EECS 16B Designing Information Devices and Systems II
Spring 2021

Note 6: Bode Plots

Overview

Having analyzed our first order filters and gone through a design example in the previous Note to show why filter-design is important, we will now plot their transfer functions $H(\omega)$ (or frequency responses) using Bode Plots. In the previous Note, we generated tables of $|H(\omega)|$, $\measuredangle H(\omega)$ at certain key values of ω_c, and while this gave some intuition, it didn't really show what happens at intermediate frequencies. There is immense value in visualizing transfer functions across a wide range of frequencies.

Throughout this section, we will use numerical approximations, which will not only prove useful in plotting a filter's frequency response but also will help us better understand its behavior.

1 Bode Plots

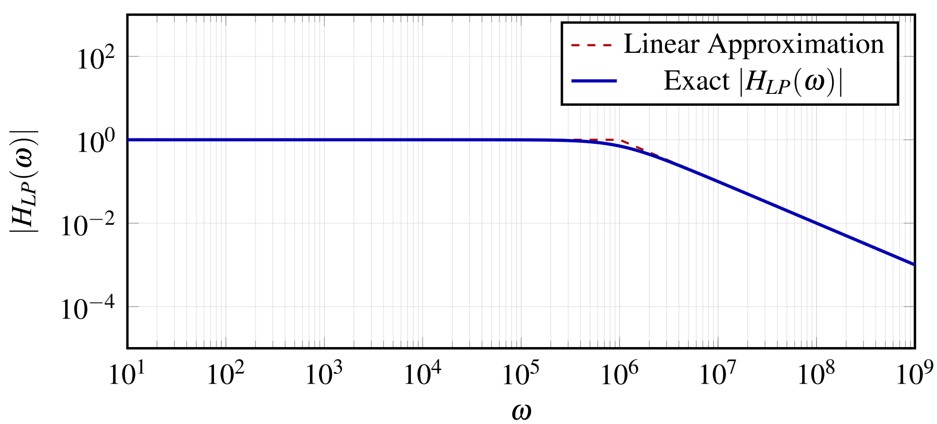
When we make Bode plots, we plot the frequency and magnitude on a logarithmic scale, and the angle in either degrees or radians. We use the logarithmic scale because it allows us to break up complex transfer functions into its constituent components. Let's start by generating Bode Plots for low-pass and high-pass first order filters, which will build our intuition. We will soon see how the analysis of more complex transfer functions can be broken down into parts.

1.1 Low-pass Filter

Recall our generalized model of a low-pass filter (perhaps RC or LR):

$$H_{LP}(\omega) = \frac{1}{1 + j\omega/\omega_c}$$

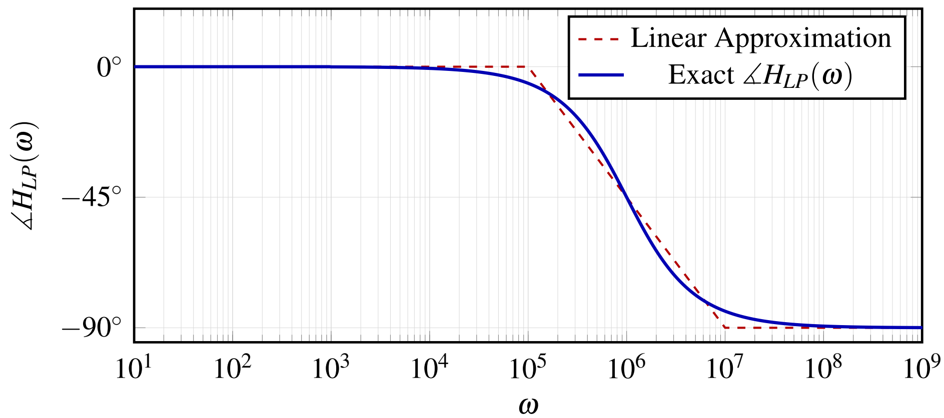
We plot the magnitude of the frequency response assuming $\omega_c = 10^6$. Note how $|H_{LP}(\omega)|$ is very close to



1 for $\omega < \omega_c$ and $|H_{LP}(\omega)|$ starts dropping off with slope -1 after ω_c. We can look at 3 distinct regions on the plot:

- $\omega \ll \omega_c$: $\implies j\omega/\omega_c \approx 0$. So, $H_{LP}(\omega) \approx 1$ and $|H_{LP}(\omega)| \approx 1$.
- $\omega = \omega_c$: $\implies H(\omega) = \frac{1}{1+j}$. So, $|H_{LP}(\omega)| = \frac{1}{\sqrt{2}}$.
- $\omega \gg \omega_c$: $\implies \omega/\omega_c \gg 1$. Therefore $H_{LP}(\omega) \approx -j\frac{\omega_c}{\omega}$. So, $|H_{LP}(\omega)| \approx \frac{\omega_c}{\omega}$. On a log scale, this means that $\log|H_{LP}(\omega)| \approx \log\omega_c - \log\omega$ explaining behavior of dropping off with slope -1.[1]

Now let's plot the phase of $H_{LP}(\omega)$: $\measuredangle H_{LP}(\omega)$ is very close to 0 for $\omega < 0.1\omega_c$ and $\measuredangle H_{LP}(\omega)$ is approxi-



mately $-\frac{\pi}{2}$ for $\omega > 10\omega_c$.

- $\omega \ll 0.1\omega_c \implies j\omega/\omega_c \approx 0$. So, $H_{LP}(\omega) \approx 1$ and $\measuredangle H_{LP}(\omega) \approx 0$.
- $\omega = 0.1\omega_c \implies H_{LP}(0.1\omega_c) = \frac{1}{1+j\,0.1}$ and $\measuredangle H_{LP}(\omega) \approx -6°$.
- $\omega = \omega_c \implies H_{LP}(\omega_c) = \frac{1}{1+j}$ and $\measuredangle H_{LP}(\omega) = -45°$.
- $\omega = 10\omega_c \implies H_{LP}(10\omega_c) = \frac{1}{1+j\,10}$ and $\measuredangle H_{LP}(\omega) \approx -84°$.
- $\omega \gg 10\omega_c \implies \omega/\omega_c \gg 10$. So, $H_{LP}(\omega) \approx -j \cdot 0$ [2] and $\measuredangle H_{LP}(\omega) \approx -90°$.

We can now better understand the values of the magnitude and phase at $0.1\omega_c$, ω_c, $10\omega_c$ (as seen in the tables of Note 5).

1.2 High-pass Filter

We can similarly analyze our generalized high-pass filter model (CR, RL):

$$H_{HP}(\omega) = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$$

we plot the magnitude of the frequency response, again assuming $\omega_c = 10^6$.

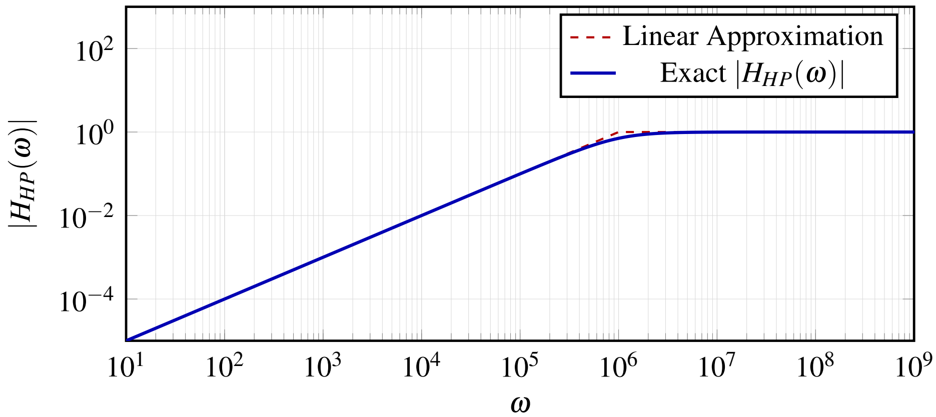
Here, $|H_{HP}(\omega)|$ rises with slope 1 for $\omega < \omega_c$ and $|H_{HP}(\omega)| \approx 1$ after ω_c. We analyze the plot:

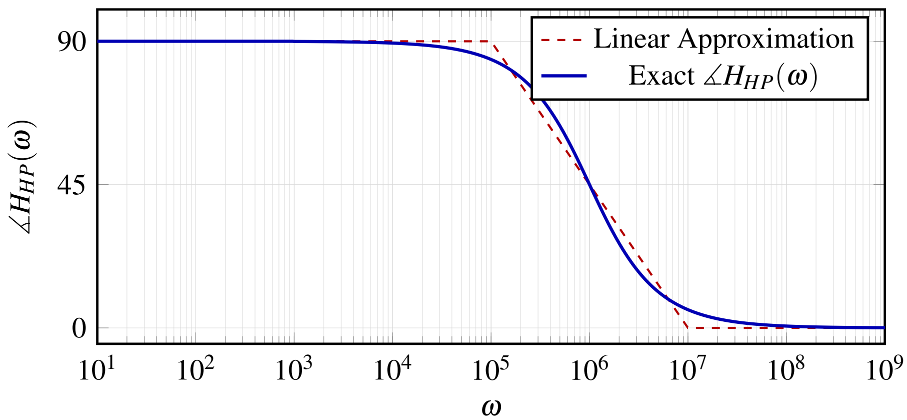
- $\omega \ll \omega_c$, then $\omega/\omega_c \ll 1$. Therefore $H_{HP}(\omega) \approx j\frac{\omega}{\omega_c}$ which implies $|H_{HP}(\omega)| \approx \frac{\omega}{\omega_c}$. On a log scale, this means that $\log|H_{HP}(\omega)| \approx \log\omega - \log\omega_c$, which explainsthe rising slope of 1.
- $\omega = \omega_c$, then $H(\omega) = \frac{j}{1+j}$ meaning $|H_{HP}(\omega)| = \frac{1}{\sqrt{2}}$
- $\omega \gg \omega_c$, then $\omega/\omega_c \gg 1$. Therefore $H_{HP}(\omega) \approx 1$ which implies $|H_{HP}(\omega)| \approx 1$.

Now let's plot the phase of the transfer function $H_{HP}(\omega)$.

[1] Recall that the line $y = mx + b$ has slope m. In this case $y = \log|H_{LP}(\omega)|$ and $x = \log|\omega|$.

[2] The magnitude will always be greater than 0, meaning its phase will still be very close to $-90°$





$\measuredangle H_{HP}(\omega)$ is very close to $\frac{\pi}{2}$ for $\omega < 0.1\omega_c$ and $\measuredangle H_{HP}(\omega)$ is approximately 0 for $\omega > 10\omega_c$.

- $\omega \ll 0.1\omega_c \implies j\omega/\omega_c \approx 0$. So, $H_{HP}(\omega) \approx 0$ (but slightly positive) and so $\measuredangle H_{HP}(\omega) \approx 90^\circ$.
- $\omega = 0.1\omega_c \implies H_{HP}(0.1\omega_c) = \frac{j\ 0.1}{1+j\ 0.1}$ and $\measuredangle H_{HP}(\omega) \approx 84^\circ$.
- $\omega = \omega_c \implies H_{HP}(\omega_c) = \frac{j}{1+j}$ and $\measuredangle H_{HP}(\omega) = 45^\circ$.
- $\omega = 10\omega_c \implies H_{HP}(10\omega_c) = \frac{j\ 10}{1+j\ 10}$ and $\measuredangle H_{HP}(\omega) \approx 6^\circ$.
- $\omega \gg 10\omega_c \implies \omega/\omega_c \gg 10$. So, $H_{HP}(\omega) \approx 1$ and $\measuredangle H_{HP}(\omega) \approx 0^\circ$.

2 Second Order Filters

We will now consider more complex systems and, in doing so, see the value in appropriate visualizations for transfer function behavior.

2.1 Band-Pass Filters

With the knowledge of low-pass filters that block out higher frequencies and high-pass filters that block out lower frequencies, how could we build a filter that lets a specific range of frequncies through? One idea could be to take the output of the low-pass filter and treat it as an input to the high-pass filter.

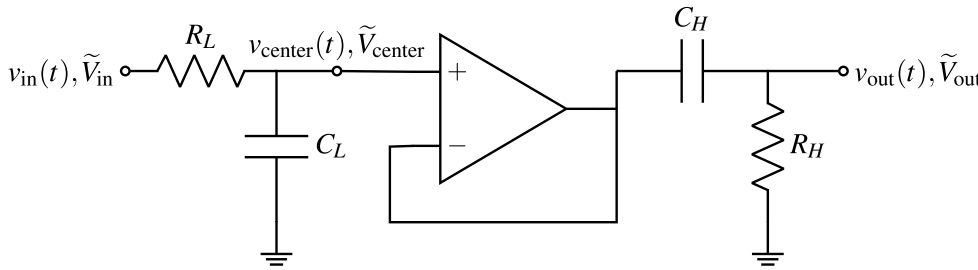


Figure 1: A Buffered Band-Pass filter, composed of low-pass and high-pass filter components.

Notice the op-amp serving as a unity gain buffer between the two filters. It is introduced to prevent the second circuit from loading the first.

The input voltage phasor is $\widetilde{V}_{\text{in}}$ at some frequency ω. Suppose the two filters have cutoff frequencies ω_{LP} and ω_{HP}, respectively. Thus, we can write that:

$$\widetilde{V}_{\text{center}} = H_{LP}(\omega)\widetilde{V}_{\text{in}}$$

Since $v_{\text{center}}(t)$ is the second filter's input, the output voltage phasor $\widetilde{V}_{\text{out}}$ is:

$$\widetilde{V}_{\text{out}} = H_{HP}(\omega)\widetilde{V}_{\text{center}} = H_{HP}(\omega)H_{LP}(\omega)\widetilde{V}_{\text{in}}$$

Thus, the net transfer function $H_{BP}(\omega)$ is:

$$H_{BP}(\omega) = H_{LP}(\omega)H_{HP}(\omega)$$

More generally, placing filters in series produces a circuit whose transfer function is the product of the individual transfer functions. From the properties of complex numbers and logarithms, we see that $\log|z_1 z_2| = \log|z_1| + \log|z_2|$. So, when plotting $|H_{BP}(\omega)|$ on a log-log plot, this corresponds to a graphical addition of the individual transfer functions. Note that we are still taking the product of the actual transfer functions, but it resembles a geomtrical sum: $|H_{LP}(\omega)| + |H_{HP}(\omega)|$. We see this idea on display in the examples at the end of this note. Similarly, for $\measuredangle H_{BP}(\omega)$, we can again plot the sum, $\measuredangle H_{LP}(\omega) + \measuredangle H_{HP}(\omega)$.[3]

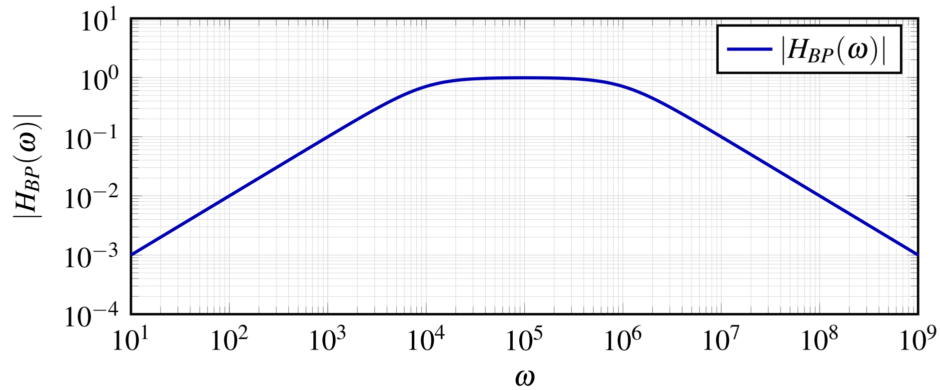
We can compute $H_{BP}(\omega)$ symbolically as:

$$H_{BP}(\omega) = H_{LP}(\omega)H_{HP}(\omega) = \frac{1}{1 + j\omega R_L C_L} \cdot \frac{j\omega R_H C_H}{1 + j\omega R_H C_H}$$

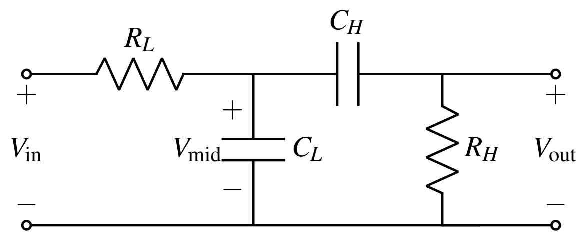
To find the cutoff frequencies of this filter, we can look at the points at which $H_{BP}(\omega_c) = \frac{1}{\sqrt{2}}$. But based on our approximations from before and from the cuttof frequencies $\omega_{LP} = \frac{1}{R_L C_L}$ and $\omega_{HP} = \frac{1}{R_H C_H}$, we can approximate $|H(\omega_{LP})| \approx \frac{1}{\sqrt{2}} \cdot 1$ and $|H(\omega_{HP})| \approx 1 \cdot \frac{1}{\sqrt{2}}$. This approximation holds best when the cuttof frequencies are spaced apart.

We've shown a convenient result! The cutoffs for the band-pass filter are identical to the individual cutoffs for the low and high-pass filters. We now plot $H_{BP}(\omega)$ with $\omega_{LP} = 10^6$ and $\omega_{HP} = 10^4$ to demonstrate the band-pass behavior.

[3] We must be careful, however, to note that in most of our plots, the x-axis does *not* correspond to 0, so we can't simply "stack" the two plots.



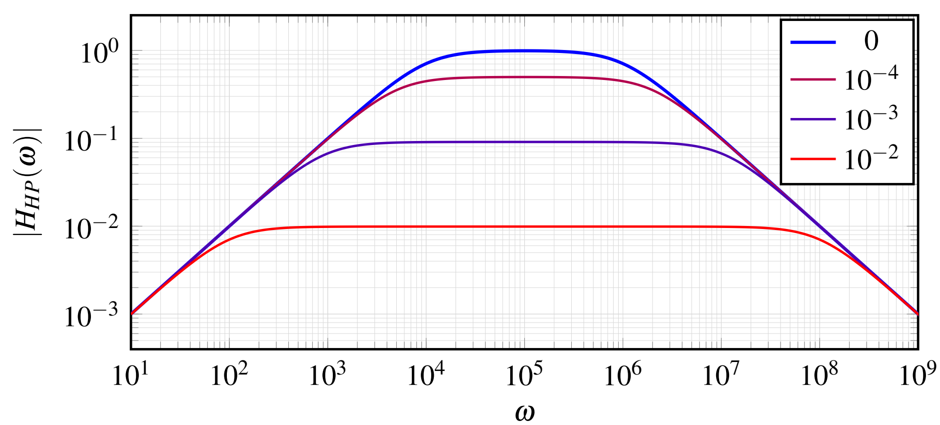
Our band-pass filter uses an op-amp, but what if we cascade our two filters, causing a loading effect?



We leave the derivation as an exercise (feel free to post your work on Piazza!), but computing the transfer function yields:

$$H(\omega) = \frac{j\omega R_H C_H}{(1 + j\omega R_L C_L)(1 + j\omega R_H C_H) + j\omega R_L C_H}$$

The loading effect introduces a term of $j\omega R_L C_H$ in the denominator. The relative size of this term determines the impact on the circuit. We plot some examples of the band-pass filter with identical low and high cutoff frequencies but different $R_L C_H$ values to show this loading effect.



Note how the maximum value of $H(\omega)$ decreases as $R_L C_H$ increases. In addition, the cutoff frequencies move further and further apart from the original $\omega_{LP} = \frac{1}{R_L C_L}$ and $\omega_{HP} = \frac{1}{R_H C_H}$.

2.2 Low-Pass Filters

From our analysis of low-pass filters, we saw that $|H(\omega)|$ dropped off by a factor of 10 for each factor-of-10 increase in frequency after ω_c. This is a desirable effect, but ideally, we would like to build a filter that drops off at a quicker rate after ω_c. Therefore, let's try cascading two low-pass filters of identical cutoff with a buffer in between. The diagram is exactly like

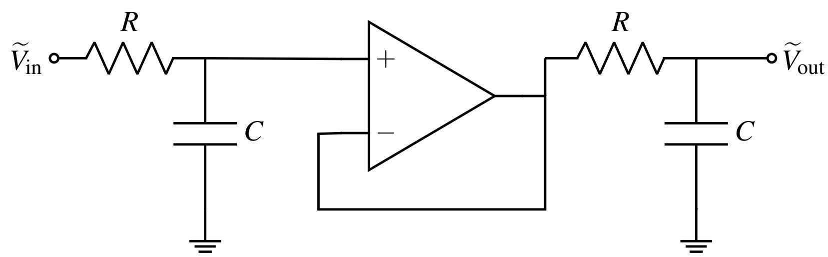


Figure 2: A Buffered Second Order Low-Pass filter. To achieve fast roll-off after ω_c, we must use the same R, C values in both filters.

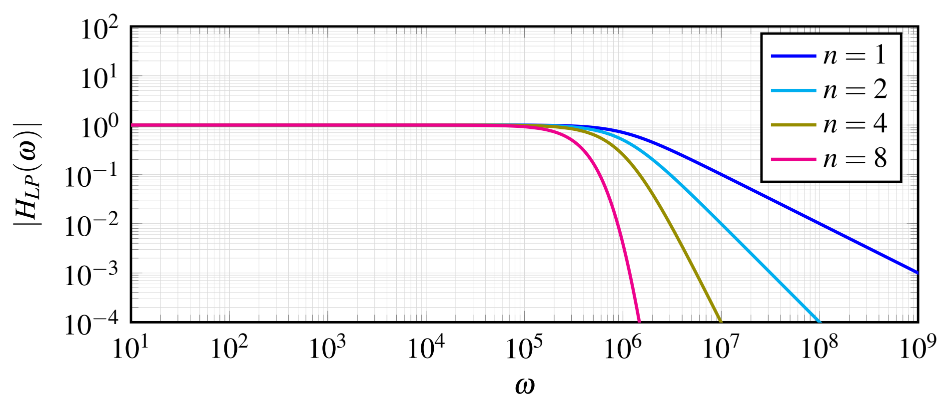
With similar analysis as for the band-pass filter:

$$H_{LP}(\omega) = \frac{1}{(1 + j\omega RC)^2}$$

We can see that it does indeed drop off at a quicker rate (with slope 2 after the cutoff ω_c.). In fact, let's just generalize the system to be an $n-$th order low-pass filter (having n such RC stages), and plot $|H_{LP}(\omega)|$ for each! For an n^{th} order filter, we see a dropoff of slope n after the cutoff. We eventually approach an ideal low-pass filter[4] in which

$$H(\omega) = \begin{cases} 1 & \omega < \omega_c \\ 0 & \omega \geq \omega_c \end{cases} \tag{1}$$

We will explore this effect in more detail in the next section.



[4]This is where our hand-approximations and reality diverge a bit. If we use straight-lines, the transition will happen exactly at ω_c but what we see in the plot is a steady shift to the left as n increases. This is the exact result, and the difference arises from the fact that our approximation error (a fractor of $\frac{1}{\sqrt{2}}$) magnifies with each additional filter stage. The Bode approximation is unable to capture this behavior. If we wanted to build something closer to the ideal low-pass filter, we need to shift $\frac{1}{RC}$ to be slightly greater than ω_c.

3 Higher Order Bode Plots

We will now see how to plot a given transfer function by using straight-line approximations, and we will notice that a specific form of transfer function will make the process of plotting more systematic and simple.

3.1 Rational Transfer Functions

When we write the transfer function of an arbitrary circuit, it always takes the following form, called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = K \cdot \frac{N(\omega)}{D(\omega)} = K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right) \left(1 + j\frac{\omega}{\omega_{z2}}\right) \cdots \left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right) \left(1 + j\frac{\omega}{\omega_{p2}}\right) \cdots \left(1 + j\frac{\omega}{\omega_{pm}}\right)} \tag{2}$$

To summarize the components, each transfer function is the product of constant gain K, one or more "origin-poles" ($(j\omega)$ in a denominator) or "origin zeros" ($(j\omega)$ in a numerator), and one or more poles ($\left(1 + j\frac{\omega}{\omega_{p_i}}\right)$ in the denominator) or zeros ($\left(1 + j\frac{\omega}{\omega_{z_i}}\right)$ in the numerator).

Here, we define the constants ω_z as "zeros" and ω_p as "poles."" The zeros are the roots of $N(\omega)$ while poles are the roots of $D(\omega)$. [5]

3.2 Composing Bode Plots

For two transfer functions $H_1(\omega)$ and $H_2(\omega)$, if $H(\omega) = H_1(\omega) \cdot H_2(\omega)$,

$$\log|H(\omega)| = \log|H_1(\omega) \cdot H_2(\omega)| = \log|H_1(\omega)| + \log|H_2(\omega)| \tag{3}$$

$$\angle H(\omega) = \angle(H_1(\omega) \cdot H_2(\omega)) = \angle H_1(\omega) + \angle H_2(\omega) \tag{4}$$

As a consequence, when plotting $|H(\omega)|$ on a log-log plot, we can simply plot $|H_1(\omega)|$ and $|H_2(\omega)|$ and add them up. This implies that we will be able to add the slopes of each pole and zero to provide a complete plot. In the next section we provide a further analysis on the meaning of zeros and poles and the idea of adding slopes.

3.3 Decibels (Optional)

We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = |H(\omega)| \text{ [dB]}$$

The origin of the decibel comes from looking at the ratio of the output and input power of the system.

$$|H(\omega)| \text{ [dB]} = 10\log\left|\frac{P_{\text{out}}}{P_{\text{in}}}\right| = 10\log\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|^2 = 20\log\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|$$

This means that 20 dB per decade is equivalent to one order of magnitude. We won't be using dB when plotting, but understanding the conversion to dB will help when reading other resources on Bode plots.

[5]Technically if $s = j\omega$, then the roots of $N(s)$ and $D(s)$ are $-\omega_z$ and $-\omega_p$. However, when plotting Bode plots, we refer to ω_z and ω_p as the zero and pole frequencies.

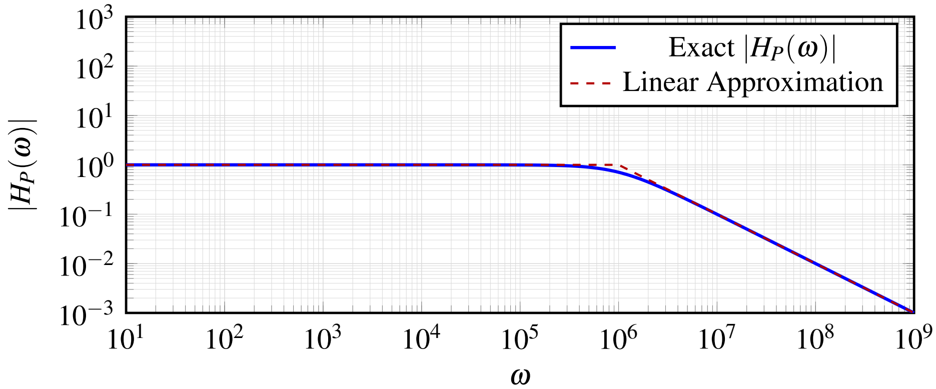
3.4 Poles, Zeros, and Constants

Single Pole, Single Zero

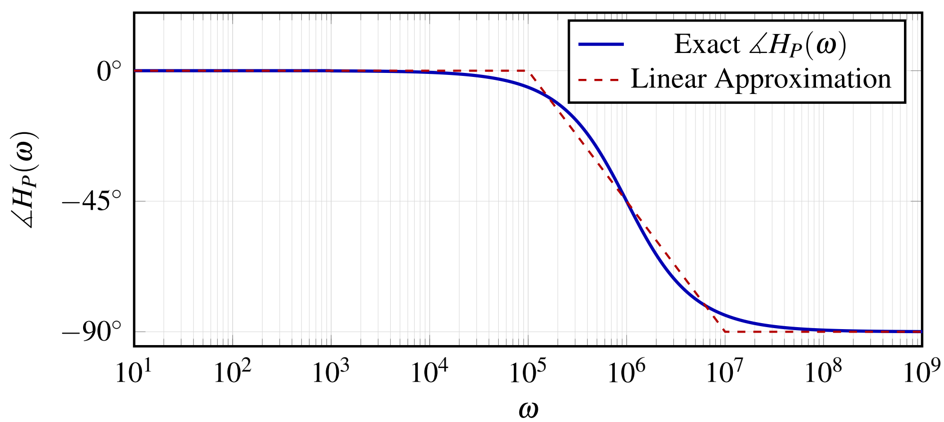
The notion of a **pole** and **zero** frequency is a generalization of the term cutoff frequency. Let's first look back at a plot of our RC low-pass filter:

$$H_P(\omega) = \frac{1}{1 + \frac{j\omega}{10^6}} \tag{5}$$

In what follows, pay special attention to the Linear Approximations! When drawing Bode plots, we claim that the plot drops off with a slope of -1 after a pole ω_p. This transfer function has a single pole at $\omega_p = 10^6$. The magnitude plot has a familiar shape (resembling a low-pass filter's transfer function!).



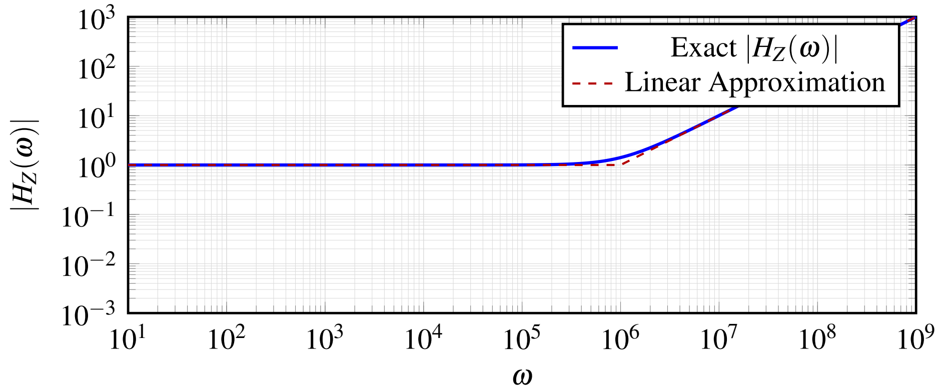
We can look at the phase plot as well:



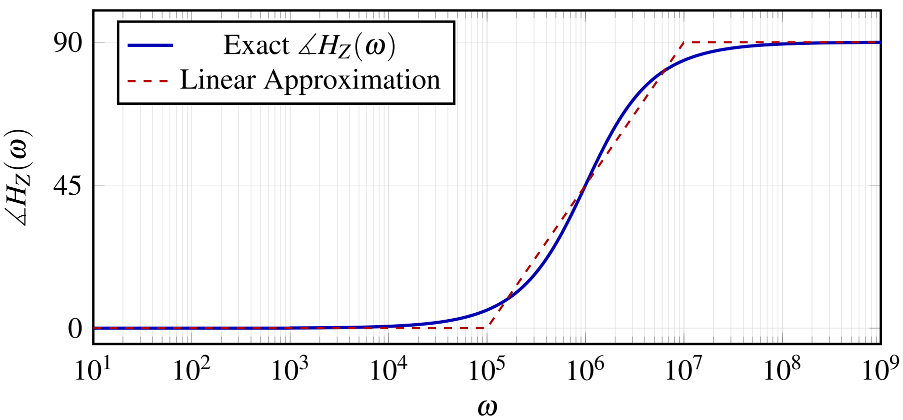
Now let's take a look at a single zero.

$$H_Z(\omega) = 1 + j\omega/\omega_z \tag{6}$$

We show that this Magnitude Bode plot rises with a slope of 1 after the zero at ω_z.

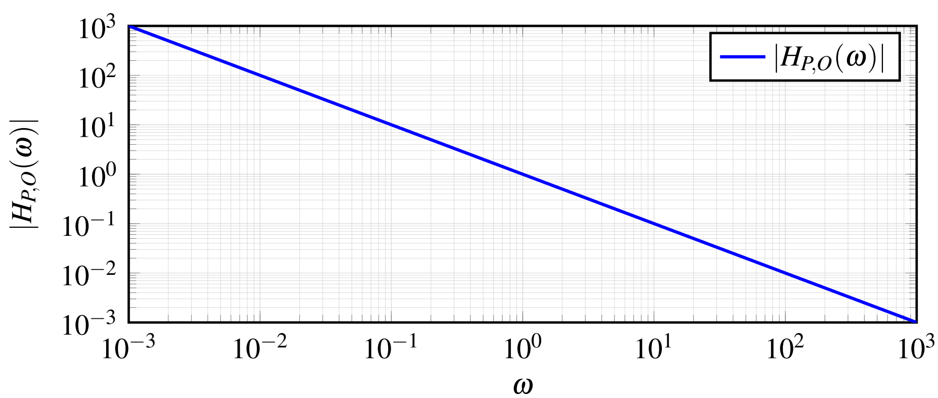


The phase plot is as follows:



Pole/Zero at the Origin

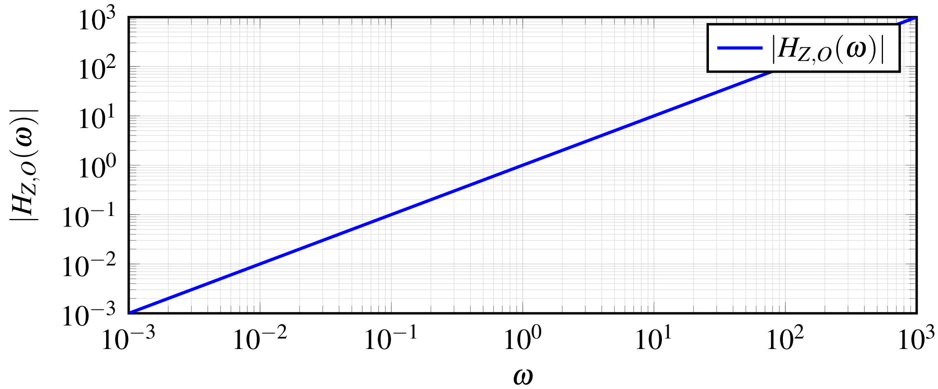
To plot a pole at the origin, recall that $H(\omega) = \frac{1}{j\omega}$ has magnitude $\frac{1}{\omega}$ and phase $-90°$.[6] If our transfer function has a pole at the origin, it will start off with a slope of -1. The phase of a pole at the origin is $-90°$ at all frequencies.



To plot a zero at the origin, recall that $H(\omega) = j\omega$ has magnitude ω and phase $90°$. If our transfer function

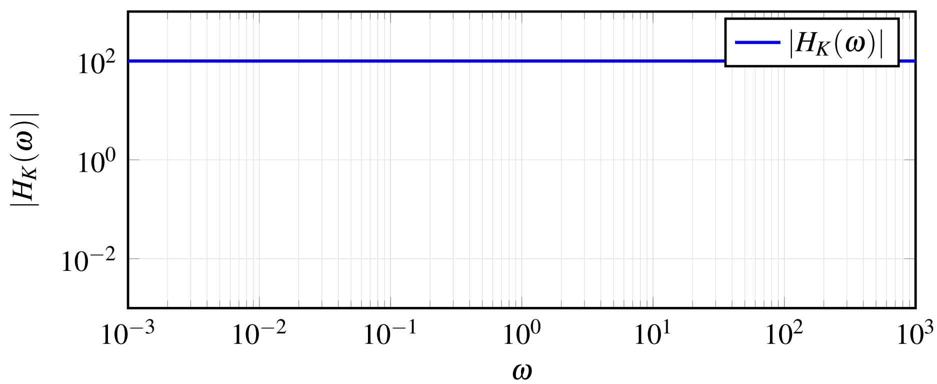
[6]For this subsection and the next, our linear approximations are actually exactly correct.

has a zero at the origin, it will start off with a slope of 1. The phase of a zero at the origin is 90° at all frequencies.



Constant Terms

Lastly, we show the plot of a constant $K = 100$. As expected, the plot remains constant. This implies that multiplication by K will shift up the entire bode plot up by K. Note that positive constants have a constant phase of 0° at all frequencies, while negative constants have a constant phase of $180^\circ \equiv -180^\circ$ at all frequencies.



3.5 Examples

We have previously seen examples of how to compute the bode plot of a bandpass filter and for an $n-$th order low-pass filter. At this point, see if you can go back and compose them yourself with the linear approximations presented above. See if you get the same results!

Transfer Function Example

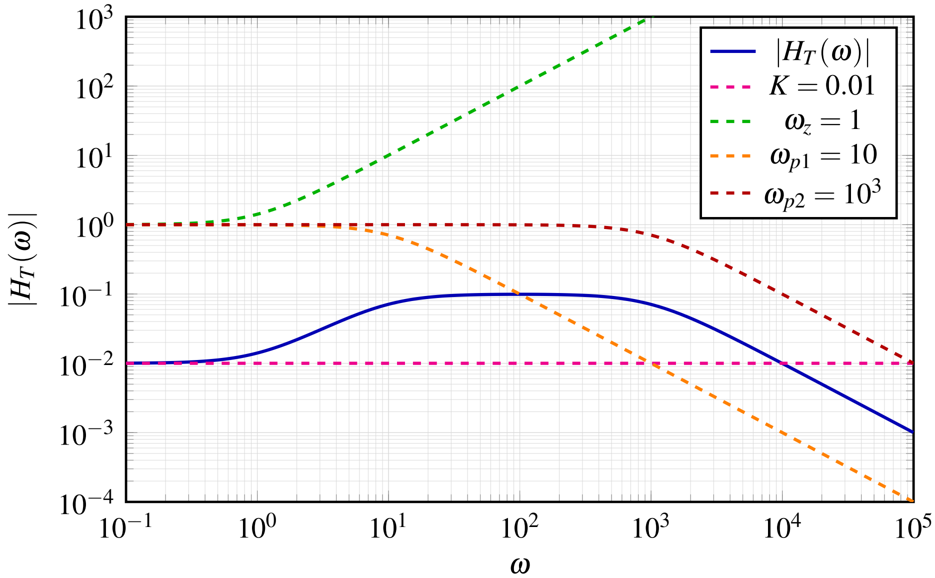
Now let's take a look at the Bode plot of a new transfer function.

$$H_T(\omega) = 100\frac{(1+j\omega)}{(j\omega)^2+1010(j\omega)+10^4}$$

Our first step is to factor this into its rational transfer function form:

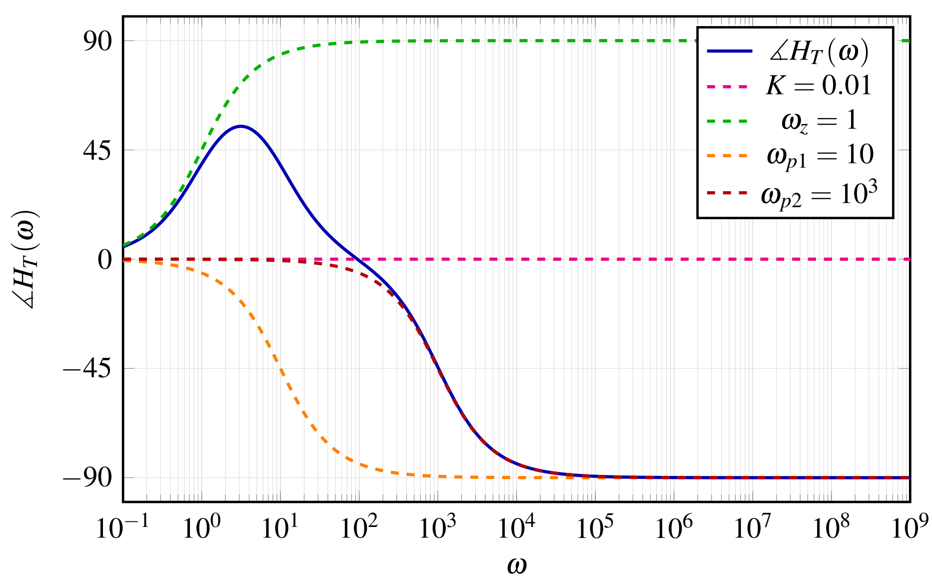
$$H_T(\omega) = 0.01 \frac{(1 + j\omega)}{(1 + j\omega/10)(1 + j\omega/10^3)}$$

With $H_T(\omega)$ in its rational form, we see that $K = 0.01, \omega_z = 1, \omega_{p1} = 10, \omega_{p2} = 10^3$. Below is a magnitude plot of each consitient component (following the building-block rules presented above), and the multiplication of all of these provides $|H_T(\omega)|$. The linear approximations are omitted to keep the plot legible, but the approximate result will very closely match the exact one.



To provide an analysis for this Bode plot, we see that the plot starts off at $K = 0.01$. Then at $\omega_z = 1$, it starts rising with slope 1. When it hits the pole at $\omega_{p1} = 10$, the slope of 1 is cancelled out by the -1 slope that the pole provides. Then the Bode plot stays constant until $\omega_{p2} = 10^3$ at which it drops off with a slope of 1. We've provided Bode plots of the individual terms to give you a sense of how we "add" Bode plots together.

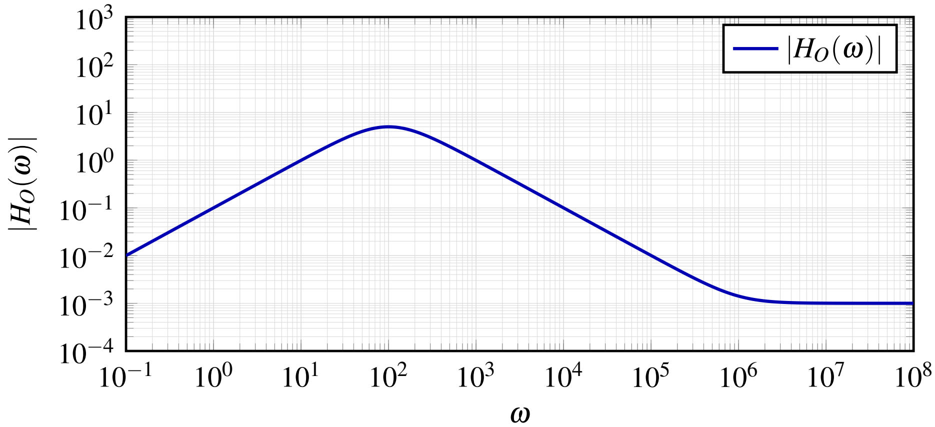
We can also plot the phase in a very similar way using our building blocks:



Zero at the Origin

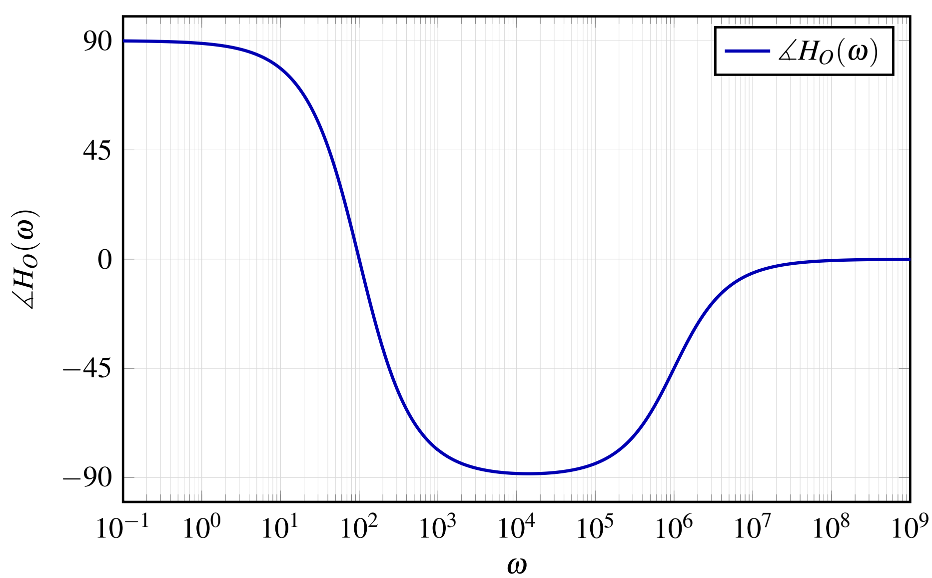
In our final example, we examine the effects of a zero at the origin. Only the final results are shown; the intermediate building blocks are left fo the reader to consider. We are given the following transfer function in rational form.

$$H_O(\omega) = 0.1\frac{(j\omega)(1+j\omega/10^6)}{(1+j\omega/10^2)^2} \tag{7}$$



Since there is a zero at the origin, the plot will initially start with a slope of 1. There are no additional zeros or poles before $\omega = 1$, so we can approximate $|H_O(1)| = K = 0.1$. Then the double pole at $\omega_p = 10^2$ provides a slope of -2 that will cancel out the slope of 1 making the overall slope after ω_p equal to -1. Lastly, there is a zero at $\omega_z = 10^6$ and we see that the addition of a slope of 1 makes $|H_O(\omega)|$ remains constant after ω_z.

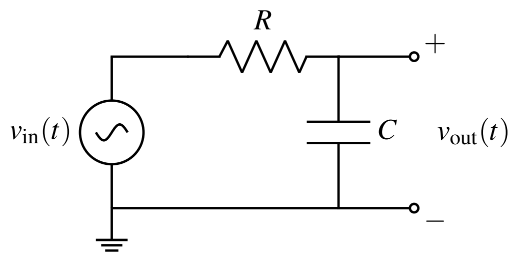
And for the phase, we have:



A Time Constant

When computing the cutoff frequency for a first order low-pass filter, we noticed that the $\omega_c = \frac{1}{RC} = \frac{1}{\tau}$. Here, we draw the connection between time constants and cutoff frequencies.

Recall from the note on differential equations that we defined the time constant of a first-order circuit to be the point at which the response $v_c(t)$ to a constant input was $1 - e^{-1}$ away from its steady state value. With this in mind, let's try plugging in an exponential input $v_{in}(t) = V_0 e^{j\omega t}$ into an RC circuit and see what happens.[7]



The differential equation for this circuit is

$$\frac{d}{dt} v_{\text{out}}(t) = \lambda (v_{\text{out}}(t) - V_0 e^{j\omega t}) \tag{8}$$

for $\lambda = -\frac{1}{\tau}$. In Note 3 we showed that the steady state value of this differential equation is

$$v_{ss}(t) = \frac{-\lambda}{j\omega - \lambda} V_0 e^{j\omega t} \tag{9}$$

Therefore, plugging in for $\lambda = -\frac{1}{\tau}$, it follows that

$$v_{ss}(t) = \frac{1}{1 + j\omega\tau} V_0 e^{j\omega t} \tag{10}$$

Notice that $H(\omega) = \frac{1}{1+j\omega\tau}$ and the cutoff arises naturally as $\omega_c = \frac{1}{\tau}$. We can also realize that at steady state, $H(\omega)$ is in fact the eigenvalue for the differential equation with eigenfunction $e^{j\omega t}$. This is a crucial connection between differential equations and the frequency response of a linear system that you will see in later half of the course and in courses like EE120.

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[7]We should be inputting $v_{in}(t) = V_0 \cos(\omega t)$ but we choose $e^{j\omega t}$ since it provides the same result while simplifying the math.