EECS 16B Designing Information Devices and Systems II
Fall 2021

Note 10: Stability

1 Overview

Given a discrete-time system, we know how to analyze its behavior if it is given inputs. We will now discuss the problem of *disturbance*, which is unpredictable and can change the properties of our system. We will study here how it affects our ability to control a system.

In general, our goal in control is going to be to get the system to do what we want. This typically involves two distinct conceptual stages: first, coming up with a plan that tells us what inputs we would like to apply in principle to achieve our objective; and second, executing our plan dynamically in the face of real-time disturbances and adapting accordingly. In the real world, plans never succeed if carried out blindly.

It turns out that mathematically, it is easier to first address the issue of dealing with real-time disturbances, and tackle the issue of coming up with plans later.

2 Stability

Stability is the most basic requirement for a system to be able to carry out our wishes successfully. It captures the sense that disturbances do not cause our system behavior to change too much. In order to capture this mathematically, we will proceed in stages. First, we'll clarify what we really mean by disturbance. Then, we'll define a simpler notion of stability. Finally, we'll extend it to what we really want.

Disturbances:

Recall that we typically model the evolution of a discrete-time linear system as follows:

$$\vec{x}_d[i+1] = A\vec{x}_d[i] + B\vec{u}_d[i]. \tag{1}$$

In reality, though, these models never capture the full behavior of a system - errors, due to nonlinearities in the system, factors not taken into account when building the model, or just random noise, will always be present. We model the total error in our model as an all-encompassing disturbance term, $\vec{w}$. We then obtain the full state equation:

$$\vec{x}_d[i+1] = A\vec{x}_d[i] + B\vec{u}_d[i] + \vec{w}[i]. \tag{2}$$

where $\vec{w}$ is unobservable and varies arbitrarily at each time step.

At this point, a natural question to ask would be: if $\vec{w}$ is unknown and varies arbitrarily, how can we possibly do anything with our system? We can apply control inputs and observe the system extremely carefully, only for everything to be thrown off by an arbitrarily large $\vec{w}$ term! To address this, we will introduce the notion of *bounded disturbance*.

Specifically, a sequence $\vec{z}$ is *bounded* if $\left\|\vec{z}[i]\right\| \leq K$ for each t and some nonzero constant K *which does not depend on* t.

For the purposes of our analysis, we assume that that the noise sequence $\vec{w}$ is bounded by ϵ – that is,

$\left\|\vec{w}[i]\right\| \leq \epsilon$ for some small nonzero constant ϵ and each t.

State stability:

For the sake of clarity, let us simplify drastically, and consider the hypothetical discrete-time system:

$$\vec{x}_d[i+1] = A\vec{x}_d[i]. \tag{3}$$

We say this system is stable in the sense of "state-stability" if, for every initial condition $\vec{x}_d[0]$ and every time t, we have that $\vec{x}_d$ is bounded, i.e., $\left\|\vec{x}_d[i]\right\| \leq K$ for all t and some constant K (which is independent of t but crucially may depend on $\vec{x}_d[0]$).

Note the qualitative behavior of state-stable systems versus non-state-stable systems. If a system is state-stable then all the states traversed by the system are bounded in magnitude, meaning that they don't stray too far from the origin. If a system is not state-stable, then (depending on the initial condition $\vec{x}_d[0]$) there are states that go arbitrarily far from the origin, and can exhibit all sorts of pathological behaviors. In short, the state trajectory of the system (i.e., the sequence of states taken by the system) could be arbitrarily complex.

Here, the system can be expressed in the form

$$\vec{x}_d[i+1] = A\vec{x}_d[i] = A^2\vec{x}_d[i-1] = \cdots = A^i\vec{x}_d[0]. \tag{4}$$

Thus, the system evolution is solely function of our initial condition $\vec{x}_d[0]$. There is only one place, then, where we may incorporate disturbances in this model – that is, in the initial condition itself. Since the state-stable condition says that $\vec{x}_d$ is bounded for every initial condition $\vec{x}_d[0]$, we know that a state-stable system is robust to any perturbation in the initial condition.

Now, let us generalize our model a little bit, and instead consider disturbances added at each time-step. That is, consider the new system

$$\vec{x}_d[i+1] = A\vec{x}_d[i] + \vec{w}[i]. \tag{5}$$

The definition of state-stability remains the same; i.e., the system is state-stable if $\vec{x}_d$ is bounded for every initial condition $\vec{x}_d[0]$.

Here, the disturbances are incorporated not-only in the initial condition $\vec{x}_d[0]$, but at every time-step through $\vec{w}[i]$. However, the qualitative behavior remains roughly the same; state-stable systems are more well-behaved than non-state-stable systems.

Bounded Input Bounded Output (BIBO) stability:

For this system, the above definition of the "state-stability" seems a bit inadequate. In particular, let us say our system's noise becomes unboundedly large (for every $K > 0$ there is a t such that $\left\|\vec{w}[i]\right\| > K$). In this case, it is not possible for the state to stay bounded, but for nothing to do with the "system" itself, per-se. The state is blowing up only because we had an unbounded disturbance. This motivates the idea of BIBO stability, which requires the external inputs to the system (including disturbances) to be bounded.

The system in eq. (5) is Bounded Input Bounded Output (BIBO) stable if for every disturbance $\vec{w}$ which is bounded (say by ϵ), and initial condition $\vec{x}_d[0]$, the state $\vec{x}_d$ remains bounded (say by K).

Formally, we say that the system in eq. (5) is BIBO stable if for every $\epsilon > 0$ there exists a K such that if $\left\|\vec{x}_d[0]\right\| < \epsilon$ and $\left\|\vec{w}[i]\right\| < \epsilon$ for each i, then $\left\|\vec{x}_d[i]\right\| < K$ for each t.

We are now ready to discuss systems with inputs and disturbances.

$$\vec{x}_d[i+1] = A\vec{x}_d[i] + B\vec{u}_d[i] + \vec{w}[i]. \tag{6}$$

The neat part of how we built up our definition is that the definition of BIBO stability for eq. (6) is almost exactly the same as for eq. (5).

In particular, the system in eq. (6) is Bounded Input Bounded Output (BIBO) stable if for every input $\vec{u}_d$ which is bounded (say by ϵ), and disturbance $\vec{w}$ which is bounded (say by ϵ), and choice of initial condition $\vec{x}_d[0]$, the state $\vec{x}_d$ remains bounded (say by K).

Formally, we say that the system in eq. (6) is BIBO stable if for every $\epsilon > 0$ there exists a K such that if $\left\|\vec{x}_d[0]\right\| < \epsilon$ and $\left\|\vec{u}_d[i]\right\| < \epsilon$ and $\left\|\vec{w}[i]\right\| < \epsilon$ for each t, then $\left\|\vec{x}_d[i]\right\| < K$ for each i.

Note that in this definition, we are treating $\vec{u}_d$ and $\vec{w}$ like inputs – in particular, the "Input" in "Bounded Input Bounded Output", where to say anything about the system, *both* $\vec{u}_d$ and $\vec{w}$ have to be bounded. Thus, this definition now removes the problem of unbounded inputs and/or disturbances that we faced with the earlier definition.

From now on, when we say a system is "stable" we mean that it is BIBO stable.

3 Scalar Stability

First, we will consider the case of system stability in one dimension ($\vec{x}_d$ is a scalar quantity x). Thus, our system's behavior can be modelled as

$$x_d[i+1] = \lambda x_d[i] + w[i]. \tag{7}$$

The use of λ may remind you of eigenvalues - we'll see why that's the case shortly. Let's say that our system is initially at rest: $x_d[0] = 0$. Therefore, we have:

$$\begin{aligned}
& x_d[0] = 0 && (8)\\
\implies\ & x_d[1] = w[0] && (9)\\
\implies\ & x_d[2] = \lambda w[0] + w[1] && (10)\\
\implies\ & x_d[3] = \lambda^2 w[0] + \lambda w[1] + w[2] && (11)\\
& \vdots && (12)\\
\implies\ & x_d[i] = \lambda^{i-1} w[0] + \lambda^{i-2} w[1] + \ldots + w[i-1]. && (13)
\end{aligned}$$

For now, let's assume that $\lambda \geq 0$. Recall that we have no control over the noise. For our system to be stable, $x_d[i]$ should be bounded no matter how w is chosen. In this scenario, we can see that $x_d[i]$ is maximized when we maximize w. Our assumption is that w has an upper bound of ϵ, so we should set $w = \epsilon$ at every timestep i. Therefore, we can write:

$$x_d[i] = \epsilon\left(1 + \lambda + \lambda^2 + \cdots + \lambda^{i-1}\right). \tag{14}$$

Since the system should remain bounded after arbitrarily many timesteps, we can take the limit $i \to \infty$ to obtain

$$\lim_{i \to \infty} x_d[i] = x(\infty) = \epsilon\left(1 + \lambda + \lambda^2 + \ldots\right). \tag{15}$$

The sum of this infinite geometric series[1] converges if and only if $|\lambda| < 1$. In other words, a scalar system where $\lambda \geq 0$ is stable exactly when $|\lambda| < 1$.

[1] Recall that $\sum_{k=0}^{\infty} a \cdot r^k = \frac{a}{1-r}$.

Unfortunately, the above analysis becomes somewhat more tricky when $\lambda < 0$, and breaks down entirely when λ is complex! However, it is still useful in that it has provided us with a conjecture – namely that a scalar discrete-time system is stable exactly when $|\lambda| < 1$. To prove this conjecture, we need to: 1) show that our system is stable when $|\lambda| < 1$ and 2) show that it is unstable when $|\lambda| \geq 1$.

Let's prove the former claim first. We can write that:

$$x_d[\ell] = \sum_{i=0}^{\ell-1} \lambda^{\ell-i-1} w[i] \tag{16}$$

even in the complex case. We can upper-bound $\left|x_d[i]\right|$ to be

$$\begin{aligned}
\left|x_d[\ell]\right| &= \left|\sum_{i=0}^{\ell-1} \lambda^{\ell-i-1} w[i]\right| && (17)\\
&\leq \sum_{i=0}^{\ell-1} \left|\lambda^{\ell-i-1} w[i]\right| && (18)\\
&= \sum_{i=0}^{\ell-1} \left|\lambda^{\ell-i-1}\right| \left|w[i]\right| && (19)\\
&\leq \epsilon \sum_{i=0}^{\ell-1} |\lambda|^{\ell-i-1}, && (20)
\end{aligned}$$

since $\left|\lambda^{\ell-i-1}\right| \geq 0$ and $\left|w[i]\right| \leq \epsilon$ by the definition of bounded noise. Now, since $|\lambda| < 1$, we can apply the geometric series formula, and we can take an upper-bound by summing to infinity, rather than stopping at $\ell - 1$. We then obtain the final bound:

$$\left|x_d[\ell]\right| \leq \epsilon \left(1 + |\lambda| + |\lambda|^2 + \cdots\right) = \epsilon \frac{1}{1 - |\lambda|} \tag{21}$$

This is a bounded quantity even as $\ell \to \infty$. Consequently, we have shown that our system is BIBO stable when $|\lambda| < 1$ (even when λ is a complex number.)

What if $|\lambda| \geq 1$? We'd like to show that our system is unstable, and to do so, we can construct any kind of error sequence we want that is *adversarial*; that is, it is the worst-case scenario for our particular system. More specifically, this adversarial error is a particular sequence of bounded $w[i]$ that takes $\left|x_d[\ell]\right|$ to infinity over time (indicating that the state is unbounded). Looking at the special case of real, positive λ for inspiration, we might conjecture that:

$$w[i] = \epsilon \tag{22}$$

is one such sequence. It makes sense that if the error is bounded by a constant, and we want to consider the worst-possible case, we just make the error as large as it can be at every timestep! This candidate sequence of $w[i]$ is indeed bounded (by ϵ).

For this candidate sequence, we see that (when $\lambda \neq 1$):[2]

$$x_d[\ell] = \sum_{i=0}^{\ell-1} \lambda^{\ell-i-1} w[i] \tag{23}$$

$$= \epsilon(\lambda^0 + \lambda^1 + \cdots + \lambda^{\ell-1}) \tag{24}$$

$$= \epsilon \frac{\lambda^\ell - 1}{\lambda - 1}, \tag{25}$$

so:

$$\left|x_d[\ell]\right| = \epsilon \frac{\left|\lambda^\ell - 1\right|}{|\lambda - 1|}. \tag{26}$$

As $\left|\lambda^\ell\right| = |\lambda|^\ell$ goes to infinity when $|\lambda| > 1$, we have shown that our system is unstable when $|\lambda| > 1$.

Unfortunately, in the above derivation, we had to exclude the case when $|\lambda| = 1$. For this remaining case, we need to generate a new candidate sequence of noise. Some experimentation yields the candidate sequence:

$$w[i] = \epsilon \lambda^i. \tag{27}$$

Note that since $|\lambda| = 1$, $\left|w[i]\right| = \epsilon 1^i = \epsilon$, so the noise is still bounded. Now, plugging into our formula for $x_d[\ell]$, we find that

$$x_d[\ell] = \epsilon \sum_{i=0}^{\ell-1} \lambda^{\ell-i-1} \lambda^i \tag{28}$$

$$= \epsilon \sum_{i=0}^{\ell-1} \lambda^{\ell-1} \tag{29}$$

$$= \epsilon \ell \lambda^{\ell-1}, \tag{30}$$

so:

$$\left|x_d[\ell]\right| = \ell|\epsilon|, \tag{31}$$

which goes to infinity over time. Thus, when $|\lambda| = 1$, we can supply bounded noise that still drives our system to infinity, so the system is unbounded.

Putting everything together, we have shown that whenever the parameter λ of our scalar discrete-time linear system has magnitude less than 1, our system is stablem, but that in all other cases, it is unstable.

4 Vector Stability

Now, we will look at the discrete-time vector case, with the state equation:

$$\vec{x}_d[i+1] = A\vec{x}_d[i] + B\vec{u}_d[i] + \vec{w}[i]. \tag{32}$$

We will assume that we start at the target state $\vec{x}_d[0] = \vec{0}$, and that we supply no inputs thereafter, so all $\vec{u}_d[i] = \vec{0}$. Note that this means that B cannot affect the stability of our system, so we will disregard it.

Now, recall that when we considered continuous-time systems with coupled differential equations, we were

[2]The sum of a finite geometric series $\sum_{k=0}^{n} a \cdot r^k = a\frac{1-r^{n+1}}{1-r}$.

able to separate them into a set of decoupled differential equations by diagonalizing their state matrix.[3] We can do something similar here, by letting $A = V\Lambda V^{-1}$. Rearranging, we obtain

$$\vec{x}_d[i+1] = V\Lambda V^{-1}\vec{x}_d[i] + \vec{w}[i] \tag{33}$$

$$\implies V^{-1}\vec{x}_d[i+1] = \Lambda V^{-1}\vec{x}_d[i] + V^{-1}\vec{w}[i] \tag{34}$$

$$\implies \vec{\breve{x}}_d[i+1] = \Lambda\vec{\breve{x}}_d[i] + \vec{\breve{w}}[i], \tag{35}$$

where $\vec{\breve{x}}_d = V^{-1}\vec{x}_d$ and $\vec{\breve{w}} = V^{-1}\vec{w}$. If a single component of $\vec{\breve{x}}$ is unbounded, then the system as a whole is unstable. So all we have to do is consider each of the components of $\vec{\breve{x}}$, each of which gives us the scalar state equation

$$\breve{x}_j[i+1] = \lambda_j\breve{x}_j[i] + \breve{w}_j. \tag{36}$$

It would seem that we can just apply the result from the previous section, to state that this component is stable exactly when $|\lambda_j| < 1$.

But recall, that result only applied when the input noise was bounded (that is, $|\breve{w}_j| < \breve{\epsilon}$ for some finite constant $\breve{\epsilon}$). We were given that our original noise $\vec{w}$ was bounded, so there existed some ϵ such that each component $\vec{w}_j < \epsilon$. But after applying V^{-1}, can we be certain that the transformed noise is still bounded? We might suspect the answer to be yes, but for the sake of completeness, we will derive a proof here. Let

$$V^{-1} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}, \tag{37}$$

and let m be the entry v_{ij} within V^{-1} with the largest magnitude. Thus,

$$V^{-1}\vec{w} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} v_{11}w_1 + v_{12}w_2 + \ldots + v_{1n}w_n \\ v_{21}w_1 + v_{22}w_2 + \ldots + v_{2n}w_n \\ \vdots \\ v_{n1}w_1 + v_{n2}w_2 + \ldots + v_{nn}w_n \end{bmatrix}. \tag{39}$$

Any given row r of $\vec{\breve{w}} = V^{-1}\vec{w}$ is bounded by:

$$v_{r1}w_1 + v_{r2}w_2 + \ldots + v_{rn}w_n \leq |m|\,(w_1 + w_2 + \ldots + w_n). \tag{40}$$

Since $|m|$ is a constant that does not depend on $\vec{w}$, and $w_1 + w_2 + \ldots + w_n$ is bounded (since $\|\vec{w}\|$ is bounded), each element of $\breve{w}$ is also bounded by some finite constant. Therefore, we can indeed apply our original result in the eigenbasis. Consequently, our system as a whole is stable exactly when $|\lambda_j| < 1$ for all the eigenvalues λ_j of A.

[3] We will discuss the case when A is not diagonalizable later in the course.

5 Continuous-Time Systems

At this point, we will take a brief aside from our discussion of discrete-time systems to consider the stability of continuous time systems (t is any valid time, not necessarily an integer/timestep). Consider a continuous-time scalar system whose behavior is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x_c(t) = \lambda x_c(t) + u(t) + w(t), \tag{41}$$

where λ and $w(t)$ are possibly complex.

As before, we are interested to see, given bounded noise $\left|w(t)\right| \leq \epsilon$ and input $u(t) = 0$, whether $\left|x_c(t)\right|$ remains bounded over time. Initially, let's assume that $\lambda \neq 0$. Then, from our knowledge of differential equations, we know that we can solve for the state:

$$x_c(t) = x_c(0)\mathrm{e}^{\lambda t} + \int_0^t \mathrm{e}^{\lambda(t-\theta)} w(\theta)\,\mathrm{d}\theta\,. \tag{42}$$

Let's consider the leading term $x_c(0)\mathrm{e}^{\lambda t}$ first. We know that $\mathrm{e}^{\lambda t}$ behaves very differently for real and imaginary choices of λ. Thus, to study it, it makes sense to break λ up into real and imaginary components, as follows:

$$\lambda = \lambda_r + \mathrm{j}\lambda_j. \tag{43}$$

Now, by Euler's identity, we can express

$$\mathrm{e}^{\lambda t} = \mathrm{e}^{\lambda_r t + \mathrm{j}\lambda_j t} = \mathrm{e}^{\lambda_r t}\left(\cos\bigl(\lambda_j t\bigr) + j\sin\bigl(\lambda_j t\bigr)\right). \tag{44}$$

Since the magnitude of $\cos(x) + j\sin(x)$ is bounded by 1 (it lies within the unit circle), we have that:

$$\left|\mathrm{e}^{\lambda t}\right| = \mathrm{e}^{\lambda_r t}, \tag{45}$$

so it becomes unbounded as $t \to \infty$ if $\lambda_r > 0 \iff \mathrm{Re}\{\lambda\} > 0$.

Going back to our expression for $x_c(t)$, we can now notice that the leading term $x_c(0)\mathrm{e}^{\lambda t}$ will go to infinity when $\mathrm{Re}\{\lambda\} > 0$, even if the noise $w = 0$ throughout, so long as $x_c(0) \neq 0$. What if $x_c(0) = 0$? Then we can adversarially apply some initial noise in order to perturb our state from 0, then set the noise to 0 and let the state blow up. So, our system is unstable.

Thus, a necessary condition for stability is $\mathrm{Re}\{\lambda\} \leq 0$. Is this condition sufficient? To find out, we now need to look at the effects of input on our state over time by considering the second term in our expression for $x_c(t)$. Observe that, when $\mathrm{Re}\{\lambda\} < 0$, we can present the upper bound:

$$\left|x_c(t)\right| = \left|x_c(0)\mathrm{e}^{\lambda t} + \int_0^t \mathrm{e}^{\lambda(t-\theta)} w(\theta)\,\mathrm{d}\theta\right| \tag{46}$$

$$\leq \left|x_c(0)\mathrm{e}^{\lambda t}\right| + \left|\int_0^t \mathrm{e}^{\lambda(t-\theta)} w(\theta)\,\mathrm{d}\theta\right| \tag{47}$$

We've just seen how the first term in the above sum is bounded when $\mathrm{Re}\{\lambda\} < 0$. Focusing on the second term, we can rearrange it as:

$$\left|\int_0^t \mathrm{e}^{\lambda(t-\theta)} w(\theta)\,\mathrm{d}\theta\right| \leq \int_0^t \left|\mathrm{e}^{\lambda(t-\theta)} w(\theta)\right| \mathrm{d}\theta \tag{48}$$

$$= \int_0^t \left|\mathrm{e}^{\lambda(t-\theta)}\right| \left|w(\theta)\right| \mathrm{d}\theta \tag{49}$$

$$= \int_0^t \mathrm{e}^{\lambda_r(t-\theta)} \left|w(\theta)\right| \mathrm{d}\theta\,. \tag{50}$$

Now, since $\mathrm{e}^{\lambda_r(t-\theta)}$ is always a positive real number, we can maximize the above expression by setting $\left|w(\theta)\right|$ to its maximum value ϵ. We then obtain the bound:

$$\left|\int_0^t \mathrm{e}^{\lambda(t-\theta)} w(\theta)\,\mathrm{d}\theta\right| \leq \epsilon \int_0^t \mathrm{e}^{\lambda_r(t-\theta)}\,\mathrm{d}\theta \tag{51}$$

$$= \epsilon \mathrm{e}^{\lambda_r t} \int_0^t \mathrm{e}^{-\lambda_r \theta}\,\mathrm{d}\theta \tag{52}$$

$$= \epsilon \mathrm{e}^{\lambda_r t} \left[-\frac{1}{\lambda_r} \mathrm{e}^{-\lambda_r \theta}\right]_{\theta=0}^{\theta=t} \tag{53}$$

$$= -\frac{\epsilon}{\lambda_r} \mathrm{e}^{\lambda_r t} \left[\mathrm{e}^{-\lambda_r \theta}\right]_{\theta=0}^{\theta=t} \tag{54}$$

$$= \frac{\epsilon}{\lambda_r} \mathrm{e}^{\lambda_r t} \left[\mathrm{e}^{-\lambda_r \theta}\right]_{\theta=t}^{\theta=0} \tag{55}$$

$$= \frac{\epsilon \mathrm{e}^{\lambda_r t}}{\lambda_r} \left(1 - \mathrm{e}^{-\lambda_r t}\right) \tag{56}$$

$$= \frac{\epsilon}{\lambda_r} \left(\mathrm{e}^{\lambda_r t} - 1\right). \tag{57}$$

Since $\lambda_r < 0$, $\mathrm{e}^{\lambda_r t}$ is bounded for all t, and so is the above expression. Therefore, we have shown that both terms in the closed-form expression of $x_c(t)$ are bounded when $\mathrm{Re}\{\lambda\} < 0$.

Looking back at what we have shown, we know that our system is stable when $\mathrm{Re}\{\lambda\} < 0$, and unstable when $\mathrm{Re}\{\lambda\} > 0$. The only remaining case is when $\mathrm{Re}\{\lambda\} = 0$, where $\lambda = \mathrm{j}\lambda_j$. In this case, consider the bounded noise $w(t) = \epsilon \mathrm{e}^{\mathrm{j}\lambda_j t}$. Observe that, even when we start with $x_c(0) = 0$, our state becomes

$$x_c(t) = \int_0^t \mathrm{e}^{\lambda(t-\theta)} w(\theta)\,\mathrm{d}\theta \tag{58}$$

$$= \int_0^t \mathrm{e}^{\lambda(t-\theta)} \epsilon \mathrm{e}^{\mathrm{j}\lambda_j \theta}\,\mathrm{d}\theta \tag{59}$$

$$= \epsilon \mathrm{e}^{\mathrm{j}\lambda_j t} \int_0^t \mathrm{e}^{-\lambda_j \theta + \lambda_j \theta}\,\mathrm{d}\theta \tag{60}$$

$$= \epsilon \mathrm{e}^{\mathrm{j}\lambda_j t} \int_0^t 1\,\mathrm{d}\theta \tag{61}$$

$$= \epsilon t \mathrm{e}^{\mathrm{j}\lambda_j t}. \tag{62}$$

Thus, $\left|x_c(t)\right| = \epsilon t$, so it goes to infinity over time despite the noise being bounded by $\left|w(t)\right| = \epsilon$ throughout. Consequently, when $\mathrm{Re}\{\lambda\} = 0$, our system is still unstable.

Putting everything together, we ultimately see that continuous-time scalar systems are stable exactly when $\mathrm{Re}\{\lambda\} < 0$.

Now, what about continuous-time vector systems? By performing diagonalization in an analogous manner to what we did earlier for discrete-time systems, it can be seen that (diagonalizable) vector systems are stable exactly when $\mathrm{Re}\{\lambda_i\} < 0$ for all the eigenvalues λ_i of the state matrix A. Thus, we now have derived a necessary and sufficient condition for the stability of diagonalizable continuous-time systems.

6 Closed-Loop Control

Now, we know (at least to some extent) when a system is stable in the absence of input. That is, we can determine when applying bounded perturbations over time to a system can never lead to unbounded deviations from the stable state. However, many real-world systems are not stable in this manner. They instead rely on continuous monitoring and intervention to keep them near a desired state. We will now begin to explore how we may choose inputs that achieve our goal of stability, returning to discrete-time systems.

In particular, imagine choosing inputs that linearly depend on the state vector. Specifically, let

$$\vec{u}_d[i] = K\vec{x}_d[i] \tag{63}$$

where K is a matrix that we can adjust. Then, our state equation becomes

$$\begin{align} \vec{x}_d[i+1] &= A\vec{x}_d[i] + B\vec{u}_d[i] + \vec{w}[i] \tag{64}\\ &= A\vec{x}_d[i] + BK\vec{x}_d[i] + \vec{w}[i] \tag{65}\\ &= (A+BK)\vec{x}_d[i] + \vec{w}[i]. \tag{66}\end{align}$$

In essence, it is as if we have replaced our original state matrix A with the new matrix $A+BK$, but continue to apply no other input. Ideally, we'd be able to choose a K that would move the eigenvalues of the state transition matrix $A+BK$ into the regime of stability. We will derive the general condition for the existence of such a K in future lectures. Right now, however, we will examine a few special cases. Consider the scalar case, with the system

$$x_d[i+1] = ax_d[i] + bu_d[i] + w[i] \tag{67}$$

If $|a| \geq 1$ (for instance, if $a = 3$), we know that this system will be unstable in the absence of input. But now, imagine that we choose $u(t) = kx(t)$ for a constant k to be determined. Substituting into the state equation, we obtain:

$$x_d[i+1] = (a+bk)x_d[i] + w[i]. \tag{68}$$

Thus, when $b \neq 0$, we can choose k to set our system's eigenvalues to whatever we want! In particular, by setting $k = -a/b$, we obtain

$$x_d[i+1] = w(t) \tag{69}$$

minimizing the system's eigenvalues and ensuring that noise cannot accumulate geometrically. Now, we will look at a 2D case, with the state equation

$$\vec{x}_d[i+1] = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \vec{x}_d[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \vec{u}_d[i] + \vec{w}[i]. \tag{70}$$

Notice that with no input, the eigenvalues of our state matrix are 3 and -1, so this system is definitely unstable in the absence of input. Notice also that the input only affects the second state directly, so we have no direct way of manipulating the first state! This is in contrast to our scalar example, where our input could affect the entire (one-dimensional) state. With that in mind, let our unknown K be

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \tag{71}$$

so we obtain the following state equation:

$$\vec{x}_d[i+1] = \left(\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \vec{x}_d[i] + \vec{w}[i]. \tag{72}$$

We are interested in minimizing the magnitudes of both eigenvectors of our state transition matrix, which may be re-expressed as

$$A + BK = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \tag{73}$$

$$= \begin{bmatrix} 0 & 1 \\ 3 + k_1 & 2 + k_2 \end{bmatrix} \tag{74}$$

Computing the eigenvalues λ, we see that they must satisfy the characteristic polynomial:

$$(-\lambda)(2 + k_2 - \lambda) - (3 + k_1) = 0 \tag{75}$$

$$\lambda^2 - (k_2 + 2)\lambda - (k_1 + 3) = 0 \tag{76}$$

Notice that we have full control over both coefficients of the characteristic polynomial, even though we can't fully control the state matrix. Therefore, we can place the state matrix's eigenvalues wherever we like, so we can still choose a feedback control law to make our system stable! In the future, we will explore the condition that determines whether a K that stabilizes a system exists.

Contributors:

- Neelesh Ramachandran.
- Gireeja Ranade.
- Rahul Arya.
- Anant Sahai.
- Jaijeet Roychowdhury.
- Druv Pai.

