Note 5: Phasors and AC Circuits

1 Overview

In 16A, we only dealt with circuits with constant voltage/current sources, called *DC (direct current) circuits*. In 16B so far, we have expanded our scope to include time-variation and used differential equations to model such systems. This style of analysis is *transient analysis*, because it captures how things change (for example, when we have time-varying voltage/current sources).

While the approaches you have seen are powerful and general, they are also cumbersome (requiring integrals) and the analysis can feel opaque. We want a simpler approach that is more design-friendly so we can start to create circuits that function how we want. This goal is a more advanced step, beyond analysis of provided circuits. To gain ease-of-design, we are willing to trade away some generality; instead of thinking about all inputs (arbitrary $u(t)$), we will restrict ourselves to only *sinusoidal* voltage/current inputs, which (for historical reasons) are called *AC (alternating current) circuits*. Furthermore, we will focus only on the steady-state behavior of such AC circuits, which is called *frequency analysis*. The steady-state behavior is how the circuit acts after a long time, and is important since it is the eventual response to our input.[1]

Sinusoidal sources are a very common category of inputs, and there's also a technique known as the Fourier Transform, which can be used to express many input signals as weighted sums of sinusoidal functions, enabling us to apply superposition.[2] And fortunately, analyzing sinusoidal functions is easy! Analyzing arbitrary input signals (like in transient analysis) requires us to solve a set of differential equations, but we can use a systematic procedure (resembling NVA from 16A) to analyze circuits with sinusoidal inputs.

2 Scalar Differential Equation with Exponential Input

We have seen that circuits with sources, resistors, capacitors, and inductors can be modeled with a system of linear, first-order differential equations. By developing techniques to determine the steady state of such systems, we can hope to apply them to the special case of circuit analysis. Let's step back from circuits for a little bit and revisit the fundamentals.

Consider the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t), \tag{1}$$

where the input $u(t)$ is of the form

$$u(t) = k\mathrm{e}^{st},\ s \neq \lambda \tag{2}$$

We've previously seen (in HW) how to solve this equation:

$$x(t) = \left(x_0 - \frac{k}{s-\lambda}\right)\mathrm{e}^{\lambda t} + \frac{k}{s-\lambda}\mathrm{e}^{st}, \tag{3}$$

where $x(0) = x_0$ is the initial value of $x(t)$. Interestingly, this is *almost* a scalar multiple of $u(t)$ — if we could ignore the initial term involving $\mathrm{e}^{\lambda t}$, then $x(t)$ would linearly depend on $u(t)$.

We can ignore the initial term when $\mathrm{e}^{\lambda t} \to 0$ as $t \to \infty$ (in the steady-state)? When might this happen? If λ were real, then the term decays to zero if and only if $\lambda < 0$. But what about for complex λ? We can try

[1] For now, we will give up on what happens in the neighborhood of a sudden change — for that, we would have to deal with the differential equations governing the system.

[2] This topic is one focus of classes like EE120.

writing a complex λ in the form $\lambda = \lambda_r + \mathrm{j}\lambda_i$:

$$\mathrm{e}^{\lambda t} = \mathrm{e}^{(\lambda_r + \mathrm{j}\lambda_i)t} \tag{4}$$
$$= \mathrm{e}^{\lambda_r t}\mathrm{e}^{\mathrm{j}\lambda_i t} \tag{5}$$

The $\mathrm{e}^{\lambda_r t}$ is exactly the real case we just saw above. But what about the $\mathrm{e}^{\mathrm{j}\lambda_i t}$ term? Applying Euler's formula and expanding our expression, we find that:

$$\mathrm{e}^{\lambda t} = \mathrm{e}^{\lambda_r t}(\cos(\lambda_i t) + \mathrm{j}\sin(\lambda_i t)) \tag{6}$$

The first term in the product is a real exponential, which decays to zero exactly when $\mathrm{Re}\{\lambda\} = \lambda_r < 0$. The second term is a sum of two sinusoids with unit amplitudes. Since the amplitude of each sinusoid is constant, their sum will not decay to 0 or grow to infinity over time. Thus, the asymptotic behavior of the overall expression is governed solely by the first term **()$\mathrm{e}^{\lambda t}$ will decay to zero exactly when e^{λ_r} does)**. Thus, looking back at our solution for $x(t)$, we now know that $\mathrm{e}^{\lambda t}$ decays to 0 whenever $\mathrm{Re}\{\lambda\} < 0$.

3 System of Differential Equations with Exponential Input

Can we apply similar techniques to a system of differential equations? Specifically, consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t), \tag{7}$$

where A is a fixed, real, matrix. As before, we will consider only control inputs of a special form, where each component is $k\mathrm{e}^{st}$ for some constant k. Said differently:

$$\vec{u}(t) = \vec{u}_0\mathrm{e}^{st}, \tag{8}$$

where $\vec{u}_0$ does not depend on t, and s is *not* an eigenvalue of the matrix A (this will be important later).

Assume, for now[3], that A has a linearly independent set of eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then we know that we can diagonalize A to be

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \tag{9}$$
$$= V\Lambda V^{-1}, \tag{10}$$

where V and Λ are the eigenvector and eigenvalue matrices in the above diagonalization.

Thus, we can construct a new coordinate system $z(t) = V^{-1}x(t)$ to rewrite our differential equation for $x(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) + \vec{u}(t) \tag{11}$$
$$V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(V^{-1}\vec{x}(t)\right) = \Lambda\left(V^{-1}\vec{x}(t)\right) + V^{-1}\vec{u}(t) \tag{12}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t) + V^{-1}\vec{u}(t) \tag{13}$$

We see that this diagonalized system of differential equations can be rewritten as a set of scalar differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}z_i(t) = \lambda_i z_i(t) + (V^{-1}\vec{u}_0\mathrm{e}^{st})_i, \tag{14}$$

[3] Later in the semester, we will learn about upper triangularization that allows the following analysis to apply to all matrices, not just diagonalizable ones.

where the subscript $_i$ represents the i^{th} component of the associated vector, and λ_i is the i^{th} eigenvalue of A. Since $\left(V^{-1}\vec{u}_0 e^{st}\right)_i = \left(V^{-1}\vec{u}_0\right)_i e^{st}$ is a multiple of e^{st} and $s \neq \lambda_i$, we know from our scalar results that the solution to $z_i(t)$ can be expressed as a linear combination of $e^{\lambda_i t}$ and e^{st}. Here, $e^{\lambda_i t}$ decays to zero over time if and only if $\text{Re}\{\lambda_i\} < 0$. In this case, this yields a steady-state solution involving only a scalar multiple of e^{st}:

$$\lim_{t\to\infty} z_i(t) = \alpha_i e^{st}. \tag{15}$$

Then, we can stack these solutions into vector form and pre-multiply by V to obtain

$$\lim_{t\to\infty} \vec{z}(t) = \vec{\alpha} e^{st} \tag{16}$$

$$\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty} V\vec{z}(t) = (V\vec{\alpha})e^{st} \tag{17}$$

So as $t \to \infty$, our steady-state solution for $\vec{x}(t)$ will also be a multiple of e^{st}.

Let us call $\vec{x}_0 = V\vec{\alpha}$ for notational simplicity. We will now try to find a formula for $\vec{x}_0$ using the fact that $\vec{x}(t) = \vec{x}_0 e^{st}$ in steady-state, for some constant vector $\vec{x}_0$. Since the differential equation must still apply for the steady-state solution (by definition of what it means to be a solution),

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t) \tag{18}$$

$$s\vec{x}_0 e^{st} = A\vec{x}_0 e^{st} + \vec{u}_0 e^{st} \tag{19}$$

$$(sI - A)\vec{x}_0 e^{st} = \vec{u}_0 e^{st} \tag{20}$$

$$(sI - A)\vec{x}_0 = \vec{u}_0 \tag{21}$$

Now, to express $\vec{x}_0$ in terms of $\vec{u}_0$, we can multiply by the inverse of $sI - A$ on both sides, since since $s \neq \lambda_i$ from before.[4]

Thus, we can rearrange eq. (21) to get

$$\vec{x}_0 = (sI - A)^{-1}\vec{u}_0 \tag{24}$$

$$\lim_{t\to\infty} \vec{x}(t) = \vec{x}_0 e^{st} = (sI - A)^{-1}\vec{u}_0 e^{st} \tag{25}$$

We have found a candidate steady-state solution given our vector of inputs $\vec{u}_0 e^{st}$. Let's verify that it satisfies the differential equation itself eq. (7). Rewriting the equation:

$$\frac{d}{dt}\vec{x}(t) - A\vec{x}(t) - \vec{u}(t) = \vec{0}. \tag{26}$$

To see whether the differential equation is satisfied, we have to plug in what we know about $\vec{u}(t) = \vec{u}_0 e^{st}$ and our candidate solution $\vec{x}(t) = (sI - A)^{-1}\vec{u}_0 e^{st}$ and see if everything cancels out.

$$\frac{d}{dt}\vec{x}(t) - A\vec{x}(t) - \vec{u}(t) = \frac{d}{dt}(sI - A)^{-1}\vec{u}_0 e^{st} - A(sI - A)^{-1}\vec{u}_0 e^{st} - \vec{u}_0 e^{st} \tag{27}$$

$$= (sI - A)^{-1}\vec{u}_0 \frac{d}{dt} e^{st} - A(sI - A)^{-1}\vec{u}_0 e^{st} - \vec{u}_0 e^{st} \tag{28}$$

[4] To show this, we will use a proof by contradiction. Imagine that $sI - A$ is not invertible, so $sI - A$ has a nonempty null space, containing some vector $\vec{y}$. We could then write:

$$(sI - A)\vec{y} = \vec{0} \tag{22}$$

$$\implies A\vec{y} = s\vec{y} \tag{23}$$

so s would be an eigenvalue of A with corresponding eigenvector $\vec{y}$. However, recall our original statement that s is not an eigenvalue of A. This result contradicts our original statement and so our assumption that $sI - A$ is not invertible must be incorrect, meaning $sI - A$ is invertible!

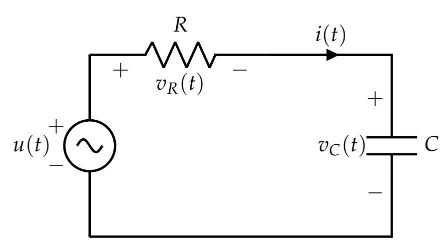
$$= (sI - A)^{-1}\vec{u}_0 s e^{st} - A(sI - A)^{-1}\vec{u}_0 e^{st} - \vec{u}_0 e^{st} \tag{29}$$
$$= e^{st} sI(sI - A)^{-1}\vec{u}_0 - e^{st} A(sI - A)^{-1}\vec{u}_0 - e^{st} I\vec{u}_0 \tag{30}$$
$$= e^{st}(sI(sI - A)^{-1}\vec{u}_0 - A(sI - A)^{-1}\vec{u}_0 - I\vec{u}_0) \tag{31}$$
$$= e^{st}\left(sI(sI - A)^{-1} - A(sI - A)^{-1} - I\right)\vec{u}_0 \tag{32}$$
$$= e^{st}\left((sI - A)(sI - A)^{-1} - I\right)\vec{u}_0 \tag{33}$$
$$= e^{st}(I - I)\vec{u}_0 \tag{34}$$
$$= \vec{0}. \tag{35}$$

So it all works out and our differential equation is satisfied for all time. We are not checking initial conditions because of our assumption that the effect of the initial conditions dies away.

4 Circuits with Exponential Inputs

Inspired by the above form of the solutions to differential equations given exponential inputs, let's take a look at what happens if our circuit was driven by inputs of the exponential function e^{st} for some constant s.

Consider a particular capacitor C within the following circuit:



From Note 1, we know the differential equation for the capacitor voltage is

$$\frac{d}{dt} v_C(t) = -\frac{v_C(t)}{RC} + \frac{u(t)}{RC} \tag{36}$$

Since $-\frac{1}{RC} < 0$, we know from our analysis of scalar differential equations in section 2 that at steady-state, the voltage will become a multiple of e^{st}:

$$v_C(t) = \widetilde{V}_C e^{st} \tag{37}$$

for some scalar $\widetilde{V}_C$.

By the known differential equation for that defines a capacitor as an element, we have that

$$i_C(t) = C\frac{d}{dt} v_C(t) \tag{38}$$
$$= C\frac{d}{dt}\left(\widetilde{V}_C e^{st}\right) \tag{39}$$
$$= Cs\widetilde{V}_C e^{st} \tag{40}$$

This means that $i_C(t)$ is also a scalar multiple of e^{st}, with this scalar factor being $\widetilde{I}_C = sC\widetilde{V}_C$. Critically, this equation resembles that of Ohm's Law! It is a purely linear equation without any time-dependence.

We will later obtain similar equations for an inductor $\widetilde{V}_L = \widetilde{I}_L \cdot sL$ and a resistor $\widetilde{V}_R = \widetilde{I}_R \cdot R$. From above, $\widetilde{V}_C = \widetilde{I}_C \cdot \left(\frac{1}{sC}\right)$. This suggests that capacitors and inductors have s-dependent resistances, called

s-impedances. The term *impedance* is a generalization of the concept of resistance to allow for different resistances at different s-values (we will soon see what governs the value of s in a circuit context). The impedance is defined as the voltage-current ratio (similar to resistance):

$$\widetilde{Z} = \frac{\widetilde{V}}{\widetilde{I}} \tag{41}$$

A capacitor has an s-impedance of $\frac{1}{sC}$. An inductor has an s-impedance of sL. And a resistor's s-impedance is just the same as its resistance R.

This reveals an approach for circuit analysis with exponential inputs, as long as all the inputs have the same s. We can replace all the independent voltage and current sources with constant voltages and currents corresponding to only the coefficients of e^{st}. We replace all capacitors and inductors with their corresponding impedances, and then analyze the entire circuit as though it only had resistances in it to get the steady-state solution. We will see examples of this in practice in the later sections.

5 Sinusoids and Phasors

So far, we stated that $\vec{u}(t)$ should be expressed as

$$\vec{u}(t) = \vec{u}_0 e^{st} \tag{42}$$

for some s. What kinds of s are we interested in for circuits? If $\mathrm{Re}\{s\} < 0$, then we know that the input approaches zero. If $\mathrm{Re}\{s\} > 0$, then our input will grow to infinity over time, so our state will blow up! This only leaves the case $\mathrm{Re}\{s\} = 0$ as neither blowing up or decaying away. If $\mathrm{Re}\{s\} = 0$, then *s must be purely imaginary, so $s = \mathrm{j}\omega$ for some real ω*. This suggests that "interesting" inputs will have the form $e^{st} = e^{\mathrm{j}\omega t}$.

It seems that any desired inputs must consist of complex exponentials of the form $e^{\mathrm{j}\omega t} = \cos(\omega t) + \mathrm{j}\sin(\omega t)$. Where are we going to find a physical waveform generator that emits imaginary voltages?

We can combine Euler's formula with complex conjugates to obtain

$$e^{\mathrm{j}\theta} + e^{-\mathrm{j}\theta} = (\cos(\theta) + \mathrm{j}\sin(\theta)) + (\cos(\theta) - \mathrm{j}\sin(\theta)) = 2\cos(\theta) \tag{43}$$

$$\cos(\theta) = \frac{e^{\mathrm{j}\theta} + e^{-\mathrm{j}\theta}}{2} \tag{44}$$

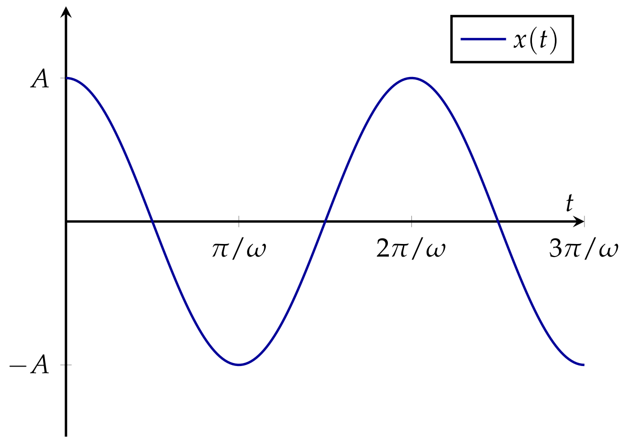
So, starting with two complex exponentials, we have pulled out a cosine function. Similarly, by subtracting complex exponentials:

$$e^{\mathrm{j}\theta} - e^{-\mathrm{j}\theta} = (\cos(\theta) + \mathrm{j}\sin(\theta)) - (\cos(\theta) - \mathrm{j}\sin(\theta)) = 2\mathrm{j}\sin(\theta) \tag{45}$$

$$\sin(\theta) = \frac{e^{\mathrm{j}\theta} - e^{-\mathrm{j}\theta}}{2\mathrm{j}} \tag{46}$$

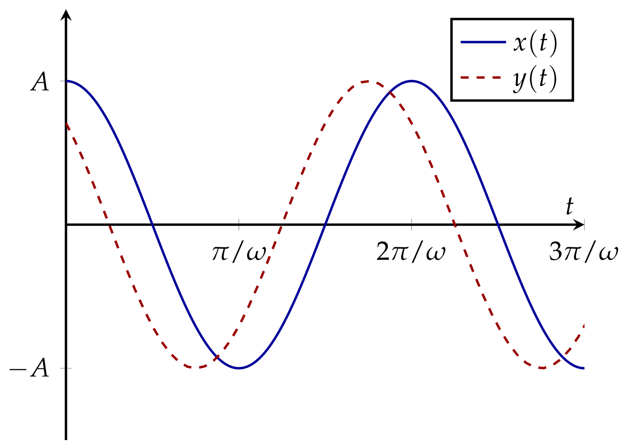
These 2 equations are often called the *inverse Euler's* formulas, allowing us to write real sinusoid functions as a sum of 2 complex exponentials, which we already know how to deal with from the previous sections! Sines and cosines are very realistic inputs to apply to a circuit.

Before we go on, we will discuss some common properties of sinusoid functions. Consider the example function $x(t) = A\cos(\omega t)$, where $x(t)$ can be thought of as representing an input in our circuit.



We call the maximum value of $x(t)$ above the mean (in this case, the x-axis) the *amplitude* (A), and the spacing between repetitions of the function the *period* ($T = 2\pi/\omega$).

Another important property of sinusoids is their *phase*. Consider $y(t) = A\cos(\omega t + \phi)$.



Here, ϕ represents the *phase shift* of $y(t)$ with respect to $x(t)$. A positive phase shift moves the function to the left by that amount. In particular, notice that the sine and cosine functions are really the same sinusoid, but just with a $\pi/2$ radian phase shift between them!

Now that we know a little about sinusoids, let's see how we can express a sinusoidal voltage input $v(t) = V_0\cos(\omega t + \phi)$ in terms of exponential functions.

From the above inverse Euler formulas,

$$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta} \tag{47}$$

$$\implies \cos(\omega t + \phi) = \frac{1}{2}e^{j\omega t + j\phi} + \frac{1}{2}e^{-j\omega t - j\phi} \tag{48}$$

$$= \frac{e^{j\phi}}{2}e^{j\omega t} + \frac{e^{-j\phi}}{2}e^{-j\omega t} \tag{49}$$

$$\implies v(t) = V_0\cos(\omega t + \phi) \tag{50}$$

$$= \frac{V_0 e^{j\phi}}{2}e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2}e^{-j\omega t}. \tag{51}$$

The coefficients of the two exponential functions are complex conjugates of one another. Then, since $e^{-j\theta} =$

$\overline{e^{j\theta}}$, we can rewrite the above as

$$v(t) = \frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \overline{\frac{V_0 e^{j\phi}}{2}} e^{-j\omega t}. \tag{52}$$

Thus, the coefficient of the $e^{j\omega t}$ can be used to represent the entire sinusoid $v(t)$ (assuming the frequency ω is known). We call this coefficient of $e^{j\omega t}$ the jω-**phasor** representing $v(t)$, denoted as

$$\widetilde{V} = \frac{V_0 e^{j\phi}}{2} \tag{53}$$

which gives us $v(t) = \widetilde{V} e^{j\omega t} + \overline{\widetilde{V}} e^{-j\omega t}$. Because we do this so often, we often omit mentioning $s = j\omega$.

With the phasor in hand, we can now try to find the steady state of systems of differential equations with sinusoidal inputs (each entry of $\vec{u}(t)$ is a cosine or sine function). First, we use the above inverse Euler transform to write the real input as a linear combination of 2 complex conjugate exponential functions

$$\vec{u}(t) = \vec{u}_0 e^{j\omega t} + \overline{\vec{u}_0} e^{-j\omega t} \tag{54}$$

where $\vec{u}_0$ is the vector of phasors. We know from our work from Section 3 how to find the steady-state solution for a single exponential input $\vec{u}_0 e^{st}$, but how do we do it for the sum of 2 exponential terms?

We claim that we can solve this by applying superposition (the solution to the differential equation of the combined input is the sum of the differential equation solutions to each individual input). We prove this claim for 2 inputs:

Proof. Assume that for 2 inputs $u_1(t), u_2(t)$, we have 2 respective solutions $\vec{x}_1(t), \vec{x}_2(t)$ for the equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + u(t). \tag{55}$$

For the combined input $u(t) = u_1(t) + u_2(t)$, we propose the candidate solution $\vec{x}(t) = \vec{x}_1(t) + \vec{x}_2(t)$. We now check whether this satisfies our differential eq. (55).

$$\begin{aligned}
\frac{d}{dt}\vec{x}(t) &= \frac{d}{dt}\vec{x}_1(t) + \frac{d}{dt}\vec{x}_2(t) && (56)\\
&= A\vec{x}_1(t) + u_1(t) + A\vec{x}_2(t) + u_2(t) && (57)\\
&= A\left(\vec{x}_1(t) + \vec{x}_2(t)\right) + \left(u_1(t) + u_2(t)\right) && (58)\\
&= A\vec{x}(t) + u(t) && (59)
\end{aligned}$$

and so, superposition does apply to our input. □

Thus, we can now solve eq. (21) from section 3 twice to get our total solution: once with $s = j\omega$ and once with $s = -j\omega$. For the first term $\vec{u}_0 e^{j\omega t}$, we need to solve

$$(j\omega I - A)\vec{x}_1 = \vec{u}_0 \tag{60}$$

for $\vec{x}_1$, where we substituted $s = j\omega$. For the second term $\overline{\vec{u}_0} e^{-j\omega t}$, we need to solve

$$(-j\omega I - A)\vec{x}_2 = \overline{\vec{u}_0} \tag{61}$$

for $\vec{x}_2$, where we substituted $s = -j\omega$. However, note that by complex conjugating both sides of eq. (60),

$$\begin{aligned}
\overline{(j\omega I - A)\vec{x}_1} &= \overline{\vec{u}_0} && (62)\\
\overline{(j\omega I - A)}\ \overline{\vec{x}_1} &= \overline{\vec{u}_0} && (63)\\
(-j\omega I - A)\overline{\vec{x}_1} &= \overline{\vec{u}_0} && (64)
\end{aligned}$$

This is exactly what we wanted to solve for in eq. (61), showing $\vec{x}_2 = \overline{\vec{x}_1}$.

So finally, we can apply superposition and sum the steady state solutions from each exponential input:

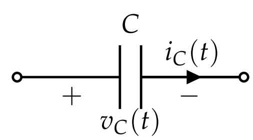
$$\lim_{t\to\infty} \vec{x}(t) = \vec{x}_1 e^{j\omega t} + \vec{x}_2 e^{-j\omega t} \tag{65}$$
$$= \vec{x}_1 e^{j\omega t} + \overline{\vec{x}_1} e^{-j\omega t} \tag{66}$$
$$= (j\omega I - A)^{-1}\vec{u}_0 e^{j\omega t} + \overline{(j\omega I - A)^{-1}\vec{u}_0} e^{-j\omega t} \tag{67}$$

6 Impedances

We now know how to find the steady-state response to a circuit with sinusoidal inputs all at the same frequency. But we can simplify it even more with some key findings about jω-*impedances*, which are a specialization of the s-impedances from section 4 when letting $s = j\omega$. The analysis will be almost the same, except now the voltage/current is a pure sinusoid and thus a combination of two exponential terms instead of just one. In addition, we will derive the jω-impedances in depth for the inductor and resistor along with the capacitor like before.

6.1 Impedance of a Capacitor

We examine a capacitor provided with the sinusoidal voltage $v_C(t) = V_0 \cos(\omega t + \phi)$, as shown:



Note that we aren't assuming anything about the origin of the $v_C(t)$ – it could come from a voltage supply or from some other circuit. From before, we know that $v_C(t)$ has a phasor representation:

$$v_C(t) = \frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2} e^{-j\omega t}, \implies \widetilde{V}_C = \frac{V_0 e^{j\phi}}{2}. \tag{68}$$

Now, by the capacitor equation, we know that:

$$i_C(t) = C\frac{d}{dt} v_C(t) \tag{69}$$
$$= C\frac{d}{dt}\left(\frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2} e^{-j\omega t}\right) \tag{70}$$
$$= C\frac{d}{dt}\left(\widetilde{V}_C e^{j\omega t} + \overline{\widetilde{V}}_C e^{-j\omega t}\right) \tag{71}$$
$$= C\left(\widetilde{V}_C \cdot j\omega \cdot e^{j\omega t} + \overline{\widetilde{V}}_C \cdot (-j\omega) \cdot e^{-j\omega t}\right) \tag{72}$$
$$= (j\omega C)\widetilde{V}_C e^{j\omega t} + (-j\omega C)\overline{\widetilde{V}}_C e^{-j\omega t} \tag{73}$$

Noting that we find phasors by taking the coefficient of the $e^{j\omega t}$ term (the coefficient of the $e^{-j\omega t}$ term is guaranteed to be the complex conjugate.) So we can represent the current as the phasor

$$\widetilde{I}_C = (j\omega C)\widetilde{V}_C. \tag{74}$$

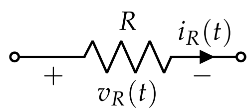
Having shown that all steady-state circuit quantities will be sinusoids with frequency ω, we can relate the phasors of a capacitor's voltage and current by a ratio depending only on the frequency and capacitance.

This ratio is called **impedance**, and can be thought of as the "AC resistance" of a capacitor since it relates the phasor representations of an element's voltage and current by a constant ratio. For a capacitor, the impedance is

$$Z_C = \frac{\widetilde{V}_C}{\widetilde{I}_C} = \frac{1}{\mathrm{j}\omega C}. \tag{75}$$

6.2 Impedance of a Resistor

Take a resistor R labeled as follows:



Let $v_R(t)$ be represented by some phasor $\widetilde{V}_R$. Thus, by Ohm's Law,

$$v_R(t) = \widetilde{V}_R e^{\mathrm{j}\omega t} + \overline{\widetilde{V}}_R e^{-\mathrm{j}\omega t} \tag{76}$$

$$i_R(t) = \frac{1}{R} v_R(t) \tag{77}$$

$$= \frac{\widetilde{V}_R}{R} e^{\mathrm{j}\omega t} + \overline{\frac{\widetilde{V}_R}{R}} e^{-\mathrm{j}\omega t}, \tag{78}$$

so we may represent the output current with the phasor

$$\widetilde{I}_R = \frac{\widetilde{V}_R}{R}, \tag{79}$$

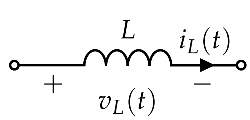
so the impedance is

$$Z_R = \frac{\widetilde{V}_R}{\widetilde{I}_R} = R. \tag{80}$$

The impedance behaves very much like resistance does, except that it generalizes to non-resistor components.

6.3 Impedance of an Inductor

Consider an inductor with voltage and current across it as follows:



Let the current $i_L(t) = I_0 \cos(\omega t + \phi)$ be represented by some phasor $\widetilde{I}_L$. By the equation of an inductor"

$$i_L(t) = \widetilde{I}_L e^{\mathrm{j}\omega t} + \overline{\widetilde{I}}_L e^{-\mathrm{j}\omega t} \tag{81}$$

$$v_L(t) = L \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} \tag{82}$$

$$= L\left((\mathrm{j}\omega)\widetilde{I}_L e^{\mathrm{j}\omega t} - (\mathrm{j}\omega)\overline{\widetilde{I}}_L e^{-\mathrm{j}\omega t}\right) \tag{83}$$

$$= (\mathrm{j}\omega L)\widetilde{I}_L e^{\mathrm{j}\omega t} + \overline{(\mathrm{j}\omega L)\widetilde{I}_L} e^{-\mathrm{j}\omega t}, \tag{84}$$

so the voltage can be represented by the phasor

$$\widetilde{V}_L = \mathrm{j}\omega L \widetilde{I}_L. \tag{85}$$

And the impedance of an inductor is

$$Z_L = \frac{\widetilde{V}_L}{\widetilde{I}_L} = \mathrm{j}\omega L. \tag{86}$$

6.4 A Remark on Conjugation

When we have an expression like $v(t) = \frac{V_0 \mathrm{e}^{\mathrm{j}\phi}}{2}\mathrm{e}^{\mathrm{j}\omega t} + \frac{V_0 \mathrm{e}^{-\mathrm{j}\phi}}{2}\mathrm{e}^{-\mathrm{j}\omega t}$, we have consistently defined the phasor as $\widetilde{V} = \frac{V_0 \mathrm{e}^{\mathrm{j}\phi}}{2}$, the coefficient of the $\mathrm{e}^{\mathrm{j}\omega t}$ term. But what about the conjugate $-\mathrm{j}\omega$-phasor $\overline{\widetilde{V}} = \frac{V_0 \mathrm{e}^{-\mathrm{j}\phi}}{2}$ associated with the complementary exponential, $\mathrm{e}^{-\mathrm{j}\omega t}$?

Since the two phasor terms form a complex-conjugate pair, knowing one automatically tells us the other one. This balance happens in such a way that *the ultimate time-domain signal stays fully real-valued*. In a sense, having the conjugate phasor is necessary to "cancel out" the imaginary parts from the original phasor. Notice that $\mathrm{e}^{\mathrm{j}\omega t}$ has a real and imaginary component; its imaginary part can only be cancelled by adding its complex conjugate (for any complex a, $a + \overline{a} = 2\,\mathrm{Re}\{a\}$, which is fully real.)

7 Circuit Analysis and Example

We now discuss how to apply the phasor and impedance techniques we just derived to an actual circuit. The major insight allowing us to do this is that with a phasor, we are representing the *entire sinusoid*. We will find this very convenient mathematically and could do this only because linear circuits (resistors, capacitors, and inductors) will never alter the frequency of a sinusoid (the input e^{st} has frequency s which is unchanged by differentiation).

7.1 Summarizing the Connection Between Time and Phasor Domains

Given a general sinusoidal time-domain input signal $u(t) = A\cos(\omega t + \phi)$, we represented $u(t)$ as a weighted sum of $\mathrm{e}^{\mathrm{j}\omega t}$ and its complex conjugate $\mathrm{e}^{-\mathrm{j}\omega t}$. This led us to define the phasor $\widetilde{U} = \frac{A\mathrm{e}^{\mathrm{j}\phi}}{2}$. We showed how the phasor captures all the critical information about $u(t)$, but without the time-dependent terms. This transformation $u(t) \to \widetilde{U}$ is often called a *Phasor-Transform*.

$$\text{Phasor Transform: } u(t) = A\cos(\omega t + \phi) \;\to\; u(t) = \frac{A\mathrm{e}^{\mathrm{j}\phi}}{2}\mathrm{e}^{\mathrm{j}\omega t} + \frac{A\mathrm{e}^{-\mathrm{j}\phi}}{2}\mathrm{e}^{-\mathrm{j}\omega t} \implies \widetilde{U} = \frac{A\mathrm{e}^{\mathrm{j}\phi}}{2} \tag{87}$$

Using $\widetilde{U}$, we have seen how to analyze the behavior of R, L, C circuit elements to find their voltages and currents. But, we ultimately want to know what the output voltage or current is as a function of time. This motivates the *Inverse Phasor Transform*. The inverse transformation takes some $\widetilde{W} = B\mathrm{e}^{\mathrm{j}\psi}$ that we've solved for, and converts it into $w(t) = B\cos(\omega t + \psi)$. Notice that the frequency term stayed the same throughout!

$$\text{Inverse Phasor Transform: } \quad \widetilde{W} = B\mathrm{e}^{\mathrm{j}\psi} \implies w(t) = 2B\cos(\omega t + \psi) \tag{88}$$

7.2 KCL and KVL with Phasors

We will now show that a sum of sinusoidal functions is zero if and only if the sum of the phasors of each of those functions equals zero as well. Let the sinusoid functions be $x_1(t), x_2(t), \ldots, x_n(t)$. Let the phasors representing the sinusoids be $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$. Then,

$$\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n = 0 \tag{89}$$

$$\iff (\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n)e^{j\omega t} = 0 \tag{90}$$

$$\iff (\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n)e^{j\omega t} + \overline{(\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n)}e^{-j\omega t} = 0 \tag{91}$$

$$\iff \sum_{k=1}^{n} \left(\widetilde{X}_k e^{j\omega t} + \overline{\widetilde{X}_k} e^{-j\omega t} \right) = 0 \tag{92}$$

$$\iff x_1(t) + x_2(t) + \ldots + x_n(t) = 0, \tag{93}$$

so we have proved that a sum of sinusoids is zero if and only if the sum of their corresponding phasors is zero as well. This proof also generalizes both KCL and KVL to phasors:

1. The sum of currents leaving a node is 0 if and only if the sum of current phasors leaving a node is 0 (this is KCL).
2. The sum of voltage differences in a loop is 0 if and only if the sum of voltage phasors in a loop is 0 (this is KVL).

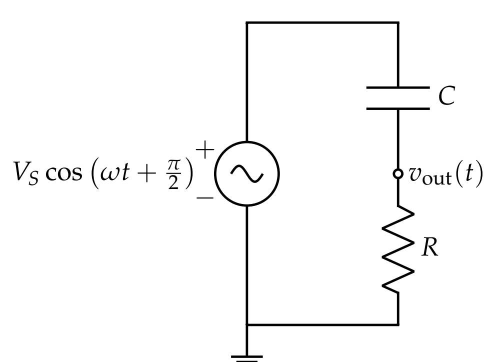
7.3 Problem-Solving Process

In previous sections, we obtained "equivalents" to Ohm's Law for inductors and capacitors, using the impedance to relate their voltage and current phasors. With KCL and KVL established to work in the phasor domain as well, we have now successfully generalized all of our techniques of DC analysis to AC/frequency analysis, and thus makes working in the phasor domain the same as analyzing resistor circuits from 16A.

We now outline the key steps to analyze and solve a general circuit using phasors:

1. Confirm that the circuit can actually be usefully analyzed using phasors. *This requires the voltages and currents to be sinusoidal!* Moreover, you must analyze each distinct frequency ω that might be present in the circuit separately.
2. Convert all $v(t), i(t)$ information to the phasor-domain $(\widetilde{V}, \widetilde{I})$.
3. Solve for the voltages and currents using EECS16A techniques (KCL, KVL, NVA, etc.)
4. Convert all phasor results back to the time-domain.

We can now consider some basic circuits to exercise this technique and verify that it works correctly. Consider a voltage divider, where we introduce a capacitor in place of one of the resistors:



We are interested in knowing how $v_{\text{out}}(t)$ varies over time in response to the input supply, $u(t) = V_S \cos\left(\omega t + \frac{\pi}{2}\right)$. Recall that we proved the voltage divider equation in the context of DC circuit analysis. However, that proof carries over to the phasor domain in a straightforward manner. Thus, the phasor $\widetilde{V}_{\text{out}}$ representing the voltage $v_{\text{out}}(t)$ can be represented in terms of the phasor $\widetilde{U}$ representing the input voltage:

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_C + Z_R}\widetilde{U}, \tag{94}$$

where Z_C and Z_R are the impedances of the capacitor and resistor, respectively. Note also that, since the input is at frequency ω, all other voltages and currents in the system will also be at the same frequency ω. Thus, using our results from earlier that $Z_C = \frac{1}{\mathrm{j}\omega C}, Z_R = R, \widetilde{U} = \frac{V_S e^{\mathrm{j}\frac{\pi}{2}}}{2} = \frac{\mathrm{j}V_S}{2}$, and substituting these values into our equation for $\widetilde{V}_{\text{out}}$, we find that:

$$\widetilde{V}_{\text{out}} = \frac{R}{\frac{1}{\mathrm{j}\omega C} + R}\frac{\mathrm{j}V_S}{2} = V_S\frac{\frac{\mathrm{j}R}{2}}{\frac{1}{\mathrm{j}\omega C} + R}. \tag{95}$$

It'll be convenient to have a magnitude-phase representation (and to multiply both top and bottom by $\mathrm{j}\omega C$ to rationalize the denominator):[5]

$$\begin{aligned}
\widetilde{V}_{\text{out}} &= \frac{V_S\left(\frac{\mathrm{j}R}{2}\right)\mathrm{j}\omega C}{1 + \mathrm{j}\omega RC} &(96)\\
&= \frac{-V_S\,\omega RC}{2(1 + \mathrm{j}\omega RC)} &(97)\\
&= \frac{V_S\,\omega RC}{2\sqrt{1 + (\omega RC)^2}}e^{\mathrm{j}(\pi - \operatorname{atan2}(\omega RC, 1))} &(98)
\end{aligned}$$

Now, we can use the Inverse Phasor Transform formula:

$$\begin{aligned}
v_{\text{out}}(t) &= \widetilde{V}_{\text{out}}e^{\mathrm{j}\omega t} + \overline{\widetilde{V}}_{\text{out}}e^{-\mathrm{j}\omega t} &(99)\\
&= 2\left|\widetilde{V}_{\text{out}}\right|\cos\left(\omega t + \measuredangle\widetilde{V}_{\text{out}}\right) &(100)\\
&= \frac{V_S\,\omega RC}{\sqrt{1 + (\omega RC)^2}}\cos(\omega t + \pi - \operatorname{atan2}(\omega RC, 1)). &(101)
\end{aligned}$$

This formula might look complicated, but it'll become significantly simpler when we have actual values of R, C, ω to plug in. An example of this will be at the start of the next note!

Appendix A Warning

Be aware that in this course phasors are defined slightly differently from how it is often done elsewhere. Essentially, there is a factor of 2 difference.

In this course, we define the phasor representation $\widetilde{X}$ of a sinusoid $x(t)$ to be such that

$$x(t) = \widetilde{X}e^{\mathrm{j}\omega t} + \overline{\widetilde{X}}e^{-\mathrm{j}\omega t}. \tag{102}$$

However, elsewhere, the phasor representation may be defined such that

$$x(t) = \frac{1}{2}(\widetilde{X}e^{\mathrm{j}\omega t} + \overline{\widetilde{X}}e^{-\mathrm{j}\omega t}). \tag{103}$$

[5] If these steps are not familiar, it is important at this point to carefully review Note j. Here, we are using the polar form of the numerator and denominator, treating the entire phasor as a ratio of complex numbers.

Our definition is more natural and aligns to what you will see in later courses when you learn about Laplace and Fourier transforms. This is because our definition arises from the mathematics, and the same spirit of definition works even when working with inputs of the form e^{st} where s is not a purely imaginary number.

But then why would anyone ever use the alternative, more common definition? Its main advantage is that the magnitude of the phasor equals the amplitude of the signal. For instance, if we have the signal $A\cos(\omega t + \phi)$, then the alternative definition yields the phasor $Ae^{j\phi}$, with magnitude A. In contrast, our definition yields the phasor $(A/2)e^{j\phi}$. The former definition can sometimes be convenient when conducting physical observations - when using an oscilloscope, one can easily see[6] the amplitude A of a signal, not the half-amplitude $A/2$.

Furthermore, it turns out that there are some slight calculation advantages (i.e. it makes some formulas arguably simpler) to the more common definition when working with power systems and power electronics, which you may see if you take the relevant upper-division EE courses. However, for the purposes of the scope of this course, our definition is simpler and easier to understand, so we will stick with it throughout.

Of course, if the mathematics is done correctly, there is no real difference between the two definitions, in that both describe the same physical behaviors.

Contributors:

- Neelesh Ramachandran.
- Rahul Arya.
- Anant Sahai.
- Jaijeet Roychowdhury.
- Ashwin Vangipuram.
- Druv Pai.

[6] Actually in practice, if there is a DC component to the circuit — i.e. there are some inputs that are constants too — then the easiest thing to see is the peak-to-peak swing of the voltage which corresponds to twice the amplitude. So even the more common definition often forces the person using it to have to divide by two.

