

# Note 3: Inductors and RLC Circuits

## 1 Inductors

### 1.1 Introduction to Inductors

Here, we introduce a new passive component, the inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

**Definition 1 (Inductor)**

An inductor is denoted as in Figure 1.

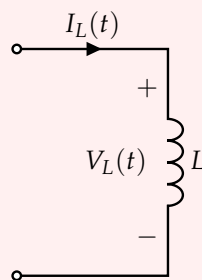


Figure 1: Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L \frac{dI_L(t)}{dt} \quad (1)$$

where  $L$  is the *inductance* of the inductor. The SI unit of inductance is the Henry (H).

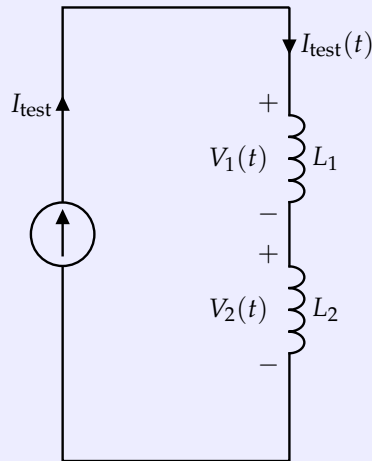
The following are important facts about inductors:

1. The current through an inductor cannot change instantaneously.
2. Immediately after a current is passed through the inductor, the inductor acts as an open circuit, but as  $t \rightarrow \infty$ , the inductor acts like a short.

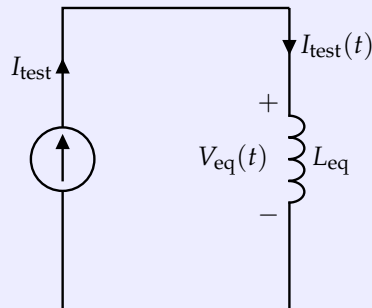
Notice that the voltage-current relationship written in eq. (1) is similar to that of a capacitor, but with voltage and current swapped. The short term and long term behavior of inductors and capacitors are also opposites of each other.

**Theorem 2 (Series Equivalence)**

Consider the two inductors in series configuration in Figure 2, and suppose we wish to find the series equivalent as in Figure 3.



**Figure 2:** Series Inductor Circuit



**Figure 3:** Equivalent Series Inductor Circuit

The equivalent series inductance is  $L_{\text{eq}} = L_1 + L_2$ .

*Proof.* We use the test current source,  $I_{\text{test}}(t)$ , depicted in Figure 2 and Figure 3 to find the equivalent voltage across both inductors, i.e.,  $V_{\text{eq}}(t)$ . Using KVL, we have

$$V_1(t) + V_2(t) = V_{\text{eq}}(t) \quad (2)$$

$$L_1 \frac{dI_L(t)}{dt} + L_2 \frac{dI_L(t)}{dt} = V_{\text{eq}}(t) \quad (3)$$

$$\underbrace{(L_1 + L_2)}_{L_{\text{eq}}} \frac{dI_L(t)}{dt} = V_{\text{eq}}(t) \quad (4)$$

as desired. □

### Theorem 3 (Parallel Equivalence)

Consider the two inductors in parallel configuration in Figure 4, and suppose we wish to find the parallel equivalent as in Figure 5.

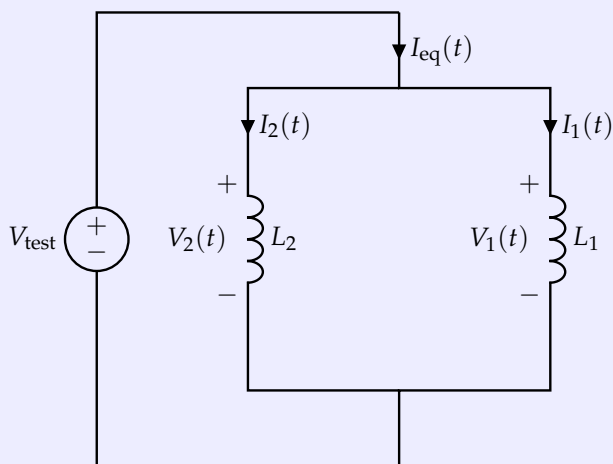


Figure 4: Parallel Inductor Circuit

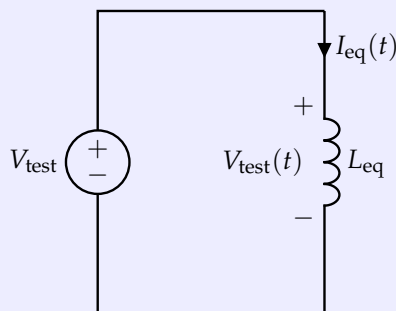


Figure 5: Equivalent Parallel Inductor Circuit

The equivalent inductance is given by  $L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$ .

*Proof.* We can apply the test voltage  $V_{test}$  as depicted in Figure 4 and Figure 5 to find the equivalent current through both inductors, i.e.,  $I_{eq}(t)$ . By NVA, we have that

$$V_1(t) = V_2(t) = V_{test}(t) \quad (5)$$

$$L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} = L_{eq} \frac{dI_{eq}}{dt} \quad (6)$$

and from KCL we have

$$I_{eq}(t) = I_1(t) + I_2(t) \quad (7)$$

$$\frac{dI_{eq}}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} \quad (8)$$

$$\frac{dI_{eq}}{dt} = \frac{L_{eq}}{L_1} \frac{dI_{eq}}{dt} + \frac{L_{eq}}{L_2} \frac{dI_{eq}}{dt} \quad (9)$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \quad (10)$$

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1} \quad (11)$$

as desired. □

#### Theorem 4 (Stored Energy)

The stored energy in an inductor can be written as

$$E = \frac{1}{2} L i_L^2 \quad (12)$$

where  $i$  is the current through the inductor.

*Proof.* The formula for power can be manipulated as follows:

$$P_L = v_L i_L \quad (13)$$

$$P_L = \left( L \frac{di_L}{dt} \right) i_L \quad (14)$$

$$P_L dt = L i_L di_L \quad (15)$$

Integrating both sides to find stored energy, we have

$$\int P_L dt = \int L i_L di_L \quad (16)$$

$$E = \frac{1}{2} L i_L^2 \quad (17)$$

□

#### Definition 5 ((OPTIONAL) Mutual Inductance)

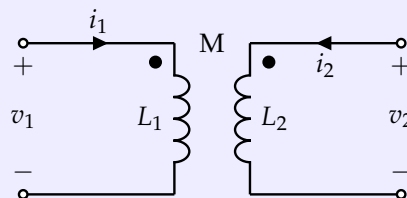
The mutual inductance between two inductors  $L_1$  and  $L_2$  is given by

$$M = \frac{N_2 \Phi_{21}}{i_1} = \frac{N_1 \Phi_{12}}{i_2} \quad (18)$$

where  $N_1$  and  $N_2$  are the number of windings in the coils for inductors  $L_1$  and  $L_2$  respectively, and  $i_1$  and  $i_2$  are the current through the respective inductors.  $\Phi_{12}$  is the flux passing through coil 1 from the magnetic field induced by coil 2, and  $\Phi_{21}$  is the flux passing through coil 2 from the magnetic field induced by coil 1.

#### Theorem 6 (Induced Voltage from Mutual Inductance)

Consider the circuit below, with two inductors  $L_1$  and  $L_2$ , with mutual inductance  $M$ .



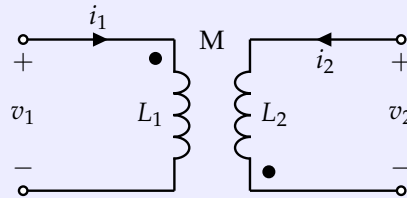
The dots in the circuit indicate the orientation of the inductors. For the given orientation, the following

equations hold:

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \quad (19)$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \quad (20)$$

If the orientation of  $L_2$  is flipped, as shown in the circuit below



then the following equations hold:

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \quad (21)$$

$$v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \quad (22)$$

*Proof.* The proof is out of scope of this course, since it requires some knowledge of Faraday's law. Here, we will only prove the first part of the theorem since the second part follows by a symmetry argument, with a negated value of EMF to account for the flipped orientation. First, we can find the induced EMF in  $L_2$  due to mutual inductance only. We can apply Faraday's law,

$$\mathcal{E}_{2,\text{mutual}} = -N_2 \frac{d\Phi_{21}}{dt} \quad (23)$$

$$= -N_2 \frac{d}{dt} \left( \frac{Mi_1}{N_2} \right) \quad (24)$$

$$= -M \frac{di_1}{dt} \quad (25)$$

where in eq. (24) we apply Definition 5. Now, notice that there is also current flowing through the second inductor, so we have an induced EMF from that. We can compute that, using Definition 1, as follows

$$\mathcal{E}_{2,\text{current}} = L_2 \frac{di_2}{dt} \quad (26)$$

Combining these two EMFs using superposition and taking care to note the orientation of  $L_2$ , we obtain

$$v_2 = -\mathcal{E}_{2,\text{mutual}} + \mathcal{E}_{2,\text{current}} = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \quad (27)$$

We can apply the exact same argument symmetrically to  $L_1$  to obtain

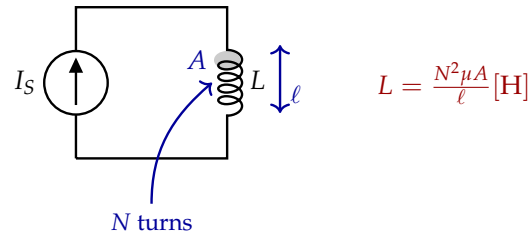
$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \quad (28)$$

□

## 1.2 OPTIONAL: Physics behind Inductors

Inductors store energy in a magnetic field. In the same way that a capacitor separates charge ( $Q$ ) and this leads to an electric field ( $\vec{E}$ ), anytime current flows down a conductor, it creates a magnetic field ( $\vec{B}$ ), and this magnetic field can store energy. Inductors' behavior can be described using **Faraday's Law of Induction**.

The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries to create useful inductances. A **solenoid** is a good example, where we wind a wire around a conductor like a copper rod:



**Figure 6:** The Inductance of a Solenoid: a wire coiled around something.

Note that the inductance ( $L$ ) depends on the **geometry** and a material property called **magnetic permeability** ( $\mu$ ) of the solenoid core material. In the case of the solenoid in fig. 6, the inductance depends on the number of turns ( $N$ ), the length of the solenoid ( $l$ ) and the area ( $A$ ) of the loops. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc. But in many high speed applications, their presence might be undesirable as they create delays in the time response of the circuit (analogous to capacitors).

## 2 Second Order Differential Equations

### Definition 7 (Second Order, Linear Differential Equation)

A second order, linear differential equation can be put into the form

$$\frac{d^2x(t)}{dt^2} + 2\frac{\zeta}{\tau} \frac{dx(t)}{dt} + \frac{1}{\tau^2}x(t) = f(t) \quad (29)$$

for some constants  $\zeta, \tau \in \mathbb{R}$  (often referred to as the *damping coefficient* and *time constant* respectively) and some function of time  $f(t)$  (this is sometimes called a *forcing function*). The solution to this differential equation can be separated into complementary and particular solutions of the form

$$x(t) = x_p(t) + x_c(t) \quad (30)$$

where  $x_c(t)$  represents the complementary solution and  $x_p(t)$  represents the particular solution.

We typically solve separately for the particular and complementary solutions. The complementary solution is the solution to

$$\frac{d^2x_c(t)}{dt^2} + 2\frac{\zeta}{\tau} \frac{dx_c(t)}{dt} + \frac{1}{\tau^2}x_c(t) = 0 \quad (31)$$

**Theorem 8** (Complementary Solution to Second Order Differential Equations)

Define  $s_1 := -\frac{\zeta}{\tau} + \frac{1}{\tau}\sqrt{\zeta^2 - 1}$  and  $s_2 := -\frac{\zeta}{\tau} - \frac{1}{\tau}\sqrt{\zeta^2 - 1}$ . The complementary solution will take on one of the following forms, depending on the value of  $\zeta$ .

1. *Overdamped case:* ( $\zeta > 1$ )

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \quad (32)$$

2. *Critically damped case:* ( $\zeta = 1$ )

$$x_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t} \quad (33)$$

Note that  $s_1 = s_2$  in this case.

3. *Underdamped case:* ( $\zeta < 1$ )

Note that  $s_1$  and  $s_2$  will be complex, so we can rewrite them as  $s_1 = -\frac{\zeta}{\tau} + j\omega_n$  and  $s_2 = -\frac{\zeta}{\tau} - j\omega_n$  where  $\omega_n := \frac{1}{\tau}\sqrt{1 - \zeta^2}$  is defined as the natural frequency. The solution is of the form

$$x_c(t) = K_1 e^{-\frac{\zeta}{\tau} t} \cos(\omega_n t) + K_2 e^{-\frac{\zeta}{\tau} t} \sin(\omega_n t) \quad (34)$$

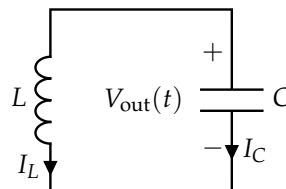
In all of the cases above,  $K_1$  and  $K_2$  are arbitrary constants that are determined by initial conditions. Note that you will need two initial conditions to completely solve a second order differential equation.

**Concept Check:** This note will not prove the solutions from first principles as that is out of scope, but as an exercise, you are encouraged to verify that the solutions satisfy eq. (31).

In general, finding the particular solution is not easy, but we can consider the specific case for a DC forcing function. In other words, we can consider the case where  $f(t) = C$  for some constant  $C \in \mathbb{R}$ . To solve for the particular solution in this case, we can replace circuit components by their DC steady-state equivalents (so a capacitor becomes an open circuit and an inductor becomes a wire) and then solve for  $x_p(t)$  using circuit analysis.

## 2.1 Example: LC Tank

Consider the following circuit.



**Figure 7:** An LC Tank.

We can model  $V_{\text{out}}(t)$  using differential equations. Suppose that  $V_{\text{out}}(0) = 0$  and  $I_L(0) = 1$  A. From KVL, we have

$$V_C(t) = V_L(t) \quad (35)$$

$$V_{\text{out}}(t) = L \frac{dI_L(t)}{dt} \quad (36)$$

Further, we have from KCL that  $I_L(t) = -I_C(t)$ . Plugging this in above, we get

$$-L \frac{d}{dt}(I_C(t)) = V_{\text{out}}(t) \quad (37)$$

For a capacitor, we have  $I_C(t) = C \frac{dV_C(t)}{dt} = C \frac{dV_{\text{out}}(t)}{dt}$ . Plugging this in above, we get

$$-L \frac{d}{dt} \left( C \frac{dV_{\text{out}}(t)}{dt} \right) = V_{\text{out}}(t) \quad (38)$$

$$-LC \frac{d}{dt} \left( \frac{dV_{\text{out}}(t)}{dt} \right) = V_{\text{out}}(t) \quad (39)$$

$$-LC \frac{d^2 V_{\text{out}}(t)}{dt^2} = V_{\text{out}}(t) \quad (40)$$

$$-\frac{d^2 V_{\text{out}}(t)}{dt^2} = \frac{1}{LC} V_{\text{out}}(t) \quad (41)$$

$$\frac{d^2 V_{\text{out}}(t)}{dt^2} + \frac{1}{LC} V_{\text{out}}(t) = 0 \quad (42)$$

Pattern matching to eq. (29), we have  $\tau^2 = LC \implies \tau = \pm\sqrt{LC}$ ,  $\frac{\zeta}{\tau} = 0 \implies \zeta = 0$ , and  $f(t) = 0$ . Since  $\tau$  is defined as a time constant, we will only consider positive values of  $\tau$ , so  $\tau = \sqrt{LC}$ . Hence, we are dealing with the underdamped case. Since  $f(t) = 0$ , we only need to solve for  $x_c(t)$  (i.e.,  $x(t) = x_c(t)$ ). Following Theorem 8, we have  $\omega_n = \sqrt{\frac{1}{LC}}$ . This means that

$$V_{\text{out}}(t) = K_1 \cos\left(\sqrt{\frac{1}{LC}}t\right) + K_2 \sin\left(\sqrt{\frac{1}{LC}}t\right) \quad (43)$$

Now, we can apply the initial conditions to solve for  $K_1$  and  $K_2$ . We are told that  $V_{\text{out}}(0) = 0$ . Plugging in  $t = 0$  to eq. (43), we have

$$V_{\text{out}}(0) = K_1 \cos\left(0 \cdot \sqrt{\frac{1}{LC}}\right) + K_2 \sin\left(0 \cdot \sqrt{\frac{1}{LC}}\right) = K_1 \quad (44)$$

so we have  $K_1 = V_{\text{out}}(0) = 0$ . Now, we can rewrite eq. (43) as

$$V_{\text{out}}(t) = K_2 \sin\left(\sqrt{\frac{1}{LC}}t\right) \quad (45)$$

We can incorporate the fact that  $I_L(0) = 1$  A. We know that  $I_L(t) = -I_C(t) = -C \frac{dV_{\text{out}}(t)}{dt}$ . Plugging in eq. (45), we have

$$I_L(t) = -C \frac{d}{dt} \left( K_2 \sin\left(\sqrt{\frac{1}{LC}}t\right) \right) = -K_2 \frac{C}{\sqrt{LC}} \cos\left(\sqrt{\frac{1}{LC}}t\right) = -K_2 \sqrt{\frac{C}{L}} \cos\left(\sqrt{\frac{1}{LC}}t\right) \quad (46)$$

So, plugging in  $t = 0$  above, we get

$$I_L(0) = -K_2 \sqrt{\frac{C}{L}} \cos\left(0 \cdot \sqrt{\frac{1}{LC}}\right) = -K_2 \sqrt{\frac{C}{L}} \quad (47)$$

Using the fact that  $I_L(0) = 1$ , we can solve for  $K_2$  to obtain  $K_2 = -\sqrt{\frac{L}{C}}$ . Thus, plugging in for  $K_2$  into eq. (45), we have

$$V_{\text{out}}(t) = -\sqrt{\frac{L}{C}} \sin\left(\sqrt{\frac{1}{LC}}t\right) \quad (48)$$

#### Contributors:



- Anish Muthali.
- Neelesh Ramachandran.
- Kristofer Pister.
- Utkarsh Singhal.
- Aditya Arun.
- Kyle Tanghe.
- Anant Sahai.
- Kareem Ahmad.
- Nikhil Shinde.