Note 3: Inductors and RL Circuits

1 Inductors

Here, we introduce a new passive component, the inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

1.1 Physics behind Inductors

Inductors store energy in a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an electric field ($\vec{E}$), anytime current flows down a conductor, it creates a magnetic field ($\vec{B}$), and this magnetic field can store energy. Inductors' behavior can be described using **Faraday's Law of Induction**.

The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries to create useful inductances. A **solenoid** is a good example, where we wind a wire around a conductor like a copper rod:

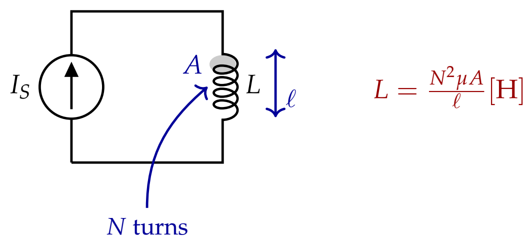


Figure 1: The Inductance of a Solenoid: a wire coiled around something.

Note that the inductance (L) depends on the **geometry** and a material property called **magnetic permeability** (μ) of the solenoid core material. In the case of the solenoid in fig. 1, the inductance depends on the number of turns (N), the length of the solenoid (l) and the area (A) of the loops. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc. But in many high speed applications, their presence might be undesirable as they create delays in the time response of the circuit (analogous to capacitors).

1.2 Introduction to Inductors

Definition 1 (Inductor)
An inductor is denoted as in Figure 7.

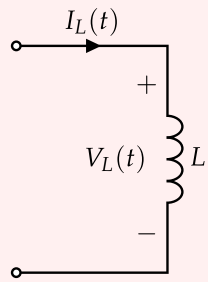


Figure 2: Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{1}$$

where L is the *inductance* of the inductor. The SI unit of inductance is the Henry (H).
The following are important facts about inductors:

1. The current through an inductor cannot change instantaneously.
2. Immediately after a current is passed through the inductor, the inductor acts as an open circuit, but as $t \to \infty$, the inductor acts like a short.
    (a) Why is this the case? Well, note that steady-state (i.e. when $t \to \infty$) is when the circuit achieves an equilibrium state. In other words, the current across the inductor will be constant. Given the inductor definition equation $V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = (L)(0)$, we can confirm that an inductor is indeed a short in steady-state.

Notice that the voltage-current relationship written in eq. (1) is similar to that of a capacitor, but with voltage and current swapped. The short term and long term behavior of inductors and capacitors are also opposites of each other.

Theorem 2 (Series Equivalence)
Consider the two inductors in series configuration in Figure 3, and suppose we wish to find the series equivalent as in Figure 4.

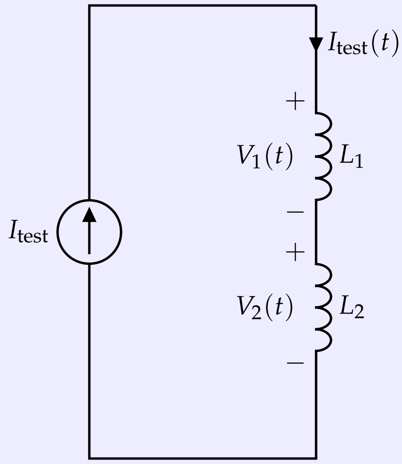


Figure 3: Series Inductor Circuit

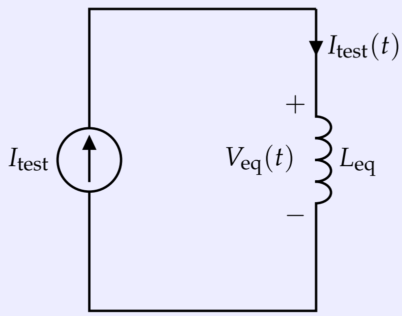


Figure 4: Equivalent Series Inductor Circuit

The equivalent series inductance is $L_{\text{eq}} = L_1 + L_2$.

Proof. We use the test current source, $I_{\text{test}}(t)$, depicted in Figure 3 and Figure 4 to find the equivalent voltage across both inductors, i.e., $V_{\text{eq}}(t)$. Using KVL, we have

$$V_1(t) + V_2(t) = V_{\text{eq}}(t) \tag{2}$$

$$L_1 \frac{\mathrm{d}I_L(t)}{\mathrm{d}t} + L_2 \frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = V_{\text{eq}}(t) \tag{3}$$

$$\underbrace{(L_1 + L_2)}_{L_{\text{eq}}} \frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = V_{\text{eq}}(t) \tag{4}$$

as desired. □

Theorem 3 (Parallel Equivalence)
Consider the two inductors in parallel configuration in Figure 5, and suppose we wish to find the parallel equivalent as in Figure 6.

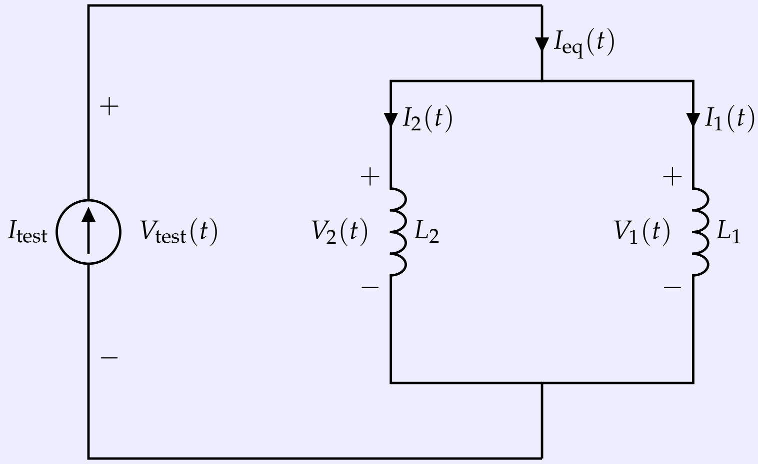


Figure 5: Parallel Inductor Circuit

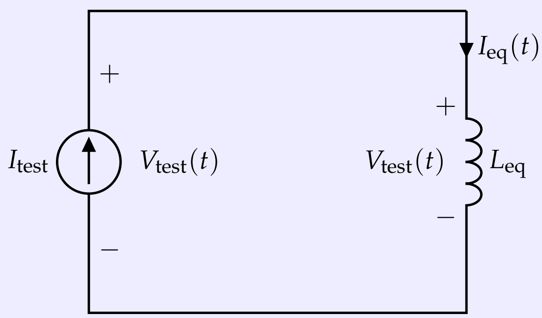


Figure 6: Equivalent Parallel Inductor Circuit

The equivalent inductance is given by $L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$.

Proof. We can apply the test current I_{test} as depicted in Figure 5. The voltage across the current generator is going to be V_{test}. Figure 6 to find the equivalent current through both inductors, i.e., $I_{eq}(t)$. By NVA, we have that

$$V_1(t) = V_2(t) = V_{test}(t) \tag{5}$$

$$L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} = L_{eq} \frac{dI_{eq}}{dt} \tag{6}$$

and from KCL we have

$$I_{eq}(t) = I_1(t) + I_2(t) \tag{7}$$

$$\frac{dI_{eq}}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} \tag{8}$$

$$\frac{dI_{eq}}{dt} = \frac{L_{eq}}{L_1}\frac{dI_{eq}}{dt} + \frac{L_{eq}}{L_2}\frac{dI_{eq}}{dt} \tag{9}$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \tag{10}$$

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1} \tag{11}$$

as desired. □

Theorem 4 (Stored Energy)
The stored energy in an inductor can be written as

$$E = \frac{1}{2} L i_L^2 \tag{12}$$

where i is the current through the inductor.

Proof. The formula for power can be manipulated as follows:

$$P_L = v_L i_L \tag{13}$$

$$P_L = \left(L \frac{\mathrm{d}i_L}{\mathrm{d}t} \right) i_L \tag{14}$$

$$P_L \,\mathrm{d}t = L i_L \,\mathrm{d}i_L \tag{15}$$

Integrating both sides to find stored energy, we have

$$\int P_L \,\mathrm{d}t = \int L i_L \,\mathrm{d}i_L \tag{16}$$

$$E = \frac{1}{2} L i_L^2 \tag{17}$$

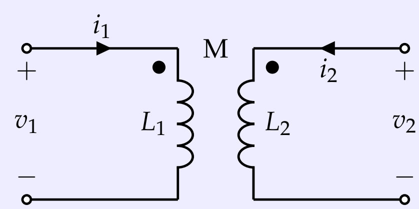
□

Definition 5 (Mutual Inductance)
The mutual inductance between two inductors L_1 and L_2 is given by

$$M = \frac{N_2 \Phi_{21}}{i_1} = \frac{N_1 \Phi_{12}}{i_2} \tag{18}$$

where N_1 and N_2 are the number of windings in the coils for inductors L_1 and L_2 respectively, and i_1 and i_2 are the current through the respective inductors. Φ_{12} is the flux passing through coil 1 from the magnetic field induced by coil 2, and Φ_{21} is the flux passing through coil 2 from the magnetic field induced by coil 1.

Theorem 6 (Induced Voltage from Mutual Inductance)
Consider the circuit below, with two inductors L_1 and L_2, with mutual inductance M.

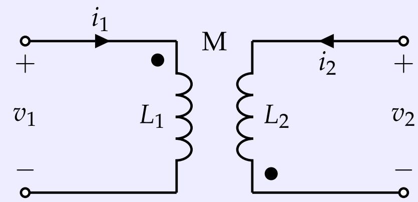


The dots in the circuit indicate the orientation of the inductors. For the given orientation, the following equations hold:

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{19}$$

$$v_2 = M\frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{20}$$

If the orientation of L_2 is flipped, as shown in the circuit below



then the following equations hold:

$$v_1 = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} - M\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{21}$$

$$v_2 = -M\frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{22}$$

Proof. Understanding of this proof is **optional**, but knowing/understanding mutual inductance is still in scope. Here, we will only prove the first part of the theorem since the second part follows by a symmetry argument, with a negated value of voltage to account for the flipped orientation. First, we can find the induced voltage in L_2 due to mutual inductance only. We can apply Faraday's law,

$$V_{2,\text{mutual}} = -N_2\frac{\mathrm{d}\Phi_{21}}{\mathrm{d}t} \tag{23}$$

$$= -N_2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Mi_1}{N_2}\right) \tag{24}$$

$$= -M\frac{\mathrm{d}i_1}{\mathrm{d}t} \tag{25}$$

where in eq. (24) we apply Definition 5. Now, notice that there is also current flowing through the second inductor, so we have an induced voltage from that. We can compute that, using Definition 1, as follows

$$V_{2,\text{current}} = L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{26}$$

Combining these two voltages using superposition and taking care to note the orientation of L_2, we obtain

$$v_2 = -V_{2,\text{mutual}} + V_{2,\text{current}} = M\frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{27}$$

We can apply the exact same argument symmetrically to L_1 to obtain

$$v_1 = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + M\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{28}$$

□

2 RL Circuits

In the same way that one resistor and one capacitor in a circuit can lead to a differential equation and a solution with an $e^{-1/RC\cdot t}$ term in it, an RL circuit results in an differential equation and $e^{-R/L\cdot t}$ terms. We will sketch an example below:

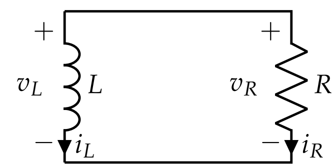


Figure 7: Example RL Circuit

Assume that $i_L(0) = I_S$.

Let's solve this circuit for $i_L(t)$ for $t > 0$ by writing a KCL equation:

$$i_L + i_R = 0 \tag{29}$$

We also know that $v_L = v_R$ since the resistor and inductor are in parallel.

$$v_L = v_R \tag{30}$$

$$\Rightarrow v_L + i_L R = 0 \tag{31}$$

Finally, substituting in the voltage-current relationship of an inductor, we see that

$$L\frac{\mathrm{d}i_L}{\mathrm{d}t} + i_L R = 0 \tag{32}$$

$$\frac{\mathrm{d}i_L}{\mathrm{d}t} + \frac{R}{L}i_L = 0 \tag{33}$$

This equation should be familiar to you! This is a first order differential equation, just like the RC circuit (but now with different coefficients). The solution is then:

$$i_L(t) = i_L(0)\mathrm{e}^{-\frac{R}{L}t} \tag{34}$$

And since $i_L(0) = I_S$, thus:

$$i_L(t) = I_S\mathrm{e}^{-\frac{R}{L}t} \tag{35}$$

If we plot and analyze the time it takes to charge up an inductor to steady state, we realize that the time constant is $\tau = \frac{L}{R}$. Just like with the RC time constant for RC circuits, this $\frac{L}{R}$ time constant is characteristic of RL circuits and is useful to remember (though it does not show up as often as the RC time constant).

Inductor and Voltage Source

Let's proceed by connecting an inductor to a perfect constant voltage source and explore what insights the equation for the inductor provides us (this is essentially the same situation as when we connect a current source to a capacitor!)

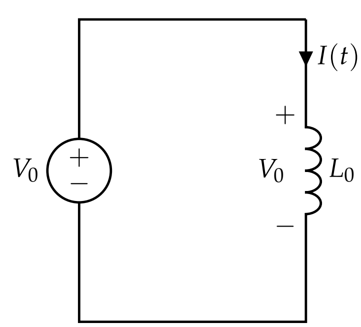


Figure 8: Voltage Source connected to Inductor

Let's be specific and say $V_0 = 3\,\mathrm{V}$ and $L_0 = 10\,\mathrm{mH}$.

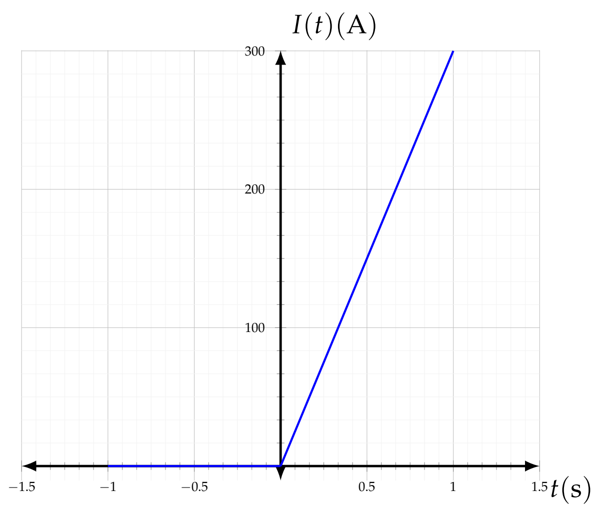
If we plug these values into the inductor equation we get:

$$V_0 = L_0 \frac{\mathrm{d}I}{\mathrm{d}t}.$$

Or, solving for $\frac{dI}{dt}$:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{V_0}{L_0} = \frac{3}{10 \times 10^{-2}} = 300\,\frac{\mathrm{A}}{\mathrm{s}}$$

This implies that the current passing through the inductor will exhibit an increasing slope equal to $\frac{V_0}{L_0}$. While this is a fascinating outcome as indicated by the equation, it's important to note that such a circuit is not feasible in practical scenarios. This concept is merely a theoretical construct to understand the behavior under a constant voltage condition. In reality, if this circuit were to be implemented, the current would continue to escalate until the limits of the actual voltage source are reached, unable to sustain the growing demand for current. However, over a brief period, this model does accurately represent the behavior of real inductors. Essentially, applying a steady voltage to an inductor leads to a uniform rate of change in the current.



3 Check Your Understanding

Read through these simple questions to check your basic understanding of the notes.

- What is an inductor?
- Describe the behavior of a inductor in steady state.
- How do we calculate equivalent inductance in series and in parallel? Is it similar to resistor equivalence or capacitor equivalence?
- Define mutual inductance. How does mutual inductance modify the inductor I-V equation when it applies?
- What is the time constant for the standard RL circuit?

Contributors:
- Anish Muthali.
- Neelesh Ramachandran.
- Kristofer Pister.
- Utkarsh Singhal.
- Aditya Arun.
- Kyle Tanghe.
- Anant Sahai.
- Kareem Ahmad.
- Nikhil Shinde.
- Chancharik Mitra.
- Nikhil Jain.
- Matteo Guarrera.

