Using the same technique as the proof of Theorem 5, we find $i_R(t)$ as follows:

$$i_R(t) = \frac{1}{R} v_R(t) = \frac{V_0}{R}\cos(\omega t + \phi) \tag{19}$$

The phasor domain representation of this is

$$\widetilde{I}_R = \frac{V_0}{R} e^{j\phi} = \frac{1}{R} V_0 e^{j\phi} \tag{20}$$

The expression for $\widetilde{V}_R$ remains the same as the expression for $\widetilde{V}_C$ in Theorem 5. Hence,

$$Z_R = \frac{\widetilde{V}_R}{\widetilde{I}_R} = R \tag{21}$$

□

Theorem 7 (Impedance of an Inductor)
Suppose we applied an input current $i_L(t) = V_0 \cos(\omega t + \phi)$ through an inductor with inductance L. Its phasor domain impedance is given by $Z_L = \mathrm{j}\omega L$.

Proof. We can find $\widetilde{V}_L$ by first finding $v_L(t)$ as follows:

$$v_L(t) = L\frac{\mathrm{d}}{\mathrm{d}t} i_L(t) \tag{22}$$
$$= L\frac{\mathrm{d}}{\mathrm{d}t}(V_0 \cos(\omega t + \phi)) \tag{23}$$
$$= -L\omega V_0 \sin(\omega t + \phi) \tag{24}$$

Now, we can use Corollary 4 to find $\widetilde{V}_L$:

$$\widetilde{V}_L = \frac{-\omega L V_0 \mathrm{e}^{\mathrm{j}\phi}}{\mathrm{j}} \tag{25}$$
$$= \mathrm{j}\omega L V_0 \mathrm{e}^{\mathrm{j}\phi} \tag{26}$$

Here, we have that $\widetilde{I}_L = V_0 \mathrm{e}^{\mathrm{j}\phi}$ so

$$Z_L = \frac{\widetilde{V}_L}{\widetilde{I}_L} = \mathrm{j}\omega L \tag{27}$$

□

Key Idea 8 (Using Phasor Impedances)
Since the phasor impedance represent an I-V relationship in phasor domain, and since the impedance is constant with respect to time, we can treat all components' phasor domain representations as time domain resistors. That is, we can apply the same rules for KCL, NVA, and parallel/series combinations of resistors.

Example:
We can solve for $v_{\text{out}}(t)$ in Figure 1 by using phasor domain conversions.

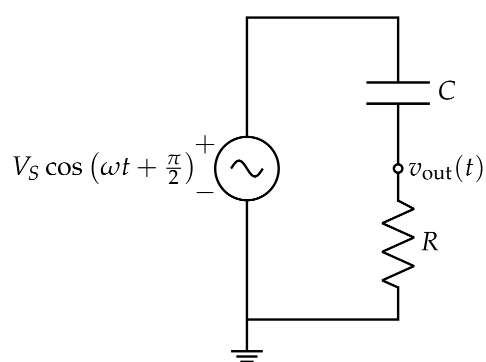

Figure 1: Example Circuit

Here, we can perform the phasor domain conversion on the input voltage since it is a single sinusoid. That is, we have that $v_{\text{in}}(t) := V_S \cos\left(\omega t + \frac{\pi}{2}\right)$ so $\widetilde{V}_{\text{in}} = V_S \mathrm{e}^{\mathrm{j}\frac{\pi}{2}}$. Using the fact that $Z_C = \frac{1}{\mathrm{j}\omega C}$ and $Z_R = R$, we can treat these components as "resistors" in phasor domain. That is, we can apply the resistor voltage divider formula to obtain

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_C + Z_R}\widetilde{V}_{\text{in}} \tag{28}$$

$$= \frac{R}{\frac{1}{\mathrm{j}\omega C} + R}\left(V_S \mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\right) \tag{29}$$

$$= \frac{\mathrm{j}\omega RC}{1 + \mathrm{j}\omega RC}\left(V_S \mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\right) \tag{30}$$

The factor of $\frac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC}$ here is important. Let's define $H = |H|\mathrm{e}^{\mathrm{j}\measuredangle H} = \frac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC}$ (the polar form of the complex number):

$$\widetilde{V}_{\text{out}} = |H|\mathrm{e}^{\mathrm{j}\measuredangle H}\left(V_S \mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\right) \tag{31}$$

$$= |H| V_S \mathrm{e}^{\mathrm{j}\left(\frac{\pi}{2} + \measuredangle H\right)} \tag{32}$$

Next, we can reverse the steps of Key idea 3 to obtain the time domain output. We can pattern match $V_0 = |H|V_S$ and $\phi = \frac{\pi}{2} + \measuredangle H$, so

$$v_{\text{out}}(t) = |H| V_S \cos\left(\omega t + \frac{\pi}{2} + \measuredangle H\right) \tag{33}$$

Notice how the amplitude is multiplied by $|H|$ and $\measuredangle H$ is added to the current phase within the sinusoidal function. For a specific case, we could calculate the exact value of H and use it to find the exact time domain output for a specific frequency.

3 Check your Understanding

Read through these simple questions to check your basic understanding of the notes.

- What is the phasor/frequency domain representation of $V_0 \cos(\omega t + \phi)$?
- Why does our phasor/frequency domain representation not need to include information about the frequency of the input, ω?
- What is impedance? What are the impedances of the resistor, capacitor, and inductor?
- How could phasors help us solve a differential equation with sinusoidal input? (Hint: The type of input corresponds to the type of particular solution. How do we like to find the particular solution for circuits and how does phasor analysis correspond to this?)

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