Today:
- Continue Stability (BIBO)
  - Discrete Time
  - Continuous Time
- Feedback Control / Stabilization /
  Eigenvalue placement
- Controllability

Discrete Time System

\[ x(t+1) = A x(t) + u(t) + w(t) \]

BIBO stable: System is BIBO stable if and only if when \( |u(t)| < 3 \) and \( |w(t)| < 3 \) for all \( t \), then there exists a \( k \) such that all \( |x(t)| < k \) for all \( t \).

For all : \( \forall \)
exists : \( \exists \)
Cases:

- $|\lambda| > 1$: unstable
- $|\lambda| < 1$: stable
- $|\lambda| = 1$: ???

\[ u(t) \leq \varepsilon \]

\[ \text{Case: } |\lambda| = 1 \]

\[ \lambda = e^{j\theta} \]

\[ x(t) = \lambda^t \cdot x[0] + \sum_{k=0}^{t-1} \lambda^k u[t-1-k] \]

\[ = e^{j\theta t} x[0] + \sum_{k=0}^{t-1} u[t-1-k] \cdot (e^{j\theta})^k \]

If $\lambda = 1$

\[ x(t) = e^{j\theta t} x[0] + \sum_{k=0}^{t-1} u[t-1-k] \]

Choose $u(t) \equiv \varepsilon \neq t$

Blows up.

Unstable!
Choose:

\[ x(t) = \sum e^{j\omega t} x[0] + \sum_{k=0}^{t-1} e^{-j\omega k} \cdot e^{j\omega k} \]

\[ u(t-i-k) = e^{-j\omega k} \]

\[ m = t-i-k \quad \Rightarrow \quad k = t-i-m \]

\[ u(m) = e^{-j\omega k} = e^{-j\omega (t-i-m)} \]

\[ x(t) = e^{j\omega t} x[0] + \sum_{k=0}^{t-1} e^{-j\omega k} \cdot e^{j\omega k} \]

\[ x(t+1) = e^{j\omega t} x[0] + \epsilon x[0] \]

\[ \Rightarrow \text{Unstable} !!! \]

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**Discrete-Time Systems:**

\[ |\lambda| > 1 \quad \text{Unstable} \]

\[ |\lambda| < 1 \quad \text{stable} \]

\[ |\lambda| = 1 \quad \text{marginally stable/unstable} \]
Vector system:

\[ \mathbf{x}[t+1] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}[t] + \mathbf{u}[t] + \mathbf{w}[t] \]

\[ \implies \text{BIBO unstable} \]

BIBO stability: All \( |\text{eigenvalues}| < 1 \).

\[ \mathbf{x}[t+1] = A \mathbf{x}[t] + \mathbf{u}[t] + \mathbf{w}[t], \]

\[ A = V \Lambda V^{-1}, \quad \Lambda = \text{diagonal matrix}. \]

Eigenvalues of \( A \) are eigenvalues of \( \Lambda \) and they are along the diagonal!

\[ \mathbf{x}[t+1] = V \Lambda V^{-1} \mathbf{x}[t] + \mathbf{u}[t] + \mathbf{w}[t] \]

\[ V^{-1} \mathbf{x}[t+1] = \Lambda V^{-1} \mathbf{x}[t] + V^{-1} \mathbf{u}[t] + V^{-1} \mathbf{w}[t] \]

\[ V^{-1} \mathbf{x}[t+1] = \mathbf{x}[t] \text{ and so on.} \]
\[ \mathbf{x}[t+1] = \Lambda \mathbf{x}[t] + \mathbf{u}[t] + \mathbf{w}[t] \]

(\Lambda diagonal!)

\[ \rightarrow \text{Find } \text{e-vals of } \Lambda \rightarrow \text{if all have } |\lambda| < 1 \]
\[ \text{then } \rightarrow \text{BIBD stable!} \]

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**Continuous-Time Systems**

\[ \frac{d}{dt} x(t) = \begin{bmatrix} A \end{bmatrix} x(t) + u(t). \]

\[ x(t) = e^{At} x(0) + \int_{0}^{t} e^{A(t-\tau)} u(\tau) d\tau. \]

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\[ \frac{d}{dt} x(t) = -2 \cdot x(t) \]

\[ x(t) = e^{-2t} x(0) \]

\[ \frac{d}{dt} x(t) = 2 \cdot x(t) \]

\[ x(t) = e^{2t} x(0) \]
Continuous time:

\[ \text{Re}\{\lambda}\} > 0 \quad \text{unstable.} \]
\[ \text{Re}\{\lambda}\} < 0 \quad \text{stable.} \]
\[ \text{Re}\{\lambda}\} = 0 \quad \text{marginally unstable.} \]

New control system.

1. Identify the system: learn parameters
   \( \to \) sys ID. \( \to \) model.

2. Once we have the model, is the system stable or not?

3. What can we do if your system is not stable? Can we make it stable?
   \( \to \) Feedback Control.

4. Can we drive the system to where we want to go?
   \( \to \) Controllability.

Feedback: Op Amps
Discrete-Time

Open Loop Dynamics

\[ \tilde{x}[t+1] = A\tilde{x}[t] + Bu[t] + \tilde{w}[t] \]

A unstable eigenvalue \( |\lambda| > 1 \)

Feedback: Choose \( u(t) \) as a function of \( \tilde{x}(t) \) so that you can adjust the system behavior.

Choose \( \tilde{u}(t) = F \cdot \tilde{x}(t) \)

Substituting:

\[ \tilde{x}[t+1] = A\tilde{x}[t] + B \cdot F \cdot \tilde{x}[t] + \tilde{w}[t] \]

Closed Loop Dynamics

\[ \tilde{x}[t+1] = (A + BF) \tilde{x}(t) + \tilde{w}(t) \]

E.g., \( \tilde{x}[t+1] = 2\tilde{x}(t) + u(t) + \tilde{w}(t) \)

\( u(t) = F \cdot \tilde{x}(t) \).

Want eigenvalue \( \lambda_0 \)
Choose: \( f = \lambda_0 - 2 \).

\[
\begin{align*}
\mathbf{x}[t+1] &= 2\mathbf{x}[t] + (\lambda_0 - 2)\cdot\mathbf{x}[t] + w[t] \\
\mathbf{x}[t+1] &= \lambda_0 \cdot \mathbf{x}[t] + w[t]
\end{align*}
\]

“state + feedback”

E.G.: Vector systems.

Closed dynamics:

\[
\overline{\mathbf{x}}[t+1] = (A + BF) \cdot \overline{\mathbf{x}}[t] + \overline{w}[t].
\]

Say we wanted \( A + BF = G \)

\[ \Rightarrow BF = G - A \]

\[ \Rightarrow f = (B^{-1})(G - A) \quad \text{if } B \text{ is invertible!} \]

What if \( B \) is not invertible?

\( B = \overrightarrow{b} \) vector.

E.G.: \[
\begin{align*}
\overline{\mathbf{x}}[t+1] &= \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \overline{\mathbf{x}}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overrightarrow{u}[t] + \overrightarrow{w}[t] \\
\overline{\mathbf{x}}(t) &\in \mathbb{R}^2, \ A \in \mathbb{R}^{2 \times 2}, \ \overrightarrow{b} \in \mathbb{R}^2, \ \overrightarrow{u}(t) \in \mathbb{R}, \ \overrightarrow{w} \in \mathbb{R}^{2 \times 1}
\end{align*}
\]
\[ A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \rightarrow \text{eigenvalues?} \]

\[ \lambda_1 = -1, \lambda_2 = 3 \]

\[ u(t) = F \cdot \dot{x}[t] \]

\[ u(t) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Want to find \( f_1, f_2 \) so that the system closed-loop is stable.

\[ A + BF = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 1 \\ 3 + f_1 & 2 + f_2 \end{bmatrix} \]

\[ \text{e-vals?} \]

\[ (A + BF - \lambda I) : \begin{bmatrix} -\lambda & 1 \\ 3 + f_1 & 2 + f_2 - \lambda \end{bmatrix} \]

Determinant: 
\[-\lambda (2 + f_2 - \lambda) - 1(3 + f_1) = 0\]

\[\lambda^2 - (2 + f_2) \cdot \lambda - (3 + f_1) = 0\]

\[
\begin{align*}
\text{Want: } \lambda_1, \lambda_2 & \text{ roots} \\
\lambda_1 \cdot \lambda_2 & = -(3 + f_1) \\
\lambda_1 + \lambda_2 & = (2 + f_2)
\end{align*}
\]

\[(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2) + \lambda_1 \lambda_2\]

\[\Rightarrow \text{ We can achieve ANY } \lambda_1, \lambda_2 \text{ by choice of } f_1, f_2 !!!\]

"Eigenvalue placement"

\[\begin{align*}
\mathbf{x}(t+1) &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
\mathbf{f} &= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\end{align*}\]
\[ A + B \cdot F = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \]

\[ = \begin{bmatrix} 2 & 0 \\ f_1 & 3 + f_2 \end{bmatrix} \]

**e-vals:**

\[
\begin{bmatrix} 2-\lambda & 0 \\ f_1 & 3+f_2-\lambda \end{bmatrix}
\]

**determinant:**

\[(2-\lambda)(3+f_2-\lambda) - 0 \cdot f_1 = 0\]

\[\Rightarrow \lambda^2 - \lambda(2+3+f_2) + 2(3+f_2) = 0\]

\[\Rightarrow \lambda^2 - \lambda(5+f_2) + 2(3+f_2) = 0\]

Want \( \lambda_1, \lambda_2 \):

\[\lambda_1 \lambda_2 = 2(3+f_2)\]

\[\lambda_1 + \lambda_2 = 5 + f_2\]