Note 5: Second-Order Differential Equations with RLC Circuits

1 Second Order Differential Equations

Definition 1 (Second Order, Linear Differential Equation)

A second order, linear differential equation can be put into the form

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 2\alpha \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega_0^2 x(t) = f(t) \tag{1}$$

for some constants $\alpha, \omega_0 \in \mathbb{R}$ (often referred to as the *damping coefficient* and *resonant frequency* respectively) and some function of time $f(t)$ (this is sometimes called a *forcing function*). The solution to this differential equation can be separated into homogeneous and particular solutions of the form

$$x(t) = x_h(t) + x_p(t) \tag{2}$$

where $x_h(t)$ represents the homogeneous solution and $x_p(t)$ represents the particular solution.

We typically solve separately for the homogeneous and particular solutions. The homogeneous solution is the solution to

$$\frac{\mathrm{d}^2 x_h(t)}{\mathrm{d}t^2} + 2\alpha \frac{\mathrm{d}x_h(t)}{\mathrm{d}t} + \omega_0^2 x_h(t) = 0 \tag{3}$$

How do we find the homogeneous solution? Could we use the same guess and check strategy we used for the 1st order case? Yes! And it turns out that our guess in this case will also be an exponential (this continues for higher order circuits as well, when there are constant coefficients).

Let's assume that $x_h(t) = \mathrm{e}^{st}$. To verify that our guess is a solution of the homogeneous, we put it into the differential equation itself:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathrm{e}^{st}) + 2\alpha \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{st}) + \omega_0^2(\mathrm{e}^{st}) = 0 \tag{4}$$

$$s^2\mathrm{e}^{st} + 2\alpha s\mathrm{e}^{st} + \omega_0^2\mathrm{e}^{st} = 0 \tag{5}$$

We know that $e^{st} \neq 0$ for all t so we can divide it out:

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{6}$$

This is just a quadratic equation! We can solve for s (the exponential coefficients) that solves the equation using the quadratic formula. The corresponding solutions of s plugged back into our guess will be the solutions to the homogeneous differential equation!

$$s = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega_0^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \tag{7}$$

Note that s could be complex (specifically if $|\frac{\alpha}{\omega_0}| < 1$).

NOTE: We could do this process directly if we had values for the differential equation, however, here we are considering all the possible cases, leaving the equation parametric. Recall that we do not have to identify the α and ω_0 coefficients, they are given.

Theorem 2 (Homogeneous Solution to Second Order Differential Equations)
Define $s_1 := -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $s_2 := -\alpha - \sqrt{\alpha^2 - \omega_0^2}$. The homogeneous solution will take on one of the following forms, depending on the value of $\zeta = \frac{\alpha}{\omega_0}$, called the *damping ratio*.

- *Overdamped Case:* ($|\zeta| > 1$)
$$x_h(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{8}$$

- *Critically Damped Case:* ($|\zeta| = 1$)
$$x_h(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t} \tag{9}$$
Note that $s_1 = s_2$ in this case.

- *Underdamped Case:* ($|\zeta| < 1$)
Note that s_1 and s_2 will be complex, so we can rewrite them as $s_1 = -\alpha + j\omega_n$ and $s_2 = -\alpha - j\omega_n$ where $\omega_n := \sqrt{\omega_0^2 - \alpha^2}$ is defined as the natural frequency. The solution is of the form
$$x_h(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t) \tag{10}$$

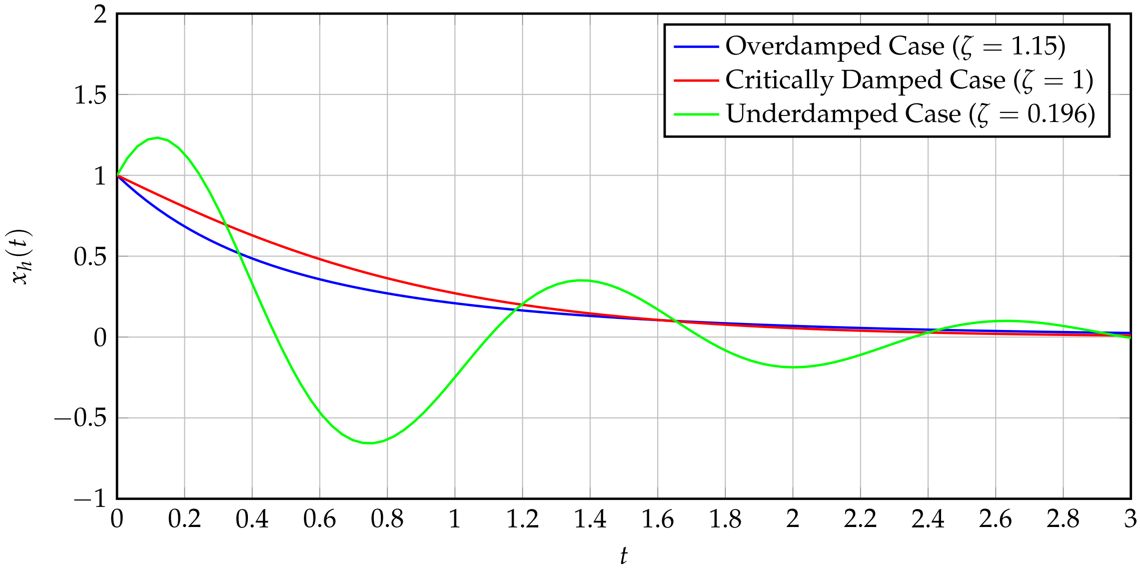
In all of the cases above, K_1 and K_2 are arbitrary constants that are determined by initial conditions. Note that, since we have to find the values of two arbitrary constants we will need **two initial conditions** to completely solve a second order differential equation.

Why do sinusoidal solutions come into play when s_1 and s_2 are complex? Remember that sinuosidal functions are linear combinations of complex exponentials:

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{11}$$
$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \tag{12}$$

Thus, since the arbitrary constants are set later by the initial conditions, the use of $\cos(\omega t)$ and $\sin(\omega t)$ is equivalent to that of $e^{j\omega t}$ and $e^{-j\omega t}$.

Here is a plot to help visualize the three cases:



Notice how the critically damped solution differs from the overdamped solution; it (eventually) decays *faster* than the overdamped solution. In some ways, if you want an overdamped system, the critically damped case is the optimal case (just on the edge of oscillations, which are seen in the underdamped case).

In general, finding the particular solution is not easy, but we can consider the specific case for a DC forcing function as we started with when looking at first-order differential equations. In other words, we can consider the case where $f(t) = C$ for some constant $C \in \mathbb{R}$. To solve for the particular solution in this case, we can replace circuit components by their DC steady-state equivalents (so a capacitor becomes an open circuit and an inductor becomes a wire) and then solve for $x_p(t)$ using circuit analysis.

1.1 Example: LC Tank

Consider the following circuit.

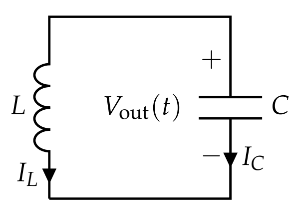


Figure 1: An LC Tank.

We can model $V_{\text{out}}(t)$ using differential equations. Suppose that $V_{\text{out}}(0) = 0$ and $I_L(0) = 1\,\text{A}$. From KVL, we have

$$V_C(t) = V_L(t) \tag{13}$$

$$V_{\text{out}}(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{14}$$

Further, we have from KCL that $I_L(t) = -I_C(t)$. Plugging this in above, we get

$$-L\frac{\mathrm{d}}{\mathrm{d}t}(I_C(t)) = V_{\text{out}}(t) \tag{15}$$

For a capacitor, we have $I_C(t) = C\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} = C\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}$. Plugging this in above, we get

$$-L\frac{\mathrm{d}}{\mathrm{d}t}\left(C\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}\right) = V_{\text{out}}(t) \tag{16}$$

$$-LC\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}\right) = V_{\text{out}}(t) \tag{17}$$

$$-LC\frac{\mathrm{d}^2V_{\text{out}}(t)}{\mathrm{d}t^2} = V_{\text{out}}(t) \tag{18}$$

$$-\frac{\mathrm{d}^2V_{\text{out}}(t)}{\mathrm{d}t^2} = \frac{1}{LC}V_{\text{out}}(t) \tag{19}$$

$$\frac{\mathrm{d}^2V_{\text{out}}(t)}{\mathrm{d}t^2} + \frac{1}{LC}V_{\text{out}}(t) = 0 \tag{20}$$

Pattern matching to eq. (1), we have $\omega_0^2 = \frac{1}{LC} \implies \omega_0 = \frac{1}{\sqrt{LC}}$ (we only consider the positive ω_0 since it represents the resonant frequency, a positive value by convention). This means that $\zeta = 0$, and $f(t) = 0$. Hence, we are dealing with the underdamped case. Since $f(t) = 0$, we only need to solve for $x_h(t)$ (i.e., $x(t) = x_h(t)$). Following Theorem 2, we have $\omega_n = \omega_0 = \sqrt{\frac{1}{LC}}$. This means that

$$V_{\text{out}}(t) = K_1\cos\left(\sqrt{\frac{1}{LC}}t\right) + K_2\sin\left(\sqrt{\frac{1}{LC}}t\right) \tag{21}$$

Now, we can apply the initial conditions to solve for K_1 and K_2. We are told that $V_{\text{out}}(0) = 0$. Plugging in $t = 0$ to eq. (21), we have

$$V_{\text{out}}(0) = K_1\cos\left(0\cdot\sqrt{\frac{1}{LC}}\right) + K_2\sin\left(0\cdot\sqrt{\frac{1}{LC}}\right) = K_1 \tag{22}$$

so we have $K_1 = V_{\text{out}}(0) = 0$. Now, we can rewrite eq. (21) as

$$V_{\text{out}}(t) = K_2\sin\left(\sqrt{\frac{1}{LC}}t\right) \tag{23}$$

We can incorporate the fact that $I_L(0) = 1\,\mathrm{A}$. We know that $I_L(t) = -I_C(t) = -C\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}$. Plugging in eq. (23), we have

$$I_L(t) = -C\frac{\mathrm{d}}{\mathrm{d}t}\left(K_2\sin\left(\sqrt{\frac{1}{LC}}t\right)\right) = -K_2\frac{C}{\sqrt{LC}}\cos\left(\sqrt{\frac{1}{LC}}t\right) = -K_2\sqrt{\frac{C}{L}}\cos\left(\sqrt{\frac{1}{LC}}t\right) \tag{24}$$

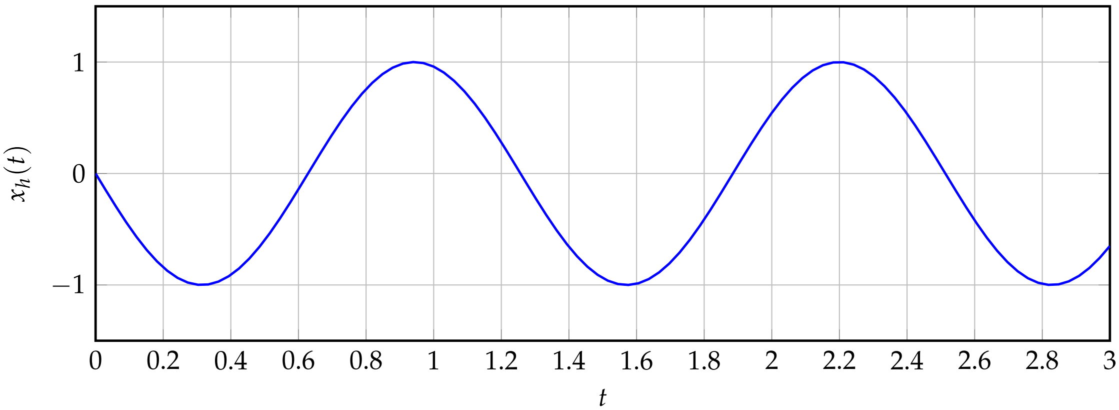
So, plugging in $t = 0$ above, we get

$$I_L(0) = -K_2\sqrt{\frac{C}{L}}\cos\left(0\cdot\sqrt{\frac{1}{LC}}\right) = -K_2\sqrt{\frac{C}{L}} \tag{25}$$

Using the fact that $I_L(0) = 1$, we can solve for K_2 to obtain $K_2 = -\sqrt{\frac{L}{C}}$. Thus, plugging in for K_2 into eq. (23), we have

$$V_{\text{out}}(t) = -\sqrt{\frac{L}{C}}\sin\left(\sqrt{\frac{1}{LC}}t\right) \tag{26}$$

Here is a plot of the shape of this output:



Notice that this is a special case of the underdamped case; there is no damping at all! This occurs because of the lack of resistance; capacitor and inductors are elements that do not dissipate power. Instead, they store energy to be moved around later (the easiest way to conceptualize this is to think about the capacitor; you could charge up a capacitor and then use that charge later as if it were a battery). For the LC tank, the energy moves back and forth between the capacitor and inductor with no loss (in reality, there is essentially always some resistance so a completely lossless passive circuit like the LC tank is essentially impossible to create, but the idea is important and relevant for devices such as oscillators).

2 Check your Understanding

Read through these simple questions to check your basic understanding of the notes.

- What solution method do we use to solve 2nd order differential equations? How is it similar and different to the 1st order differential equation solution method?
- What are the three possible homogeneous solution cases and when do they occur?
- What element (resistor, inductor, capacitor) corresponds to the damping coefficient in the 2nd order differential equation?
- What makes the LC tank a special case? Why might the LC tank also be called an oscillator circuit?

Contributors:
- Anish Muthali.
- Neelesh Ramachandran.
- Rahul Arya.
- Anant Sahai.
- Jaijeet Roychowdhury.
- Nikhil Jain.
- Matteo Guarrera.

