Note 1: Capacitors, RC Circuits, and Differential Equations

1 Differential Equations

Differential equations are important tools that help us mathematically describe physical systems (such as circuits). We will learn how to solve some common differential equations and apply them to real examples.

Definition 1 (Differential Equation)
A differential equation is an equation which includes any kind of derivative (ordinary derivative or partial derivative) of any order (e.g. first order, second order, etc.).

Definition 2 (Scalar Constant Differential Equation)
A scalar constant differential equation is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = b \tag{1}$$

To solve this differential equation and find $x(t)$, we need a few key components.

Key Idea 3 (Components of Differential Equations)
When solving differential equations, we need two main components:

1. The differential equation itself. An example that we are going to see is the mathematical description of a RC circuit $\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{1}{RC}v(t)$.
2. An initial condition. This will tell us what the solution to our differential equation is at a specific time. In the example above, we would need to know a concrete value for $v(t_0)$, for some time t_0.

Recall that differential equations describe the evolution of a phenomenon in time. So both the *physical laws* that describe the phenomenon itself, and the *starting state* are required to uniquely characterize this evolution.

Theorem 4 (Scalar Constant Differential Equation Solution)
The scalar constant differential equation defined in Definition 2 admits a solution of the form

$$x(t) = k + b(t - t_0) \tag{2}$$

with the initial condition being $x(t_0) = k$.

Proof. We can integrate both sides of eq. (1). To solve this integral, we can introduce a dummy variable τ and integrate with respect to it as follows:

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} b\,\mathrm{d}\tau \tag{3}$$

Applying the fundamental theorem of calculus,

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} b\,\mathrm{d}\tau \tag{4}$$

$$x(t) - x(t_0) = b(t - t_0) \tag{5}$$

$$x(t) = k + b(t - t_0) \tag{6}$$

□

For the start of this course, we will focus on first order differential equations with constant coefficients, which are a very common form of differential equation.

Definition 5 (First order differential equation)
A first order differential equation with constant coefficients can be written in the following form:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t) \tag{7}$$

where $u(t)$ is some "input" function of t. Eq 7 is also known as a first order non-homogeneous differential equation. The reason for this name will become clearer later in the notes.
Note also that the constant in front of the $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ term is always 1; if you want to replicate this form, make sure to divide out the constant in front of this term if it is not 1.

This type of differential equation cannot be solved by direct integration due to the dependence of the derivative on the dependent variable (the function $x(t)$ you are trying to solve for). We need different methods to solve these; thus, we introduce the homogeneous and particular solution method.

1.1 First Order Homogeneous Differential Equations

To start out, we solve the first order differential equation problem for the case where $u(t) = 0$; this will turn out to be helpful even in the case that $u(t) \neq 0$ but more on that later.

Definition 6 (First Order Homogeneous Differential Equation)
A first order differential equation with constant coefficients can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) \tag{8}$$

for some $\lambda \in \mathbb{R}$.

We will only consider the case where $\lambda \neq 0$ for this subsection, since if $\lambda = 0$, then the differential equation is exactly as in eq. (1) with $b = 0$. To solve this equation, we employ a method of guessing the solution.

Key Idea 7 (Homogeneous Differential Equation Solution Form)
We guess that the solution to the homogeneous differential equation is

$$x(t) = A\mathrm{e}^{bt} \tag{9}$$

for some constants $A, b \in \mathbb{R}$.

The intuition behind this guessing is that equation 6 has on the left term a derivative and on the right one the function that is being differentiated. Which function has the property of being almost the same after we differentiate it?

We can typically use the initial condition to find A. Given an initial condition for $x(t_0)$, we can apply this in eq. (9) as follows:

$$x(t_0) = Ae^{bt_0} \tag{10}$$

We typically use the differential equation itself to find b.

Theorem 8 (Homogeneous Differential Equation Solution)
If the initial condition is $x(t_0) = k \neq 0$, we obtain a solution of the form

$$x(t) = ke^{\lambda(t-t_0)} \tag{11}$$

for the same λ defined in Definition 6.
If the initial condition is $x(t_0) = 0$, the solution will be $x(t) = 0$ for all $t \geq 0$.

Proof. Case 1: Suppose $x(t_0) = k \neq 0$. We can start by solving for A as follows:

$$x(t_0) = Ae^{bt_0} \tag{12}$$
$$k = Ae^{bt_0} \tag{13}$$
$$\implies A = ke^{-bt_0} \tag{14}$$

Plugging this back into the differential equation, we see

$$\frac{d}{dt}\left(ke^{b(t-t_0)}\right) = \lambda\left(ke^{b(t-t_0)}\right) \tag{15}$$
$$b\left(ke^{b(t-t_0)}\right) = \lambda\left(ke^{b(t-t_0)}\right) \tag{16}$$
$$\implies b = \lambda \tag{17}$$

which concludes that $x(t) = ke^{\lambda(t-t_0)}$ when $k \neq 0$. Crucially, we used the fact that $k \neq 0$ in eq. (16), along with the fact that e^{anything} is nonzero.

Case 2: Suppose $k = 0$. Hence,

$$x(t_0) = Ae^{bt_0} \tag{18}$$
$$\underbrace{k}_{0} = Ae^{bt_0} \tag{19}$$
$$\implies A = 0 \tag{20}$$

Thus, $x(t) = 0e^{bt} = 0$ for all $t \geq 0$. □

1.2 Uniqueness

Now that we have found a set of potential solutions, the other question that arises is whether there is a unique solution to the differential equation that we are solving.

Theorem 9 (Uniqueness of Homogeneous Differential Equations)
Given a differential equation of the form in Definition 6 and given an initial condition, the solution of the form

$$x(t) = A\mathrm{e}^{bt} \tag{21}$$

satisfying the differential equation and initial condition is unique.

Let's prove the Theorem 9. First, let us show that a solution to our differential equation in Definition 6 exists. We have to verify that a given solution $x_d(t) := x_0\mathrm{e}^{at}$ satisfies the homogeneous differential equation and its initial condition. This has already been proved in Theorem 8. To show that x_d is the unique solution, we will take an arbitrary solution y and show that $x_d(t) = y(t)$ for every t. Our strategy is to show that $\frac{y(t)}{x_d(t)} = 1$ for all t. However, this particular differential equation poses a problem: if $x_0 = 0$, then $x_d(t) = 0$ for all t, so that the quotient is not well-defined. To patch this method, we would like to avoid using any function with x_0 in the denominator. One way we can do this is consider a modification of the quotient $\frac{y(t)}{x_d(t)} = \frac{y(t)}{x_0\mathrm{e}^{\alpha t}}$; in particular, we consider the function $z(t) := \frac{y(t)}{\mathrm{e}^{\alpha t}}$.

The proof goes in four stages:

Step 1. We show that $z(0) = x_0$.

$$z(0) = \frac{y(0)}{\mathrm{e}^{\alpha \cdot 0}} = \frac{x_0}{\mathrm{e}^0} = \frac{x_0}{1} = x_0. \tag{22}$$

Step 2. We show that $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$. Indeed, using the quotient rule from calculus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}z(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{\mathrm{e}^{\alpha t}} &(23)\\
&= \frac{\mathrm{e}^{\alpha t}\left(\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right) - y(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\alpha t}\right)}{\mathrm{e}^{2\alpha t}} &(24)\\
&= \frac{\mathrm{e}^{\alpha t}(\alpha y(t)) - y(t)(\alpha \mathrm{e}^{\alpha t})}{\mathrm{e}^{2\alpha t}} &(25)\\
&= \frac{\alpha \mathrm{e}^{\alpha t}y(t) - \alpha \mathrm{e}^{\alpha t}y(t)}{\mathrm{e}^{2\alpha t}} &(26)\\
&= \frac{0}{\mathrm{e}^{2\alpha t}} &(27)\\
&= 0. &(28)
\end{aligned}$$

Step 3. We show that $z(t) = x_0$ for all t. Indeed, since $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$, we know that $z(t)$ is a constant. Since $z(0) = x_0$, this gives that $z(t)$ is the constant value x_0, and hence $z(t) = x_0$ for all t.

Step 4. We show that $y(t) = x_d(t)$ for all t. Indeed, since $z(t) = x_0$ and $z(t) = \frac{y(t)}{\mathrm{e}^{\alpha t}}$, we have $x_0 = \frac{y(t)}{\mathrm{e}^{\alpha t}}$. We multiply both sides by $\mathrm{e}^{\alpha t}$ to get $y(t) = x_0\mathrm{e}^{\alpha t}$. But this is just $x_d(t)$, so $y(t) = x_d(t)$ for all t.

1.3 Particular Solutions

Key Idea 10 (Homogeneous And Particular Solution:)
We have already shown that $x_{\mathrm{h}}(t) = Ae^{bt}$ is the form of the solution for the homogeneous differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{h}}(t) = \lambda x_{\mathrm{h}}(t)$. Thus, a reasonable (and indeed correct) intuition to have is that the solution to the non-homogeneous equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t) \tag{29}$$

would be comprised of the same *homogeneous* solution, $x_{\mathrm{h}}(t)$, along with an additional *particular* solution, $x_{\mathrm{p}}(t)$ to account for the $u(t)$ term. Thus, the entire solution to the above equation is of the form:

$$x(t) = x_{\mathrm{h}}(t) + x_{\mathrm{p}}(t) \tag{30}$$

We can analyze the solution for this differential equation.

What does this particular solution represent though? One way to see this is by putting $x(t) = x_{\mathrm{h}}(t) + x_{\mathrm{p}}(t)$ into the differential equation that it is defined as a solution for.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_{\mathrm{h}}(t) + x_{\mathrm{p}}(t)\right) \tag{31}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{h}}(t) + \frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{p}}(t) \tag{32}$$

$$= \lambda x_{\mathrm{h}}(t) + \frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{p}}(t) \tag{33}$$

For the last step, we use the fact that $x_{\mathrm{h}}(t)$ is, by definition, a solution to the homogeneous differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t)$.

Using the guess for the solution, $x(t) = x_{\mathrm{h}}(t) + x_{\mathrm{p}}(t)$, and putting it into the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t) = \lambda x_{\mathrm{h}}(t) + \lambda x_{\mathrm{p}}(t) + u(t) \tag{34}$$

Using 33, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{p}}(t) = \lambda x_{\mathrm{p}}(t) + u(t) \tag{35}$$

which is simply the original differential equation we wanted to solve, with $x_{\mathrm{p}}(t)$ instead of $x(t)$!

Essentially, the particular solution $x_{\mathrm{p}}(t)$ can be any function that satisfies the original differential equation; any variation between solutions (which comes due to different initial conditions) is handled by the arbitrary constant present in the homogeneous solution. There exist many different particular solutions, though we usually try to find the simplest one when solving differential equations.

We state here (without proof) that the particular solution for a differential equation is often related to the input function $u(t)$. For example, if $u(t)$ is constant, a simple particular solution will usually be constant as well, and if $u(t)$ is sinusoidal, a simple particular solution will usually be sinusoidal as well. For now, we will mostly focus on the case where $u(t)$ is constant (constant input).

Key Idea 11 (Particular Solutions with Constant Input:)
Suppose we have the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u \tag{36}$$

where u is a constant input term.
The particular solution we will use in this case is a constant value, $x_{\mathrm{p}}(t) = A$.

For this general form of the differential equation, we can show how to solve for the value of A in terms of the given variables.

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{p}}(t) = \lambda x_{\mathrm{p}}(t) + u \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(A) = \lambda(A) + u \tag{38}$$

$$0 = \lambda A + u \tag{39}$$

Thus, $A = -\frac{u}{\lambda}$ would be a solution for the differential equation, and thus a particular solution $x_{\mathrm{p}}(t) = -\frac{u}{\lambda}$.

1.4 Generalized Solution to Linear Differential Equation with Constant Input

Key idea 11 shows us how to solve for the particular solution of a linear differential equation with a constant input. Here, we will combine what we have learned to derive a general solution for a differential equation with constant input. (Note: You should not have to memorize the general solution; it will simply be a tool to help with future derivations where we already have the differential equation and need to solve it. It is more important to understand the process of solving a differential equation using the homogeneous and particular solution method as established in the previous sections.)

1.4.1 Derivation:

We are given two key pieces of information:

1. Our system is defined by the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u$.
2. This system has some known initial condition (or state) $x(t_0) = k$.

Based on what we have already established, the solution is of the form $x(t) = x_{\mathrm{h}}(t) + x_{\mathrm{p}}(t)$.
The homogeneous solution will be the solution to the differential equation without input:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{h}}(t) = \lambda x_{\mathrm{h}}(t) \tag{40}$$

From Key idea 7, we can determine that the solution to this homogeneous differential equation will be:

$$x_{\mathrm{h}}(t) = A\mathrm{e}^{\lambda t} \tag{41}$$

with A as an arbitrary constant that we will solve for later.
We have already shown from Key idea 11 that the particular solution will be $x_{\mathrm{p}}(t) = -\frac{u}{\lambda}$.
Thus, the combined solution (prior to applying the initial condition), will be:

$$x(t) = A\mathrm{e}^{\lambda t} - \frac{u}{\lambda} \tag{42}$$

Now, we can use our initial condition to solve for A:

$$x(t_0) = Ae^{\lambda t_0} - \frac{u}{\lambda} \tag{43}$$

$$k = Ae^{\lambda t_0} - \frac{u}{\lambda} \tag{44}$$

$$A = (k + \frac{u}{\lambda})e^{-\lambda t_0} \tag{45}$$

With this, we can establish our full solution in the following theorem.

Theorem 12 (Solution to Differential Equations with Constant Nonhomogeneous Term)
Consider a differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u \tag{46}$$

for real constants $\lambda, u \in \mathbb{R}$, where $\lambda \neq 0$ and $u \neq 0$[a]. This differential equation admits a solution of the form

$$x(t) = \left(k + \frac{u}{\lambda}\right)e^{\lambda(t-t_0)} - \frac{u}{\lambda} \tag{47}$$

where the initial condition is $x(t_0) = k$. The homogeneous solution is given by $x_h(t) = \left(k + \frac{u}{\lambda}\right)e^{\lambda(t-t_0)}$ and the particular solution is given by $x_p(t) = -\frac{u}{\lambda}$.

[a] If $u = 0$, then the differential equation is of the form in Definition 6, and if $\lambda = 0$, then the differential equation is of the form in eq. (1).

2 OPTIONAL: Change of Variables Method for Solving Differential Equations

Key Idea 13 (Change of Variables)
A change of variables is the technique of defining a new $\widetilde{x}(t)$ such that we are able to transform a new type of differential equation into a differential equation for $\widetilde{x}(t)$ that we already know how to solve.

We can analyze the solution for this differential equation. Again, consider a differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u \tag{48}$$

for real constants $\lambda, u \in \mathbb{R}$, where $\lambda \neq 0$ and $u \neq 0$[1]. Recall from 12 that this differential equation admits a solution of the form

$$x(t) = \left(k + \frac{u}{\lambda}\right)e^{\lambda(t-t_0)} - \frac{u}{\lambda} \tag{49}$$

where the initial condition is $x(t_0) = k$.

Proof. Define

$$\widetilde{x}(t) = x(t) + \frac{u}{\lambda} \iff x(t) = \widetilde{x}(t) - \frac{u}{\lambda} \tag{50}$$

We can use this change of variables to define a new differential equation:

$$\frac{d}{dt}\left(\widetilde{x}(t) - \frac{u}{\lambda}\right) = \lambda\left(\widetilde{x}(t) - \frac{u}{\lambda}\right) + u \tag{51}$$

[1] If $u = 0$, then the differential equation is of the form in Definition 6, and if $\lambda = 0$, then the differential equation is of the form in eq. (1).

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}(t) = \lambda\widetilde{x}(t) \tag{52}$$

We have to define the initial condition. We are given that $x(t_0) = k$ for some constant $k \neq 0$. This means that $\widetilde{x}(t_0) = x(t_0) + \frac{u}{\lambda} = k + \frac{u}{\lambda}$.

Case 1: Suppose $k + \frac{u}{\lambda} \neq 0$. Hence, the solution for $\widetilde{x}(t)$ will follow the form

$$\widetilde{x}(t) = A\mathrm{e}^{\lambda t} \tag{53}$$

for some A. We can find A by plugging in the initial condition:

$$\widetilde{x}(t_0) = A\mathrm{e}^{\lambda t_0} \tag{54}$$

$$k + \frac{u}{\lambda} = A\mathrm{e}^{\lambda t_0} \tag{55}$$

$$\left(k + \frac{u}{\lambda}\right)\mathrm{e}^{-\lambda t_0} = A \tag{56}$$

So, the solution for $\widetilde{x}(t)$ is

$$\widetilde{x}(t) = \left(k + \frac{u}{\lambda}\right)\mathrm{e}^{\lambda(t-t_0)} \tag{57}$$

Plugging this back in to the change of variables defined in eq. (50), we can find the solution for $x(t)$:

$$x(t) = \left(k + \frac{u}{\lambda}\right)\mathrm{e}^{\lambda(t-t_0)} - \frac{u}{\lambda} \tag{58}$$

Case 2: Suppose $k + \frac{u}{\lambda} = 0$. This means the value of the solution for $\widetilde{x}(t)$ is 0 at t_0, which means that $\widetilde{x}(t) = 0$ for all $t \geq 0$. This is because $\mathrm{e}^{\lambda t}$ will always be positive. Plugging in $\widetilde{x}(t) = 0$ into eq. (50), we can find $x(t)$:

$$x(t) = -\frac{u}{\lambda} \tag{59}$$

□

3 OPTIONAL: Nonlinear Differential Equations

In the above sections, we only talked about linear differential equations where $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + b$. However, you may encounter differential equations like $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = x(t)^2$ and other such nonlinear functions of $x(t)$. In general, there are various "techniques" that can be used to attempt to guess potential solutions for such equations. At the end of the day, all of these guesses need to be checked and the appropriate uniqueness theorems proved to make sure that we have got the single true solution. Only then can this solution used for any predictive purposes.

Without a uniqueness theorem, such solutions cannot be trusted for prediction. In the homework, you will see an example that illustrates how a seemingly innocuous differential equation can have non-unique solutions. In that homework, we will also share another technique that can be used to guess solutions to nonlinear differential equations — a technique known as "separation of variables." There are many such techniques out there, and different ones tend to work for different types of equations. You will encounter these techniques in later courses alongside the kinds of differential equations for which they tend to work.

4 Mathematical Approach to RC Circuits

With some knowledge of differential equations, we will now approach some problems that involve differential equations. We know from EECS 16A that $q = Cv$ describes the charge in a capacitor as a function of the voltage across the capacitor and capacitance. We also know that the voltage across the capacitor might change over time and we can write charge as a function of time:

$$q(t) = Cv(t) \tag{60}$$

We can assume that capacitance is a constant with respect to time, since this is a quantity inherent to the physical nature of the component. This will allow us to come up with a *differential equation*.

We can derive a differential equation for capacitors based on eq. (60).

Theorem 14 (Capacitor Differential Equation)
A differential equation relating the time evolution of current through and voltage across a capacitor is given by

$$I(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{61}$$

Proof. Current is the rate of flow of charge over time, so we may write $\frac{\mathrm{d}q(t)}{\mathrm{d}t} = I(t)$. Taking time derivatives on both sides of eq. (60) yields

$$I(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{62}$$

□

4.1 RC Circuit Example

Consider the following circuit.

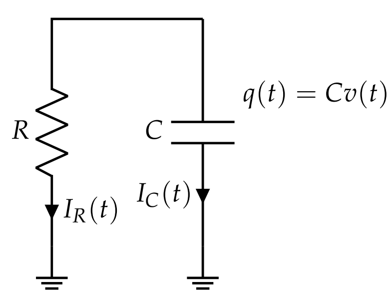

Figure 1: Capacitor discharging through circuit

Using the results from the previous section and KVL, we have the following differential equation governing the behavior of this circuit

$$C\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{v(t)}{R} \tag{63}$$

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{v(t)}{RC} \tag{64}$$

Suppose we are told that $V(0) = V_0$. With this initial condition, we now have a full differential equation problem that can be solved.

We can see that our differential equation is a homogeneous differential equation, with $\lambda = -\frac{1}{RC}$. Thus, from the previous section, our solution will be in the form of:

$$v(t) = Ae^{\lambda t} = Ae^{-\frac{1}{RC}t} \tag{65}$$

Now we just need to find the arbitrary constant with our initial condition!

$$v(0) = Ae^{-\frac{1}{RC}\cdot 0} = V_0 \tag{66}$$

$$A = V_0 \tag{67}$$

Thus, the overall solution is:

$$v(t) = V_0 e^{-\frac{1}{RC}t} \tag{68}$$

The plot below shows how this solution looks:

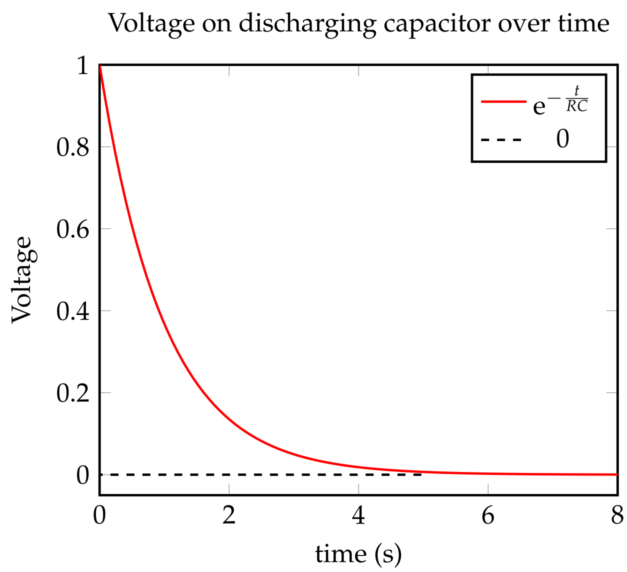

Figure 2

Suppose that now we have an input into this RC circuit:

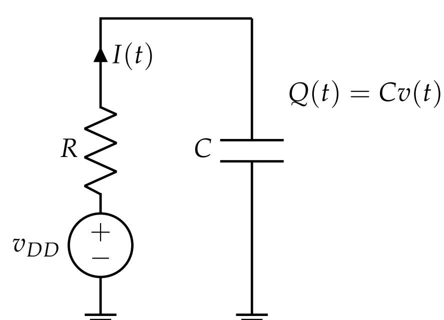

Figure 3: Capacitor charging through resistor circuit

Using Kirchhoff's voltage Law (KVL), we can see that

$$v_{DD} = RI(t) + v(t) \tag{69}$$

where $v(t)$ is the voltage across the capacitor. Using the fact that $I(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v(t)$, our resulting differential equation is

$$RC\frac{\mathrm{d}}{\mathrm{d}t}v(t) + v(t) = v_{DD} \tag{70}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\frac{1}{RC}v(t) + \frac{v_{DD}}{RC} \tag{71}$$

Suppose the capacitor is initially uncharged at time $t = 0$, i.e. $v(0) = 0$.

First, let's find the homogeneous solution, which is the solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_h(t) = -\frac{1}{RC}v_h(t) \tag{72}$$

From the homogeneous solutions section (and the previous example), we know that the solution will be

$$v_h(t) = A\mathrm{e}^{-\frac{1}{RC}t} \tag{73}$$

with A as a constant we will solve for later with our initial condition.

Since there is an input function present ($u(t) = \frac{v_{DD}}{RC}$), we will need to also find a particular solution for the full solution.

Since the input function is constant, we could solve this by assuming that the particular solution is constant as well: $v_p(t) = V_p$.

Inserting this into the differential equation, we can obtain the value of V_p:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_p(t) = -\frac{1}{RC}v_p(t) + \frac{v_{DD}}{RC} \tag{74}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_p) = -\frac{1}{RC}V_p + \frac{v_{DD}}{RC} \tag{75}$$

$$0 = -\frac{1}{RC}V_p + \frac{v_{DD}}{RC} \tag{76}$$

$$V_p = v_{DD} \tag{77}$$

With both the homogeneous and particular solutions, we can construct the full solution:

$$v(t) = v_h(t) + v_p(t) = A\mathrm{e}^{-\frac{1}{RC}t} + v_{DD} \tag{78}$$

With the initial condition $v(0) = 0$, we can find that $A = -v_{DD}$ and the solution is:

$$v(t) = v_h(t) + v_p(t) = v_{DD}(1 - \mathrm{e}^{-\frac{1}{RC}t}) \tag{79}$$

which is a relatively common solution structure for an RC circuit charging a capacitor (and one to become familiar with over time). A plot of $v(t)$ will follow the shape of the graph in fig. 4.

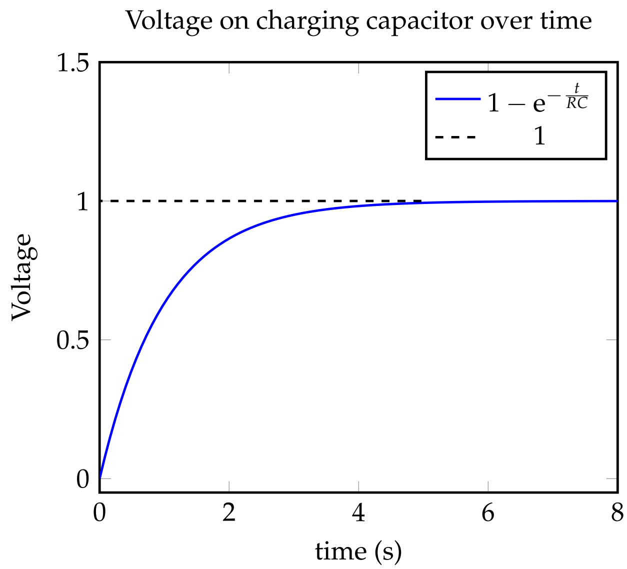

Figure 4

4.2 [IMPORTANT] Solution Interpretation: Time Constant, Steady State

What do the homogeneous solution and particular solution actually mean in the context of circuits? Are they just mathematical tools we use to solve differential equations? You may notice that the structure of the homogeneous solution in both examples was identical (same λ) even though the two circuits have different input voltages (the first has no input voltage while the second has v_{DD}). This is not a coincidence! The core elements of the RC circuit (the resistor and capacitor themselves) determine the homogeneous solution regardless of input (which makes sense since by definition, the homogeneous solution is that without the input present). This is why the homogeneous solution is also sometimes called the *natural response*; it represents the characteristic response of the circuit itself.

The detail that $\lambda = -\frac{1}{RC}$ is crucial as well; the $\tau = RC$ value present here is called the **time constant** and is a crucial metric of the time varying behavior of a circuit; even when we start to discuss more complicated RC circuits, the time constant usually is able to be simplified into some form of $\tau = R_{eq}C$, where R_{eq} is the equivalent resistance seen by the capacitor (more about this in later discussions).

How about the particular solution? The particular solution is actually related to the **steady state** behavior of the circuit, which is the behavior of the circuit as $t \to \infty$. In this limit, the homogeneous solution disappears (this happens because for stable/real systems, $\lambda < 0$ so the $e^{\lambda t}$ term characteristic of the homogeneous solution approaches 0 for $t \to \infty$) and just the particular solution is left.

The important part here is that to analyze steady state, we do not need to derive the full differential equation for the system; we can draw an equivalent circuit at steady state and analyze with KCL/KVL. This process will involve replacing the capacitors with the equivalent steady state element.

For constant inputs (called DC inputs), steady state analysis tells us that all of the variables become constant (as $t \to \infty$, the input does not change so the voltages and currents throughout the circuit will not change as well). This implies that all derivatives become 0 (since all the variables are now constant). For a capacitor, this implies that:

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} = 0 \tag{80}$$

What element has no current flow through it? Open circuit! Thus, at DC steady state, a capacitor behaves like an open circuit.

> **Key Idea 15** (Capacitor DC Steady-State:)
> In DC steady-state (as $t \to \infty$ with only DC/constant inputs), a capacitor behaves like an open circuit.

Let's apply this to the RC circuit example with a constant input voltage v_{DD}. The equivalent circuit, where the capacitor is replaced with an open circuit, is:

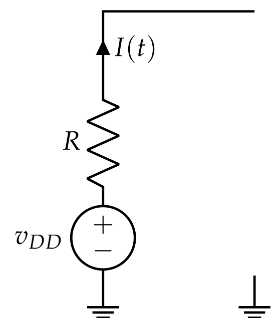

Figure 5: Capacitor charging through resistor circuit

Due to the open circuit, there can be no current through resistor, and thus no voltage drop across it. Thus, the steady-state voltage across the capacitor (which is an open circuit in the current diagram) is $v_p(t) = v_{DD}$.

This is the same particular solution as obtained with the mathematical approach, which helps validate the claim that the particular solution and steady state solution are the same.

To summarize, the homogeneous solution represents the *natural response* of the system, while the particular solution represents the *steady state response* of the system. As your experience with circuits increases, you will start to be able to solve circuit problems without deriving the differential equation! Find the homogeneous solution with the knowledge of the time constant, find the particular solution with steady state analysis, and use the initial condition to determine any arbitrary constants!

5 Check your basic understanding

Read through this simple questions to check your basic understanding of the notes.

- What is a differential equation? What are its two main components?
- What is the difference between a homogeneous and non-homogeneous differential equation?
- Why did we guess that the solution to a first order homogeneous differential equation was an exponential function?
- What is the equation that describes the time evolution of a capacitor (the I-V relation for a capacitor)?

- How the particular solution is related to the steady state of the circuit?
- What is a time constant in the context of RC circuit?

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