EE16B
Designing Information Devices and Systems II

Lecture 4A
Phasors, Freq. Response / Transfer function / AC Power

Announcements

- Last time:
  - Second order systems
  - Intro to VDE
  - RLC circuits
- Today:
  - Resonant Tank
  - AC Response / Phasors

Recap:

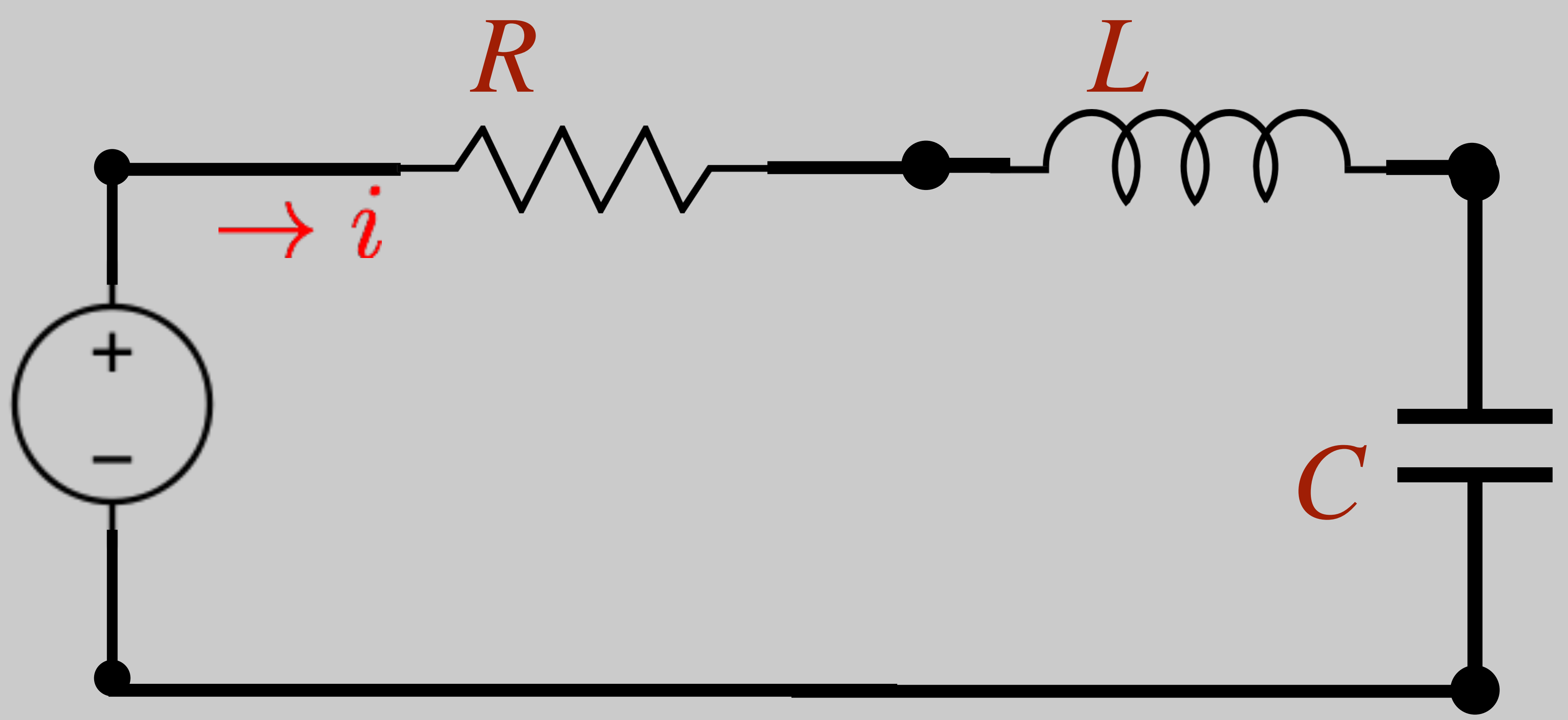
$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = u(t)$$

$a = \dfrac{R}{2L}$ Damping coefficient (associated with decay)

$\omega_0 = \dfrac{1}{\sqrt{LC}}$ Resonance frequency

$\zeta = \dfrac{\alpha}{\omega_0}$ Damping Ratio

$Q = \dfrac{1}{2\zeta} = \dfrac{\omega_0 L}{R} = \sqrt{\dfrac{L}{CR^2}}$ Quality factor



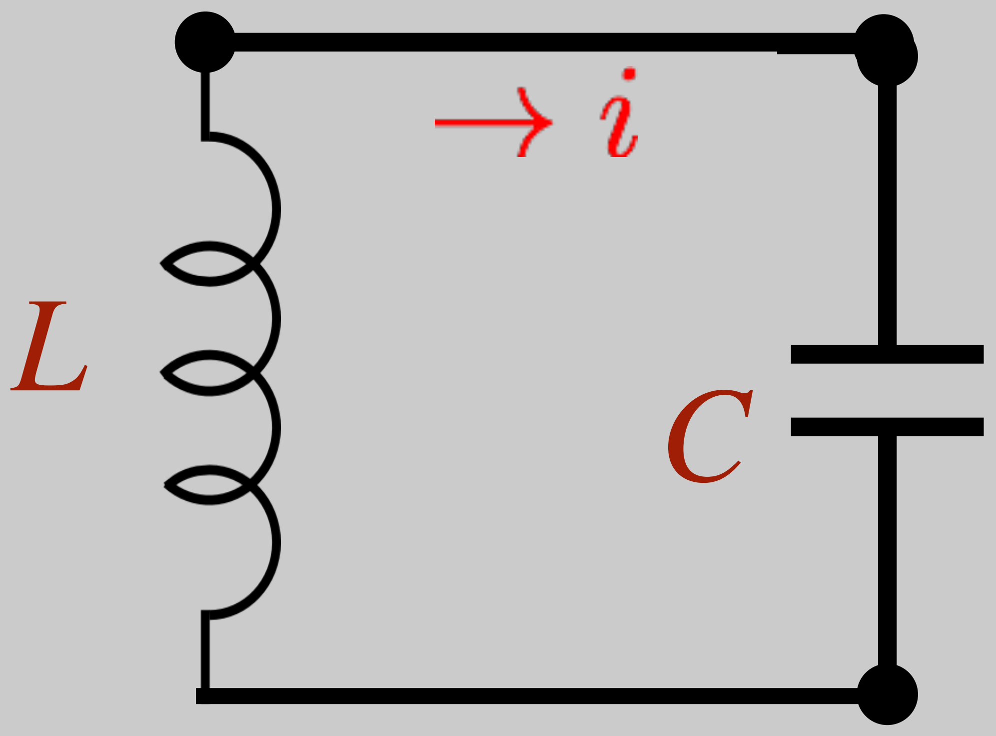
$\alpha > \omega_0, \quad \zeta > 1 \Rightarrow$ Overdamped $\Rightarrow$ Exponential decays

$\alpha = \omega_0, \quad \zeta = 1 \Rightarrow$ Critically damped $\Rightarrow$ Fastest decay without oscillations

$\alpha < \omega_0, \quad \zeta < 1 \Rightarrow$ Underdamped $\Rightarrow$ decay+oscillations

$\omega_n = \sqrt{\alpha^2 - \omega^2}$ Oscillation frequency

Tank Circuit



RLC with $R \rightarrow 0$

LC "Tank" will store energy

$$V_c(t) = V_{DD} e^{-\alpha t} \left(\cos(\omega_n t) + \frac{\alpha}{\omega_n} \sin(\omega_n t) \right)$$

$$V_c(t) = V_{DD} \cos(\omega_0 t) \qquad i(t) = -CV_{DD}\omega_0 \sin(\omega_0 t)$$

Energy in Cap:

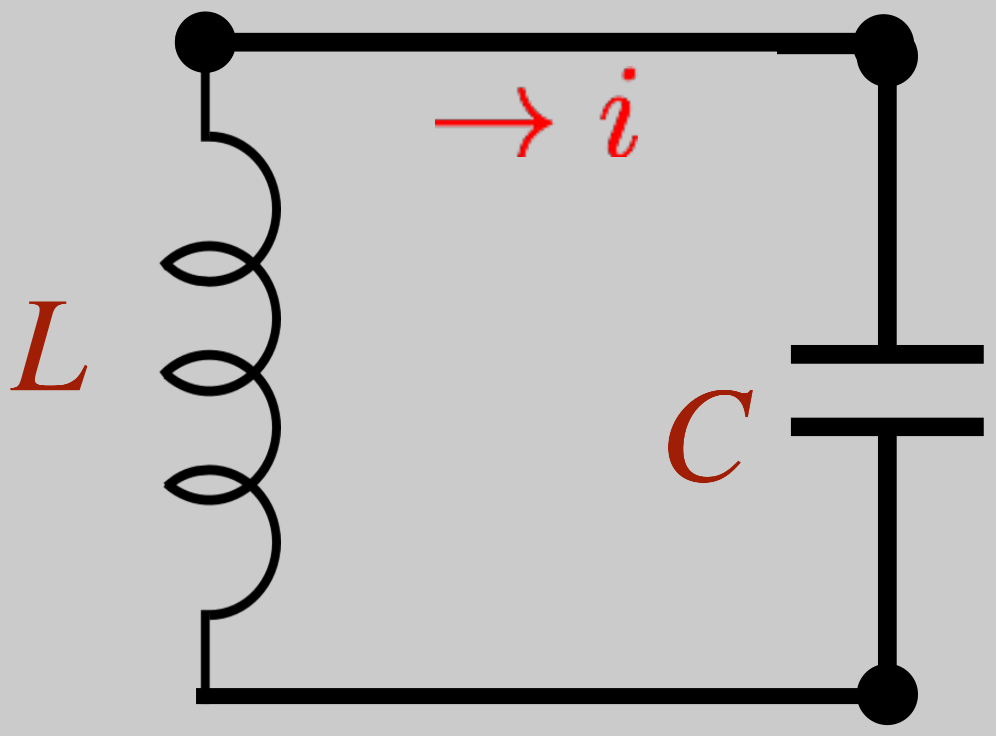
$$W_c = \frac{1}{2}CV_c^2 = \frac{1}{2}CV_{DD}^2 \cos^2(\omega_0 t)$$

Energy in Inductor:

$$W_L = \frac{1}{2}Li^2 = \frac{1}{2}V_{DD}^2 LC^2 \omega_0^2 \sin^2(\omega_0 t)$$

$$= \frac{1}{2}CV_{DD}^2 \sin^2(\omega_0 t)$$

Tank Circuit

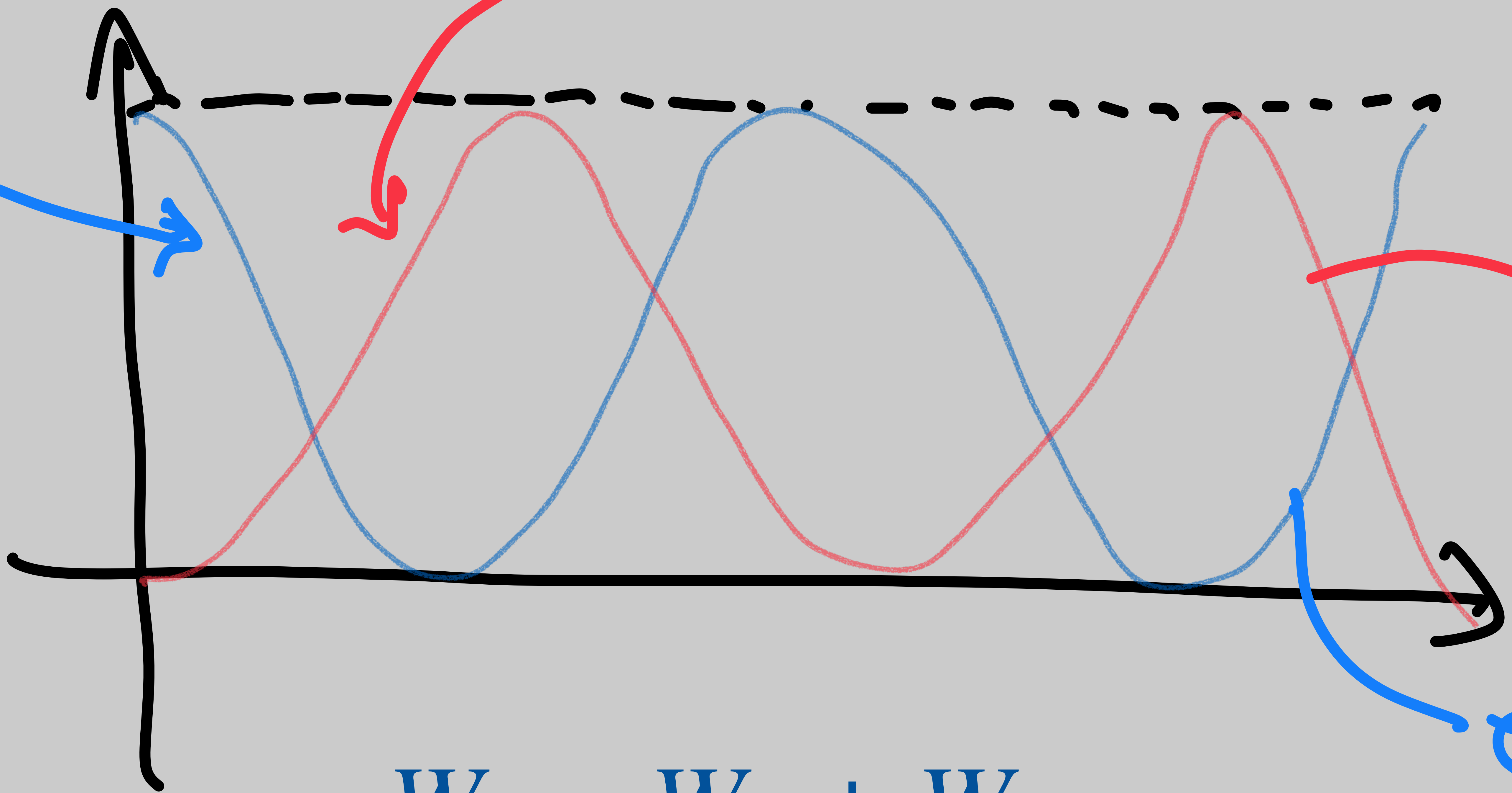


Energy in Cap:

$$W_c = \frac{1}{2} C V_{DD}^2 \cos^2(\omega_0 t)$$

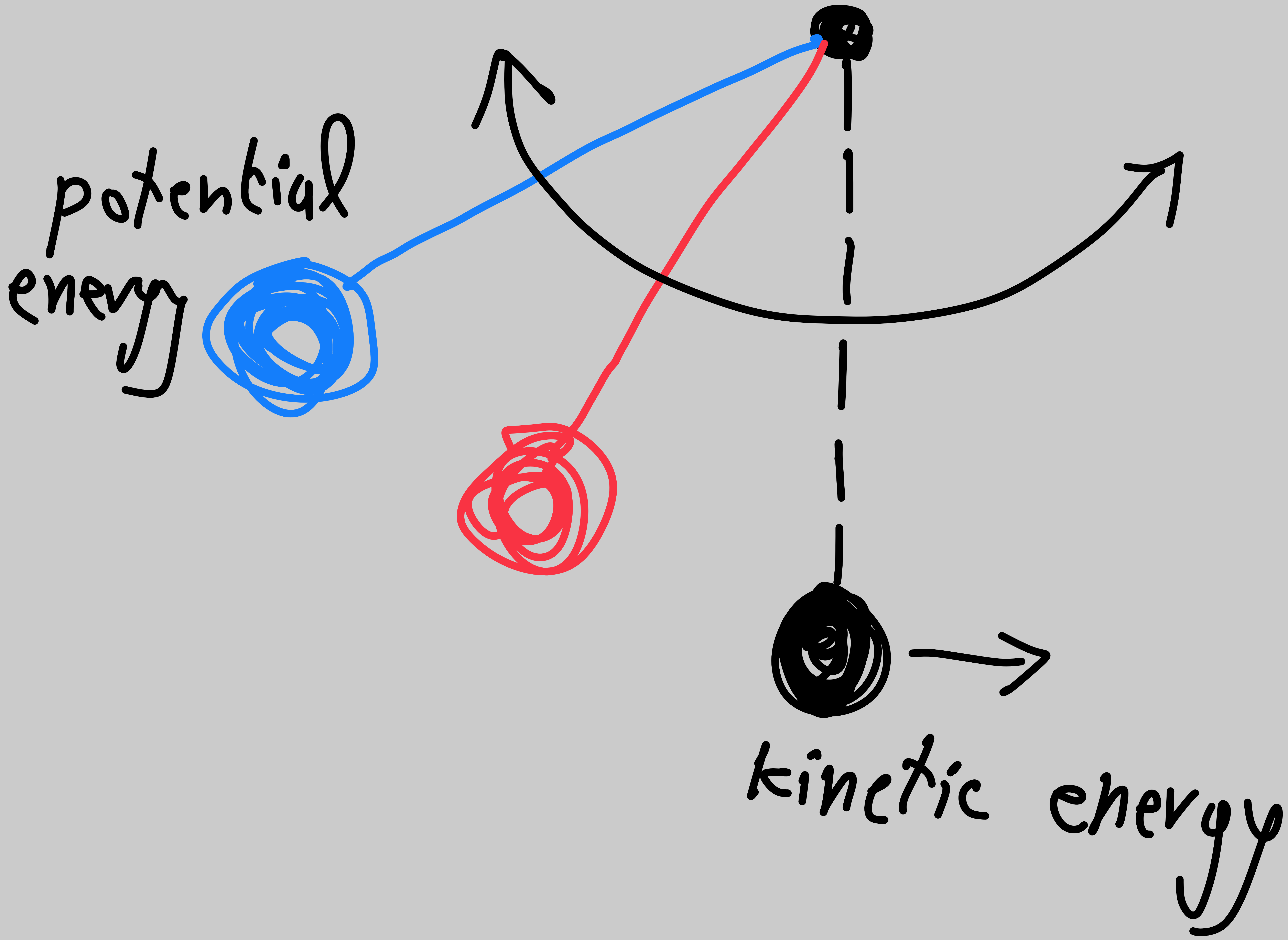
Energy in Inductor:

$$W_L = \frac{1}{2} C V_{DD}^2 \sin^2(\omega_0 t)$$

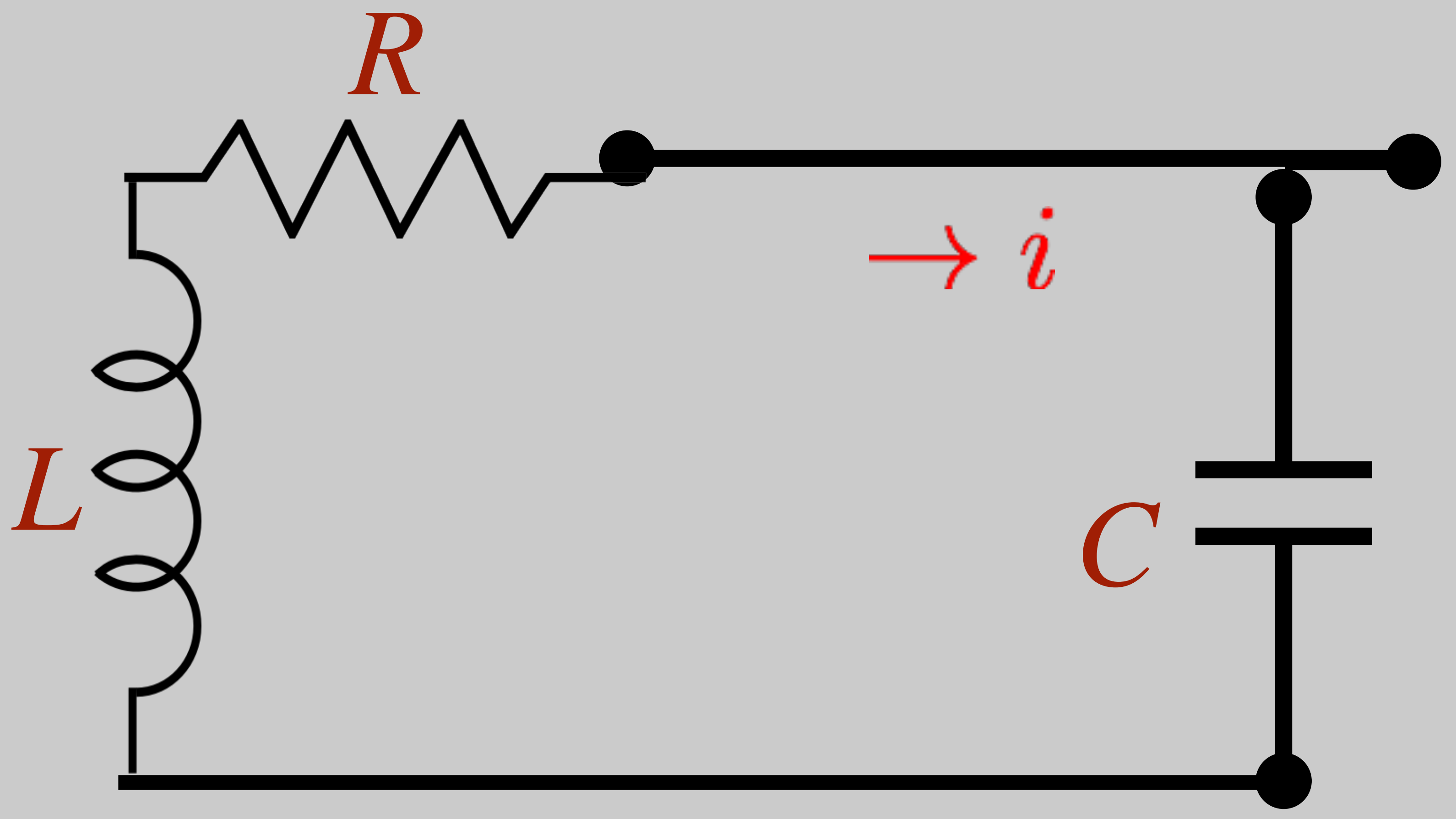


$$W_s = W_L + W_C$$

Pendulum



Lossy Tank Circuit



Deal with "small" loss in perturbation

Energy dissipated in resistor per cycle:

$$W_d = \int_{\text{cycle}} P(t)dt = \int_0^T R i^2(t)dt$$

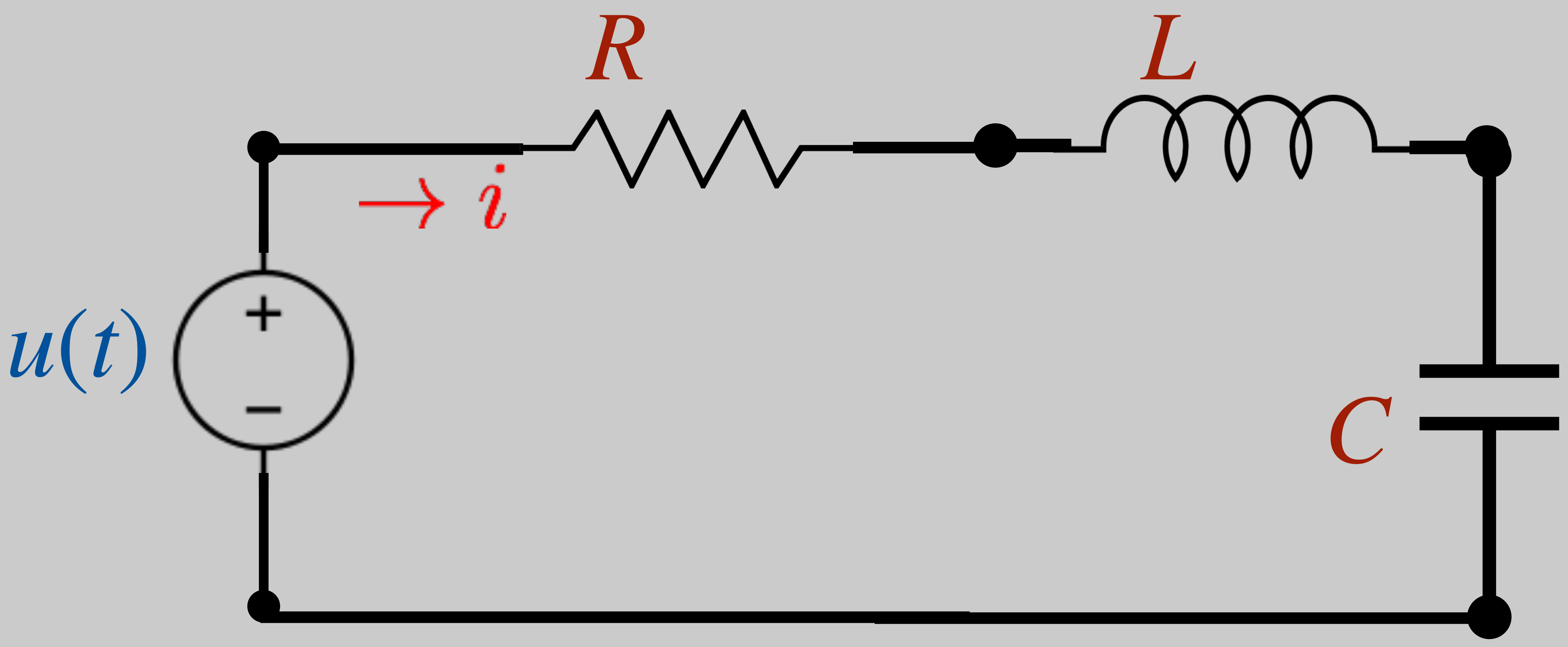
$$i(t) = I_0 \sin(\omega_0 t)$$

$$= \int_0^T R I_0^2 \sin^2(\omega_0 t)dt = \frac{1}{2} R I_0^2 T = \frac{1}{2} R I_0^2 \frac{2\pi}{\omega_0}$$

$$\frac{W_s}{W_d} = \frac{\frac{1}{2} I_0^2 L}{\frac{1}{2} I_0^2 R \frac{2\pi}{\omega_0}} = \left(\frac{\omega_0 L}{R}\right) \frac{1}{2\pi} \approx \frac{Q}{2\pi}$$

$= Q$

Non Homogeneous Solution



$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = u(t)$$

$$V_c(t) = V_h(t) + V_p(t)$$

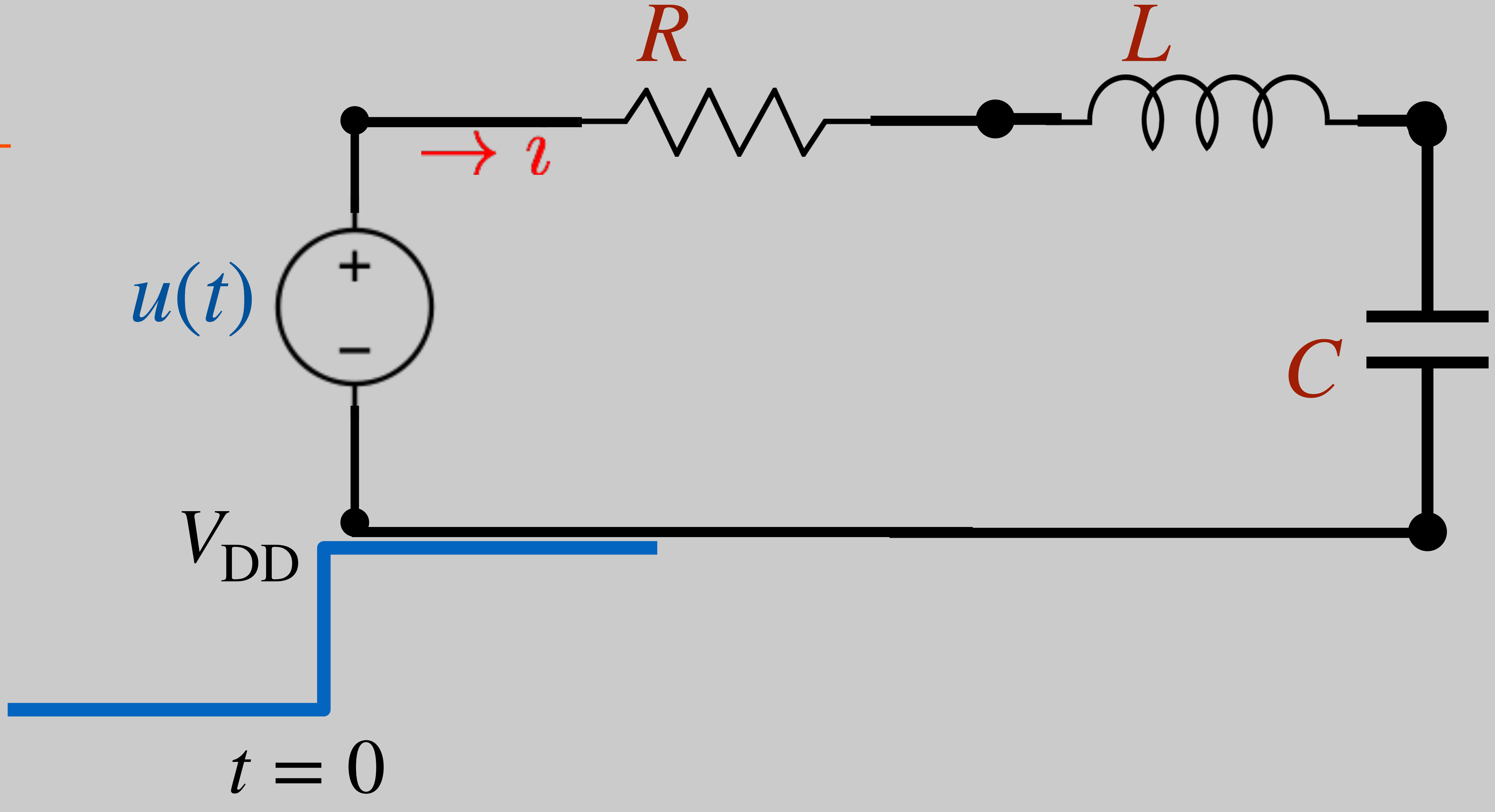
$$\alpha > \omega_0,\ \ \zeta > 1 \Rightarrow V_c(t) = Ae^{s_1 t} + Be^{s_2 t} + V_p(t)$$

$$\alpha = \omega_0,\ \ \zeta = 1 \Rightarrow V_c(t) = Ae^{s_1 t} + Bte^{s_1 t} + V_p(t)$$

$$\alpha < \omega_0,\ \ \zeta < 1 \Rightarrow V_c(t) = e^{s_1 t}\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right) + V_p(t)$$

Steady-State DC response

$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = V_{DD}$$

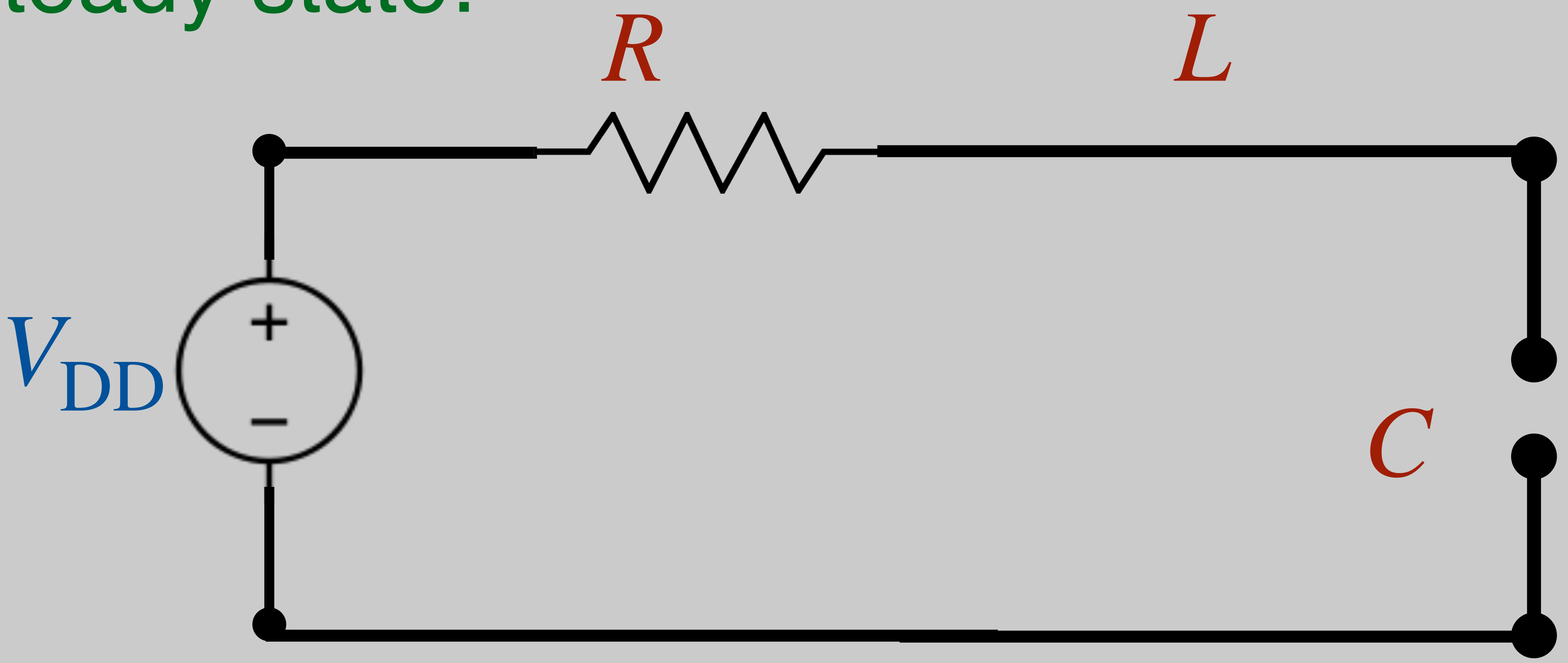


$$V_c(t) = V_h(t) + V_p(t)$$

$$\lim_{t\to\infty} V_h(t) = 0$$

$$\lim_{t\to\infty} V_c(t) = V_p(t) = V_{DD}$$

In steady state:



Solve for:

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t} + V_{DD}$$

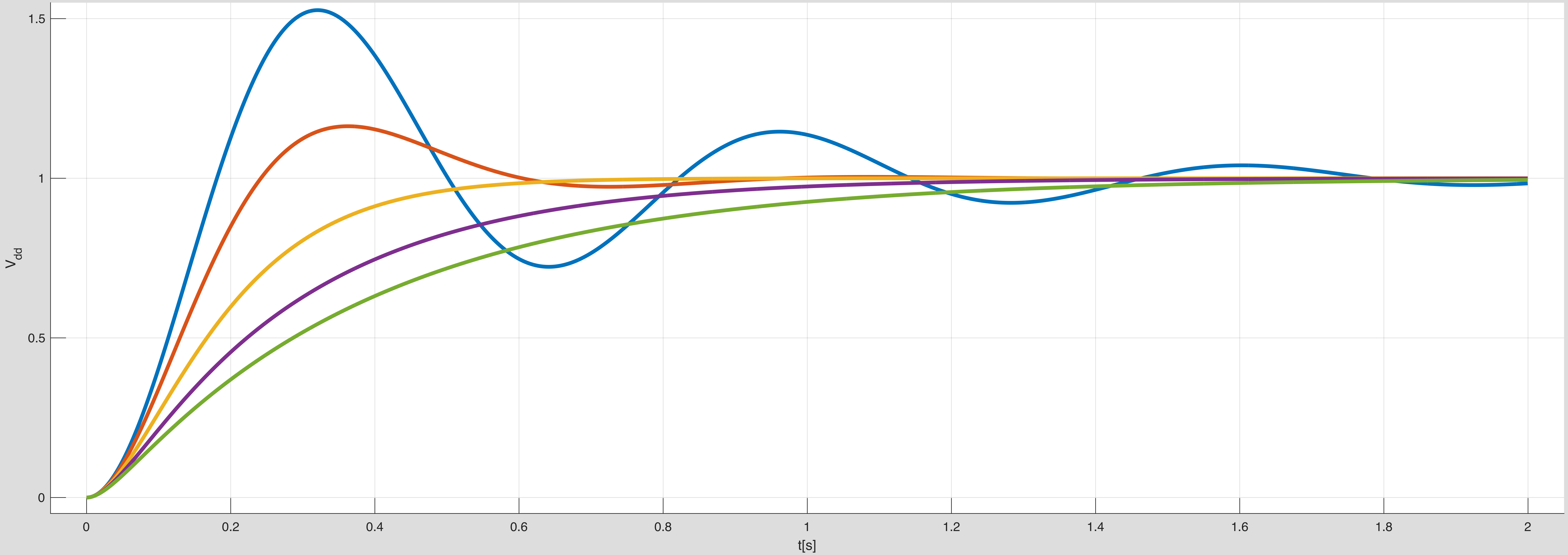
Or

$$V_c(t) = Ae^{s_1 t} + Bte^{s_1 t} + V_{DD}$$

Or

$$V_c(t) = e^{s_1 t}\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right) + V_{DD}$$

Steady-State DC response

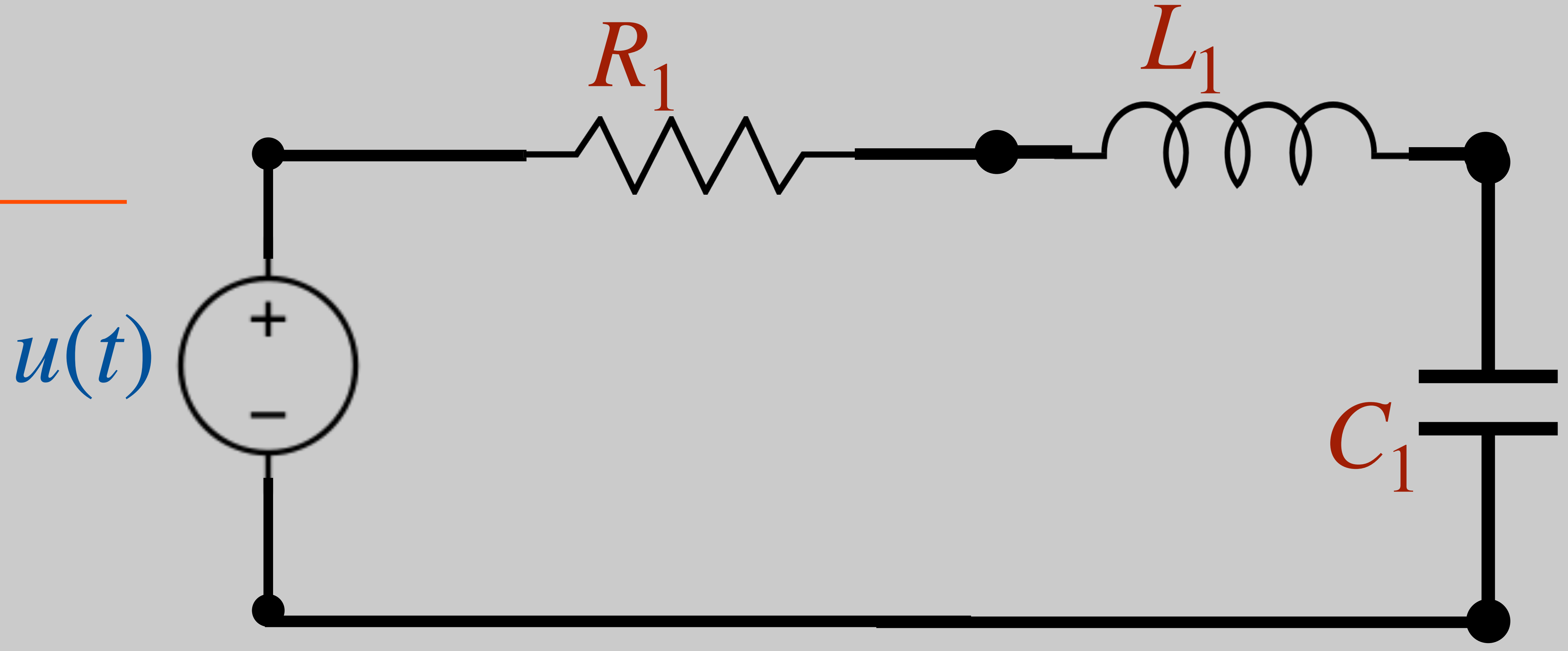


Steady State Sinusoidal Input

Q: What are we going to show?

$$H(j\omega) = \frac{A_\omega e^{j\omega t + j\theta_\omega}}{e^{j\omega t}} = A_\omega e^{j\theta_\omega}$$

Back to: VDE / State Space



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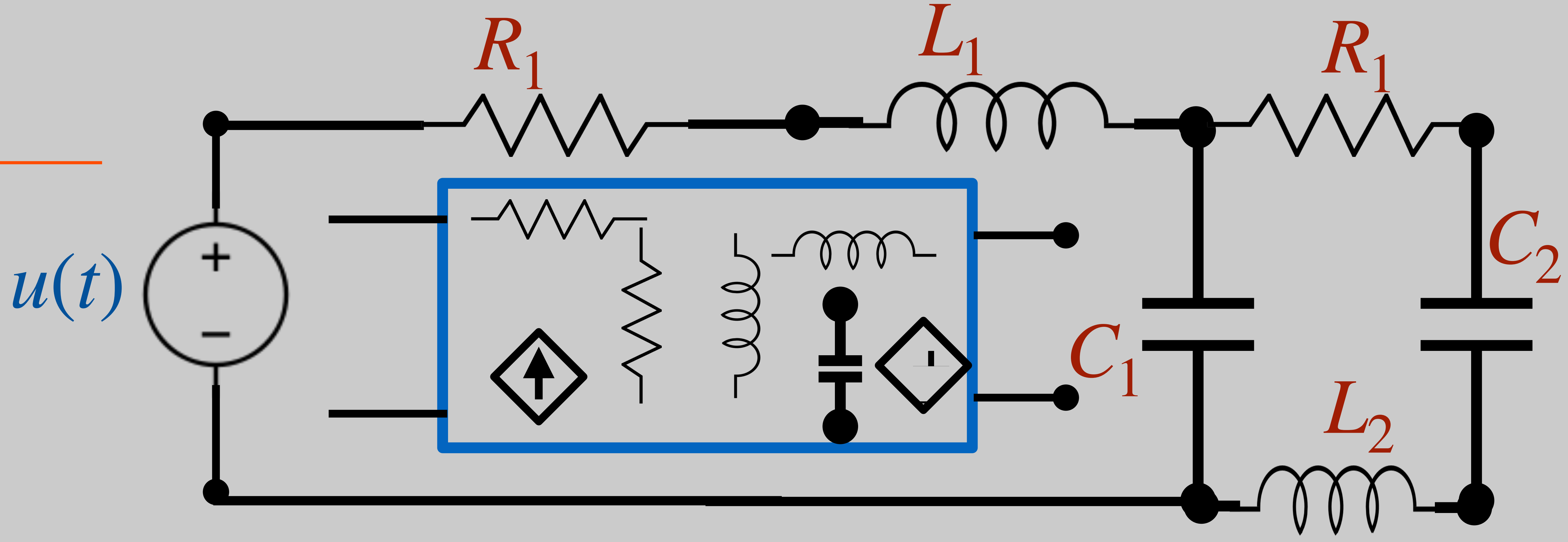
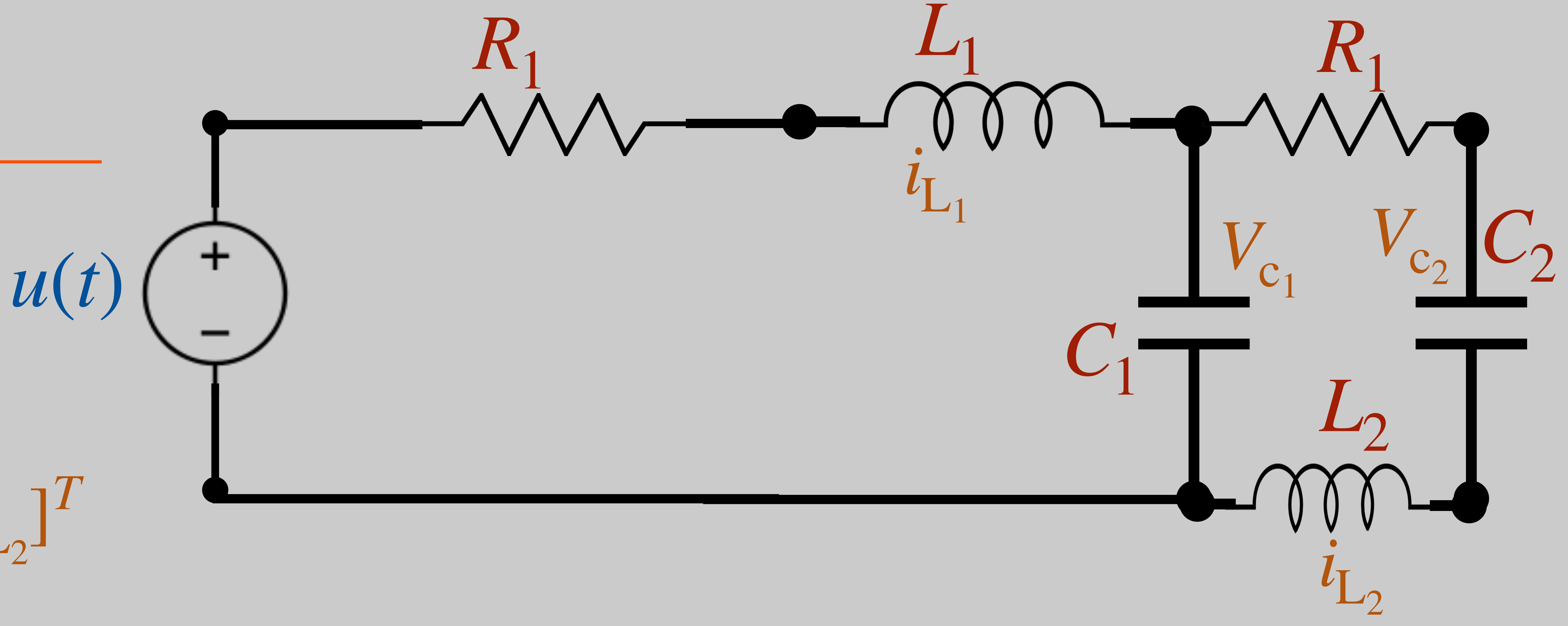


Back to: VDE / State Space



$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

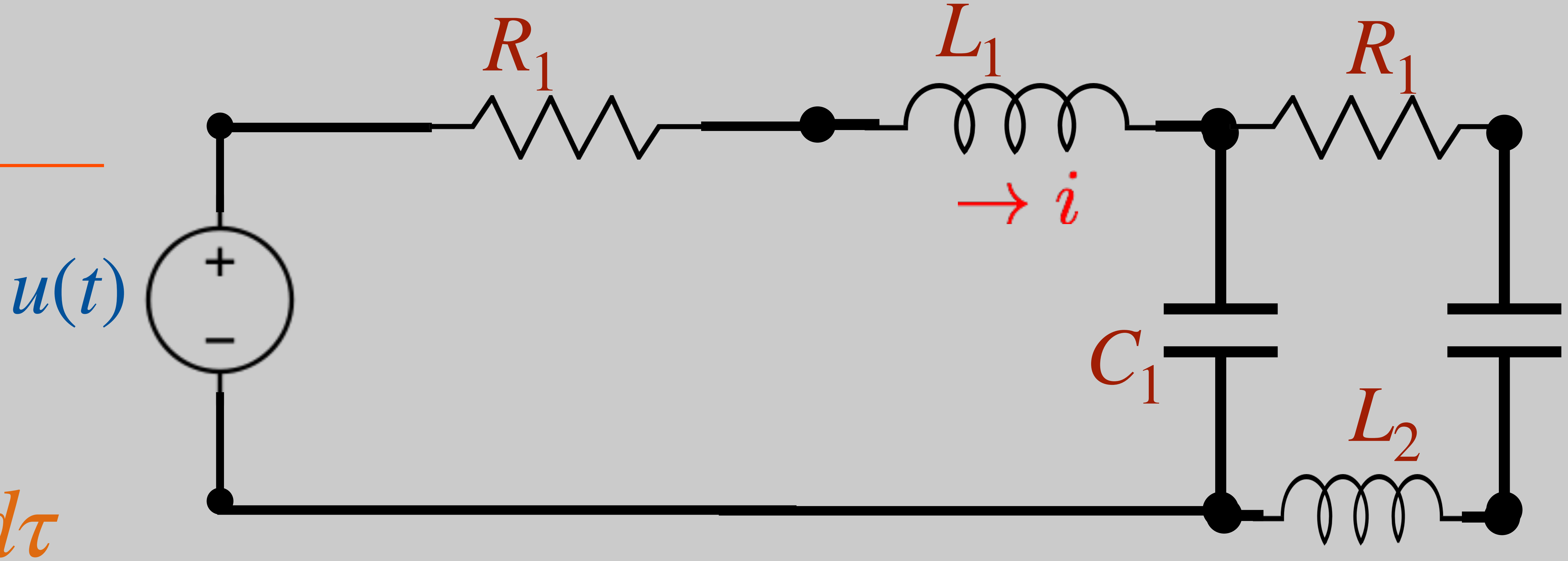
$$\vec{x} = [V_{c_1} \quad V_{c_2} \quad i_{L_1} \quad i_{L_2}]^T$$

$$\vec{z}(t) = T\vec{x}(t)$$

$$\frac{d}{dt}\vec{z}(t) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \vec{z}(t) + B_{\text{new}} u(t)$$

$$\frac{d}{dt} z_1(t) = \lambda_1 z_1(t) + \tilde{b}_1 u(t) \quad \Rightarrow z_1(t) = Ke^{\lambda_1 t} + \tilde{b}_1 e^{\lambda_1 t} \int_{-\infty}^{t} u(\tau) e^{-\lambda_1 \tau} d\tau$$

Back to: VDE / State Space



$$z_1(t) = Ke^{\lambda_1 t} + \tilde{b}_1 e^{\lambda_1 t}\int_{-\infty}^{t} u(\tau)e^{-\lambda_1 \tau}d\tau$$

$$\vdots$$

$$z_N(t) = Ke^{\lambda_N t} + \tilde{b}_N e^{\lambda_N t}\int_{-\infty}^{t} u(\tau)e^{-\lambda_N \tau}d\tau$$

If there is a complex λ, there's also λ^*!

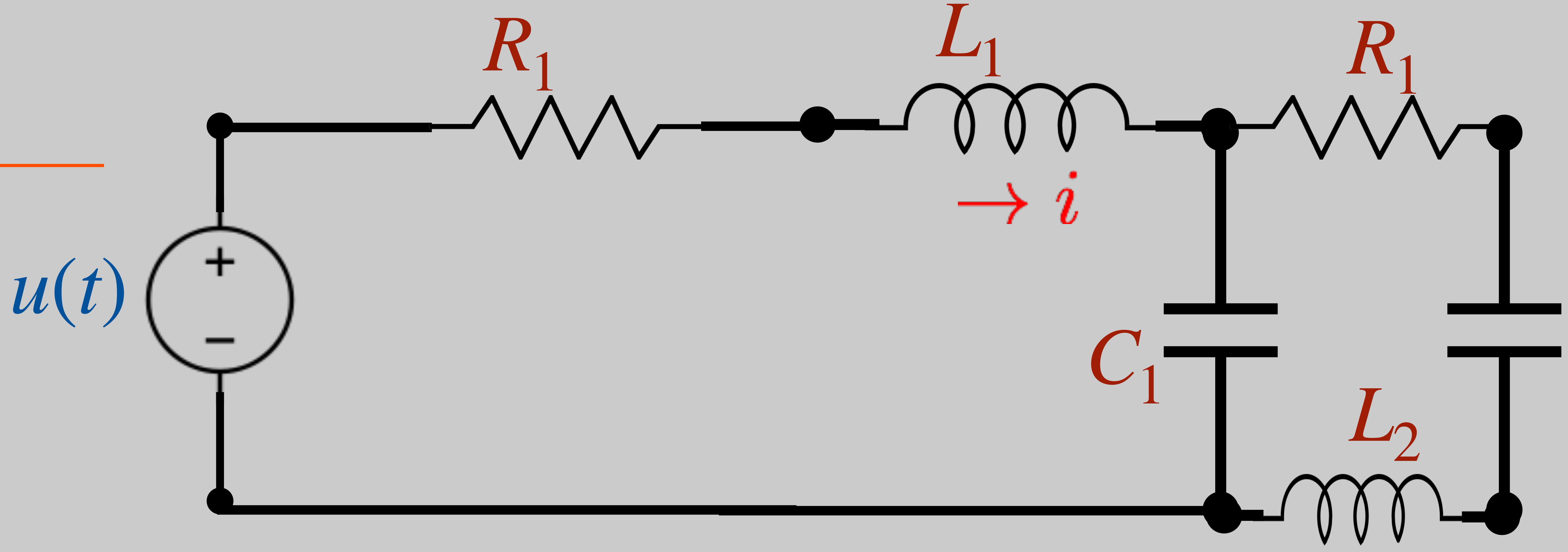
Q: Why? A: Keep it real!

Go back to original variable by: $\vec{x}(t) = T^{-1}\vec{z}(t)$

So, solution for $x_n(t) = \sum_{k=1}^{N} T^{-1}_{nk} z_k(t)$ is a linear combination of the "Eigen" responses

n[th] row elements of T^{-1}

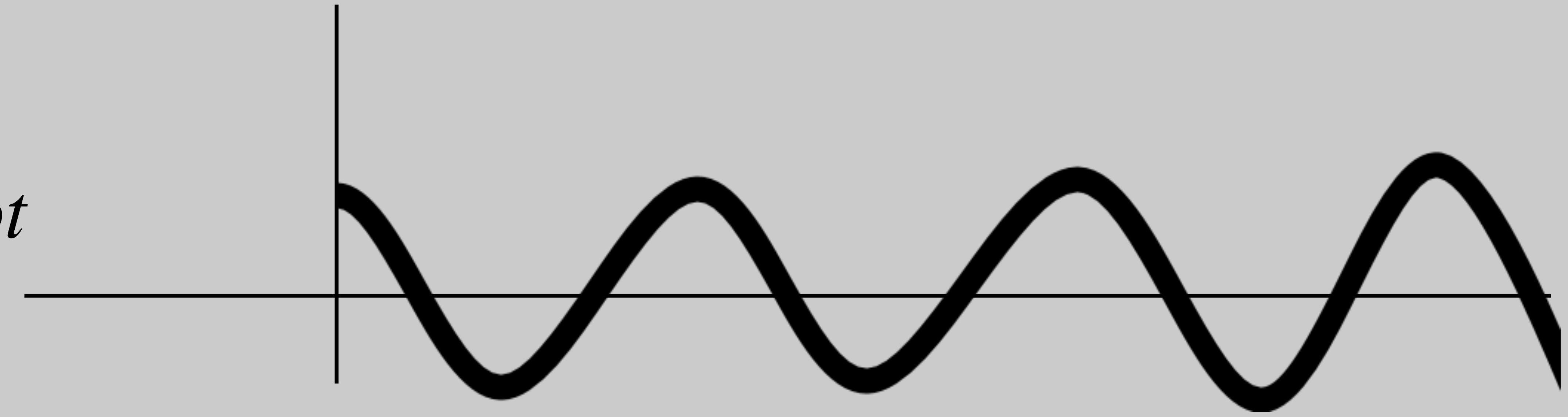
Sinusoidal input



$$z(t) = Ke^{\lambda t} + \tilde{b}e^{\lambda t}\int_{-\infty}^{t} u(\tau)e^{-\lambda\tau}d\tau$$

$$\lambda = -\alpha + j\omega_0$$

$$u(t > 0) = 2\cos(\omega t) \quad \Rightarrow u(t) = e^{j\omega t} + e^{-j\omega t}$$



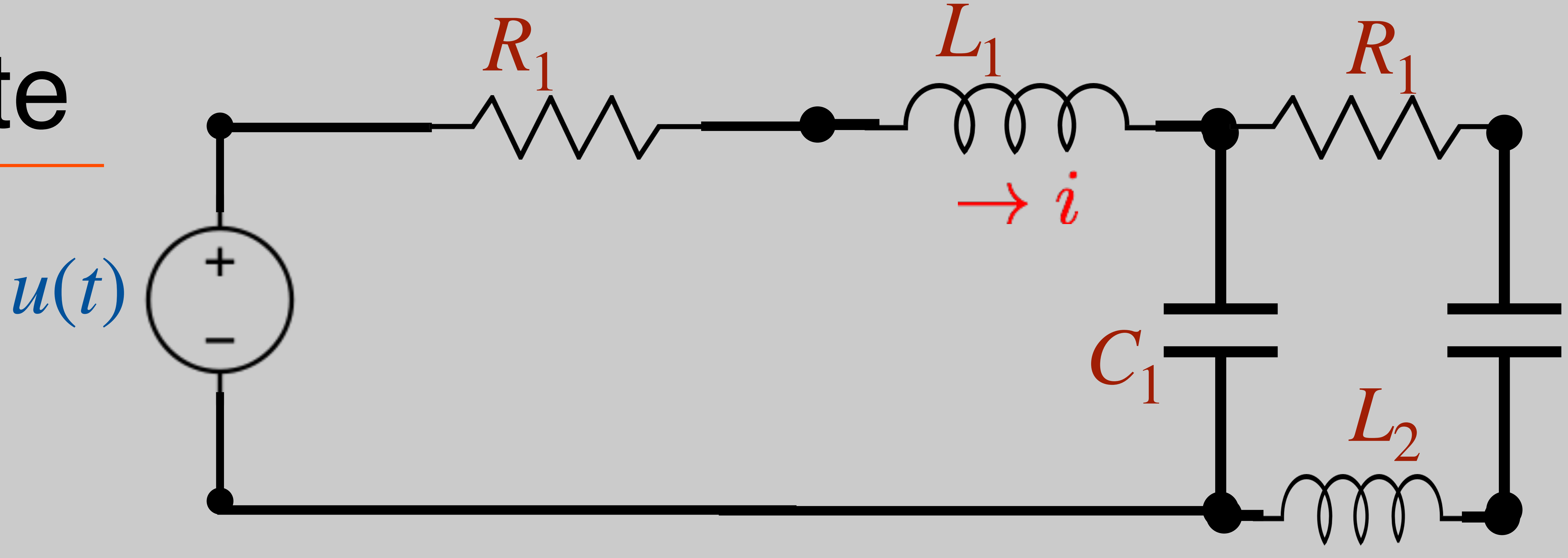
$$z(t) = K \cdot e^{\lambda t} + \tilde{b}e^{\lambda t}\int_{0}^{t} e^{(j\omega - \lambda)\tau}d\tau$$

Transient if real(λ)<0

$$= \frac{\tilde{b}e^{\lambda t}}{j\omega - \lambda}\left[e^{(j\omega-\lambda)t} - 1\right] = \frac{\tilde{b}e^{j\omega t}}{j\omega - \lambda} - \frac{\tilde{b}e^{\lambda t}}{j\omega - \lambda} =_{t\rightarrow\infty} \frac{\tilde{b}e^{j\omega t}}{j\omega - \lambda}$$

Transient if real(λ)<0

Sinusoidal input in steady state

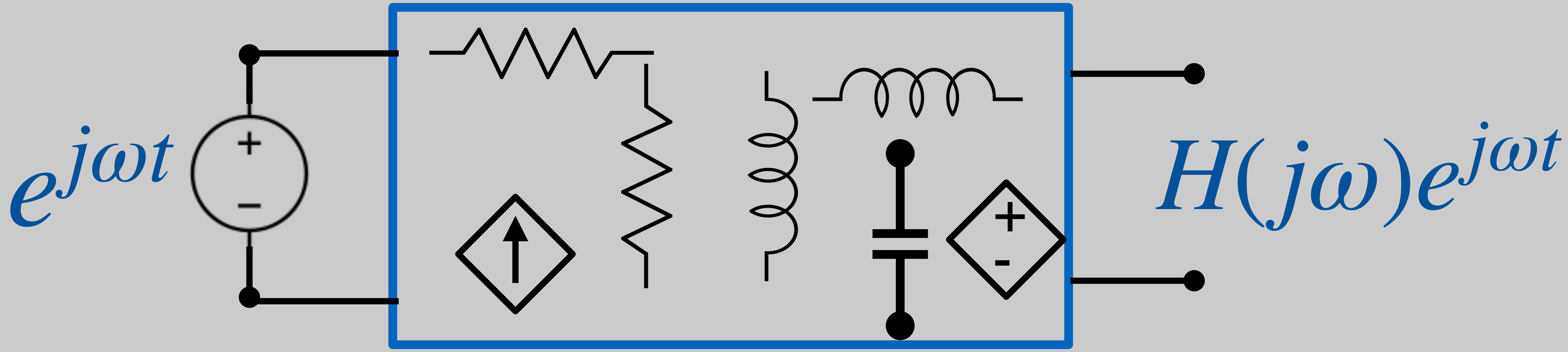


$u(t) = e^{j\omega t}$

$$z_k(t) = \frac{\tilde{b}_k}{j\omega - \lambda_k} e^{j\omega t}$$

$$x_n(t) = \sum_k \frac{T^{-1}_{nk} \tilde{b}_k}{j\omega - \lambda_k} e^{j\omega t} \quad \Rightarrow x_n(t) = H(j\omega) e^{j\omega t}$$

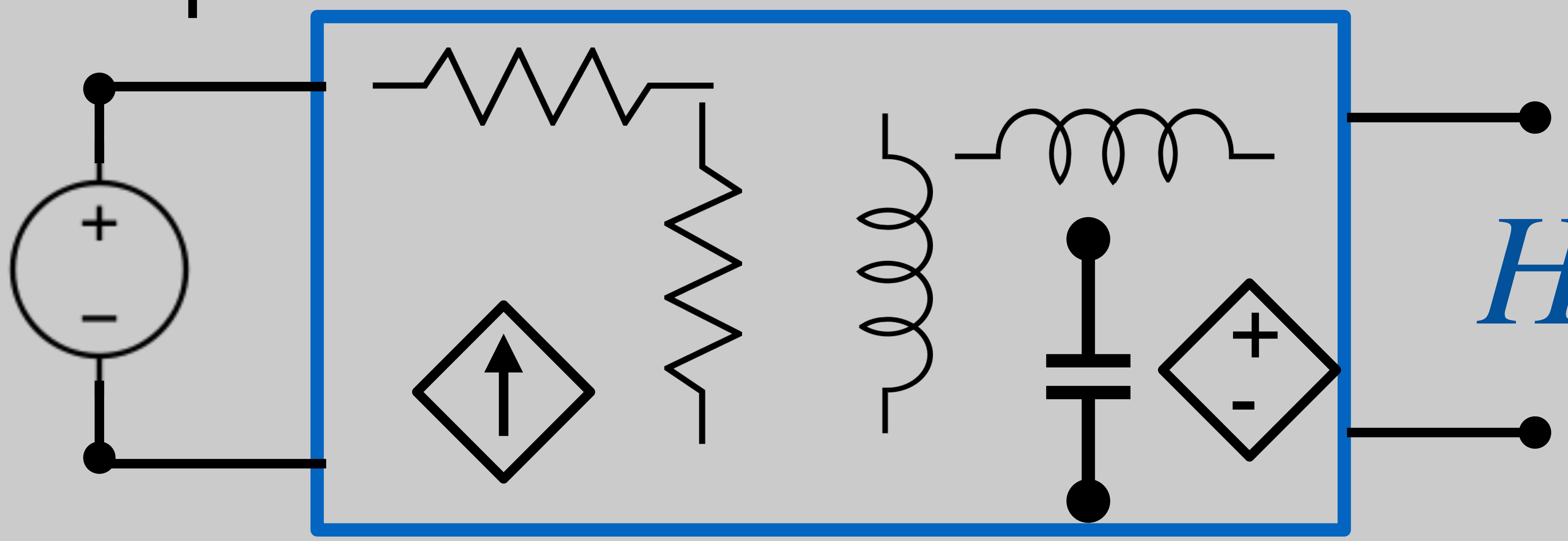
$H(j\omega)$



Frequency Response/ Transfer function

Frequency response:

$e^{j\omega t}$



$H(j\omega)e^{j\omega t}$

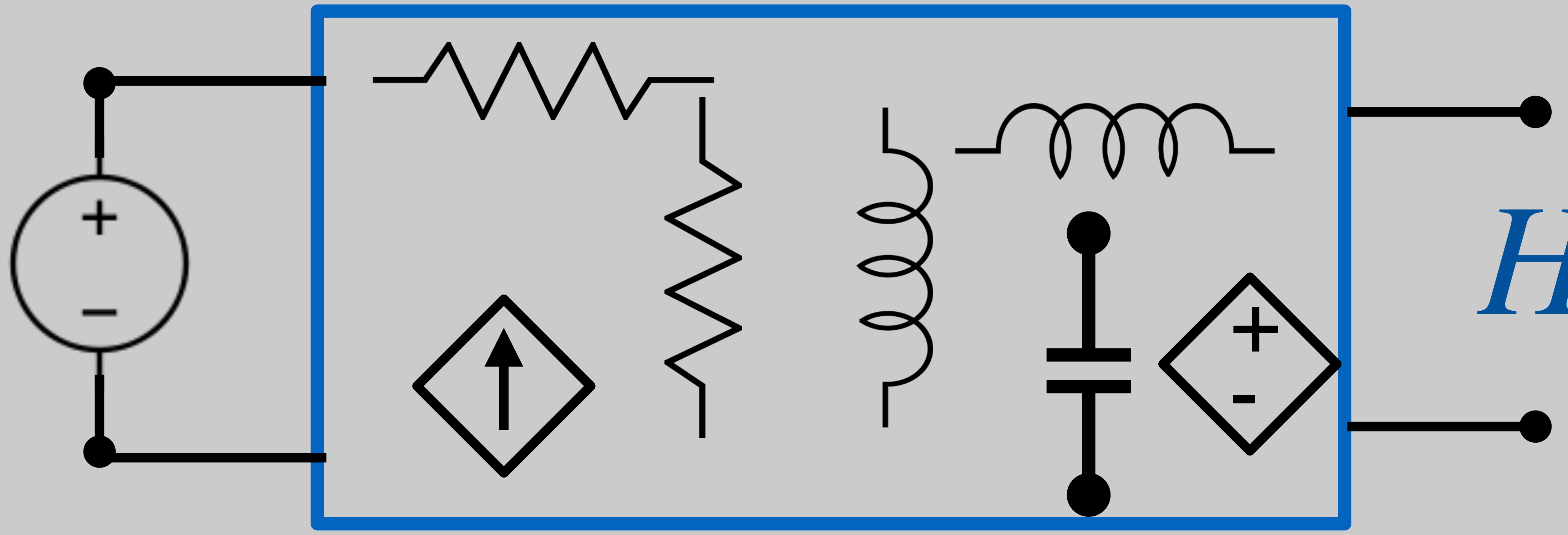
Phase resp.

$= |H(j\omega)| e^{j\omega t + j\angle H(j\omega)}$

Magnitude resp.

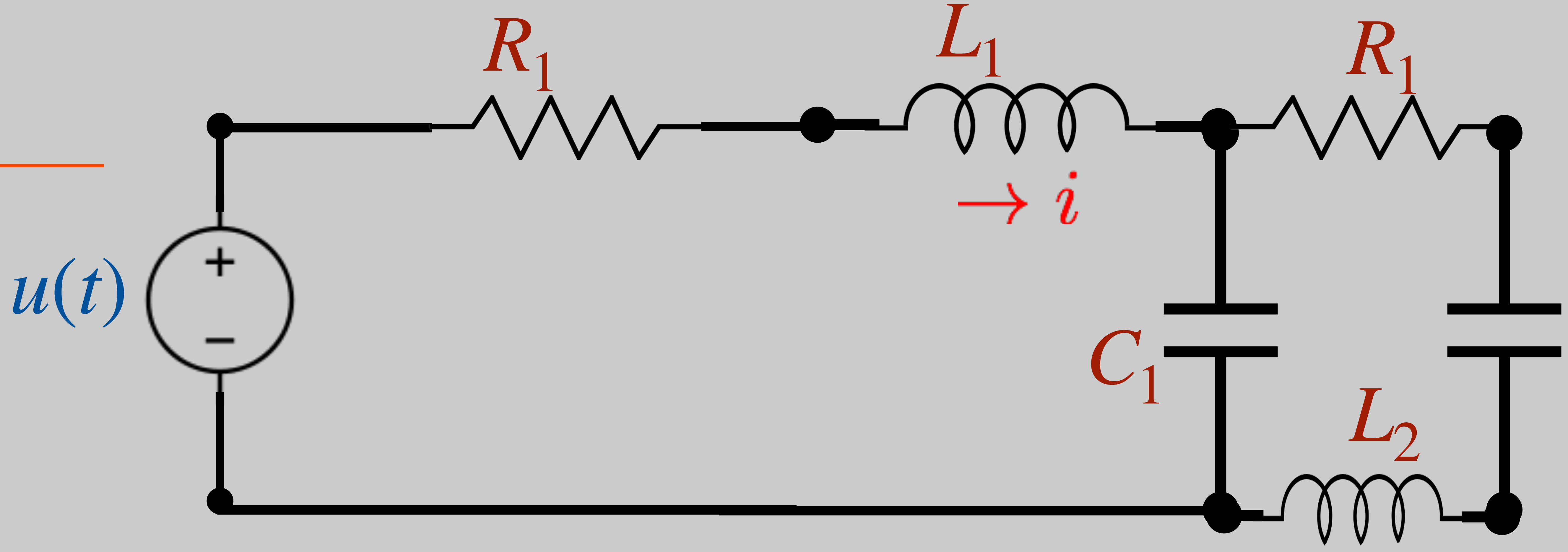
Transfer function:

$e^{(\alpha + j\omega)t} = e^{st}$



$H(s)e^{st}$

AC Analysis in steady state

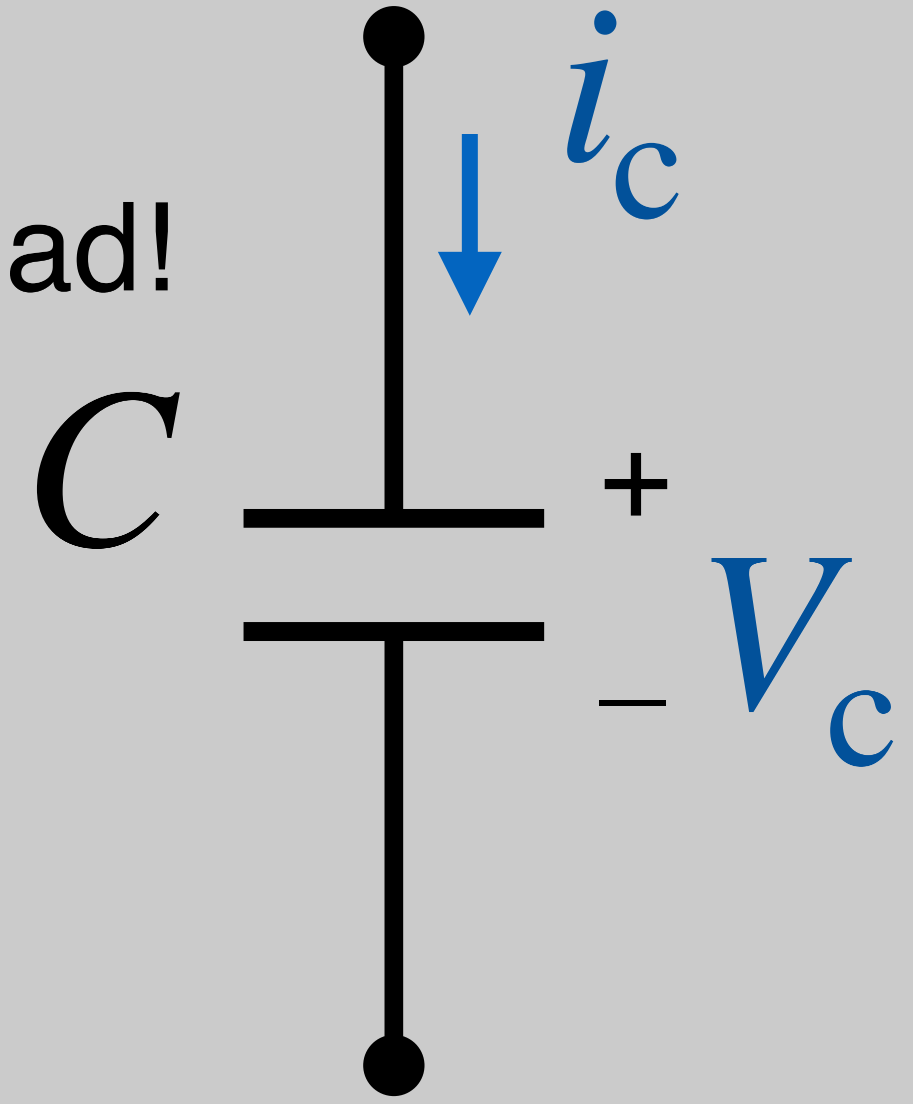


Want to avoid solving VDE every time!

But, we know for sinusoidal input, we get a sinusoidal output (in steady state)

So, simplify analysis by looking at individual components

Capacitors in AC circuits



$$V_c(t) = V_0 \cos(\omega t + \theta)$$ Use, complex exponentials instead!

$$V_c(t) = V_0 e^{j\omega t + \theta} = V_0 e^{j\theta} e^{j\omega t} = \tilde{V}_c e^{j\omega t}$$

$$i_c(t) = C \frac{d}{dt} V_c(t) = j\omega C \tilde{V}_c e^{j\omega t} = \tilde{I}_c e^{j\omega t}$$

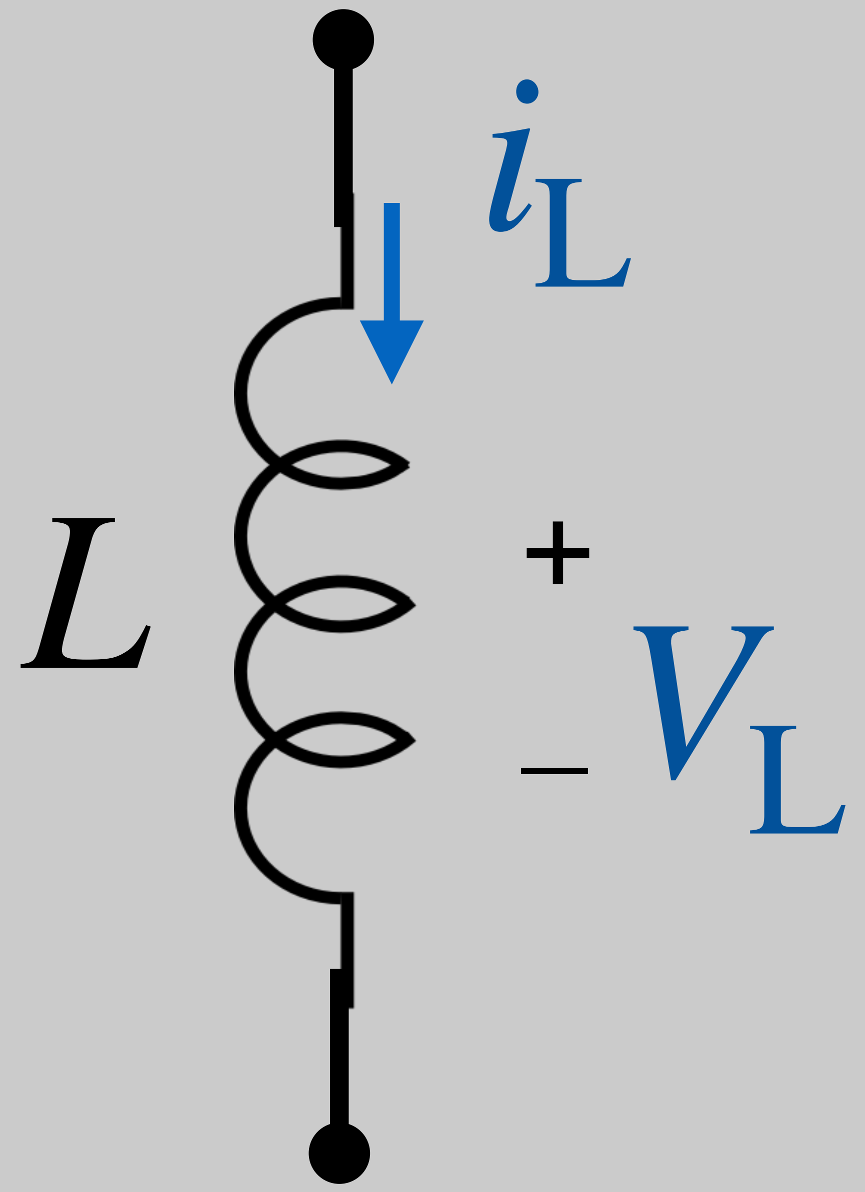
$$\Rightarrow i_c(t) = \omega C V_0 \cos(\omega t + \theta + \pi/2)$$

$$\tilde{I}_c = j\omega C \tilde{V}_c$$

$$\frac{V_c(t)}{i_c(t)} = \frac{\tilde{V}_c}{\tilde{I}_c} = \frac{1}{j\omega C}$$

In steady state, a capacitor IV relationship looks like a “imaginary valued” resistor!

Inductors in AC circuits



$$i_L(t) = I_0 e^{j\omega t + \theta} = \tilde{I}_L e^{j\omega t}$$

$$V_L(t) = L\frac{d}{dt} i_L(t) = j\omega L \tilde{I}_L e^{j\omega t}$$

$$\tilde{V}_L = j\omega L \tilde{I}_L$$

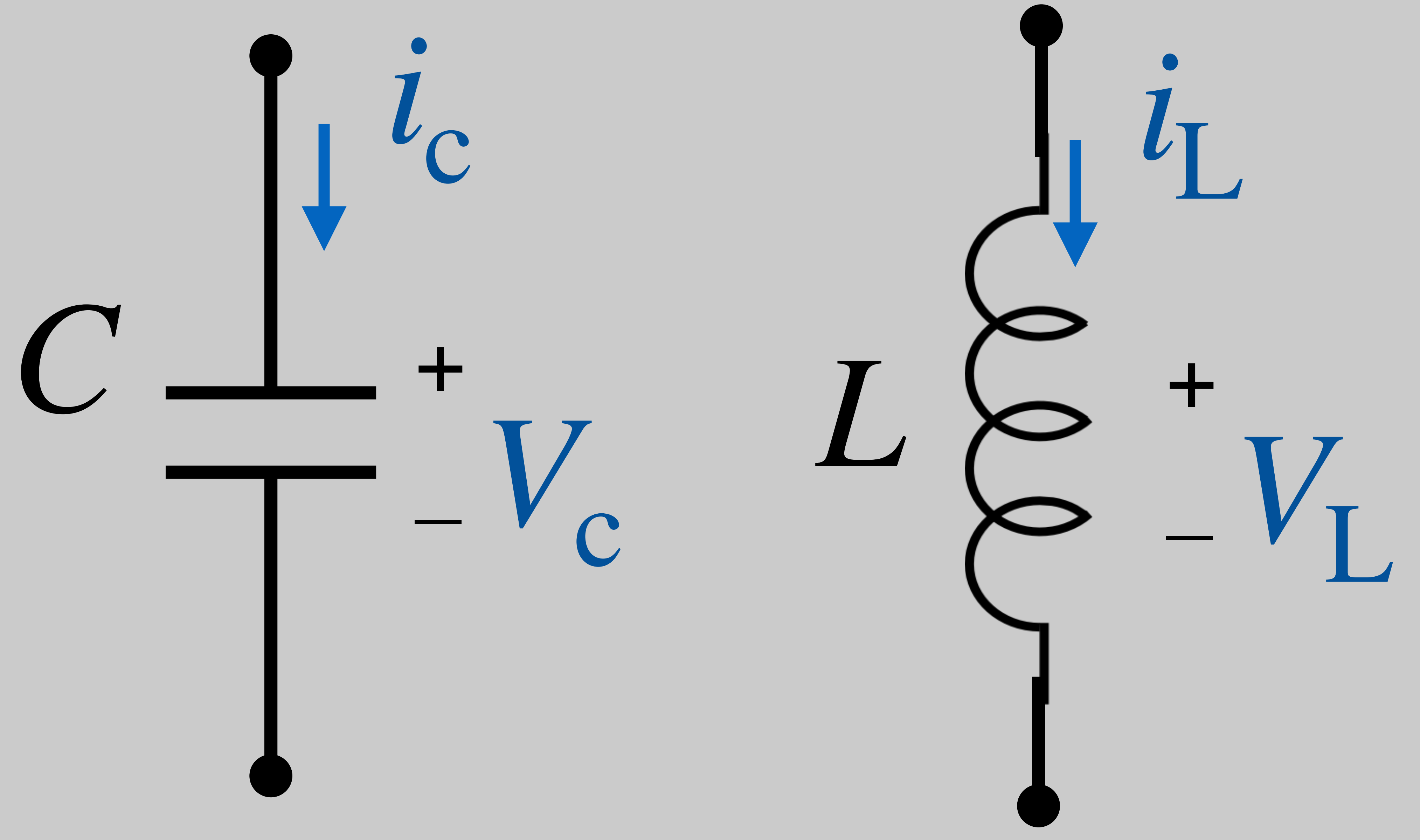
$$\frac{V_L(t)}{i_L(t)} = \frac{\tilde{V}_L}{\tilde{I}_L} = j\omega L$$

In steady state, a Inductor IV relationship looks like a “imaginary valued” resistor!

Phasors

$$i_L(t) = I_0 e^{j\omega t + j\theta} = \tilde{I}_L e^{j\omega t}$$

$$V_c(t) = V_0 e^{j\omega t + j\theta} = \tilde{V}_c e^{j\omega t}$$



$\tilde{I}_L$ and $\tilde{V}_c$ are called "phasors", complex numbers

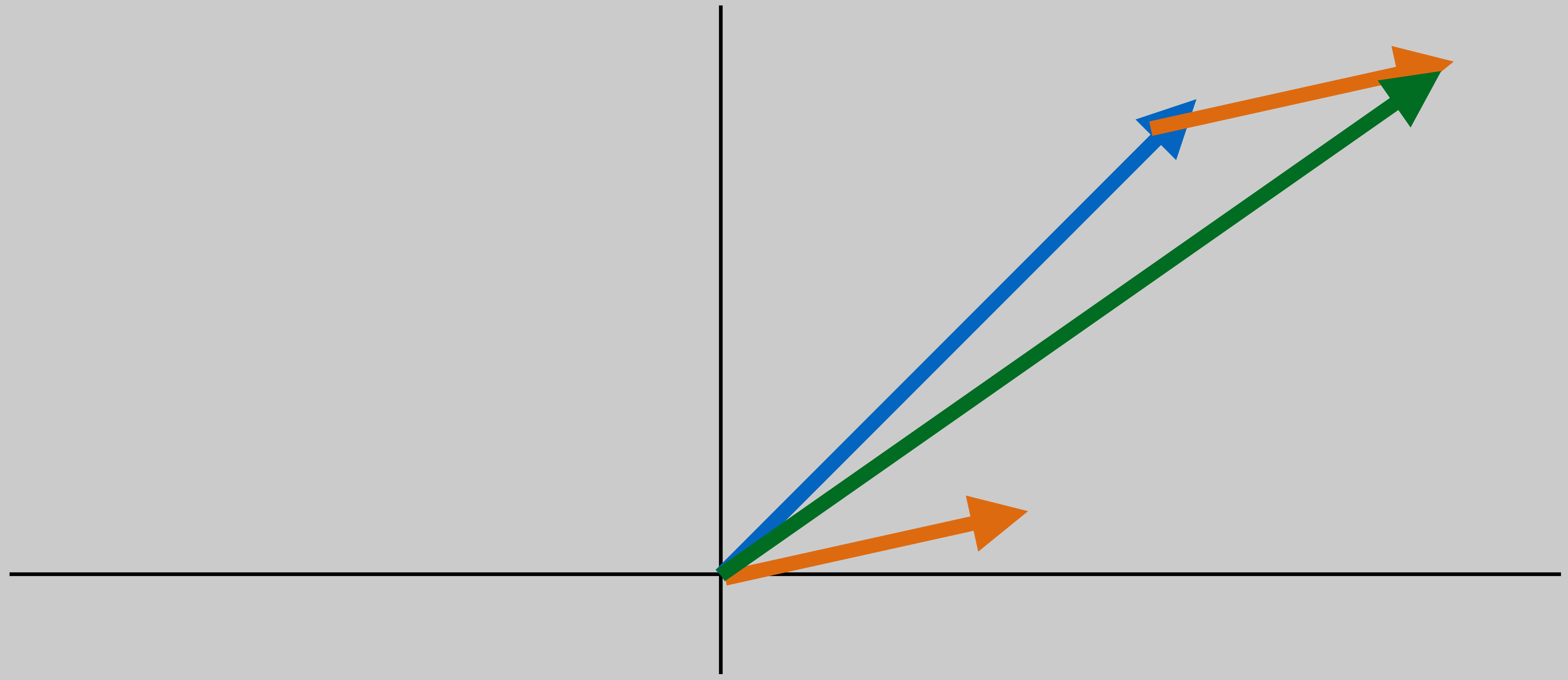
representing amplitude and phase of a sinusoidal time signal

In real time domain, $V_c(t) = |\tilde{V}_c| \cos(\omega t + \angle \tilde{V}_c) = V_0 \cos(\omega t + \theta)$

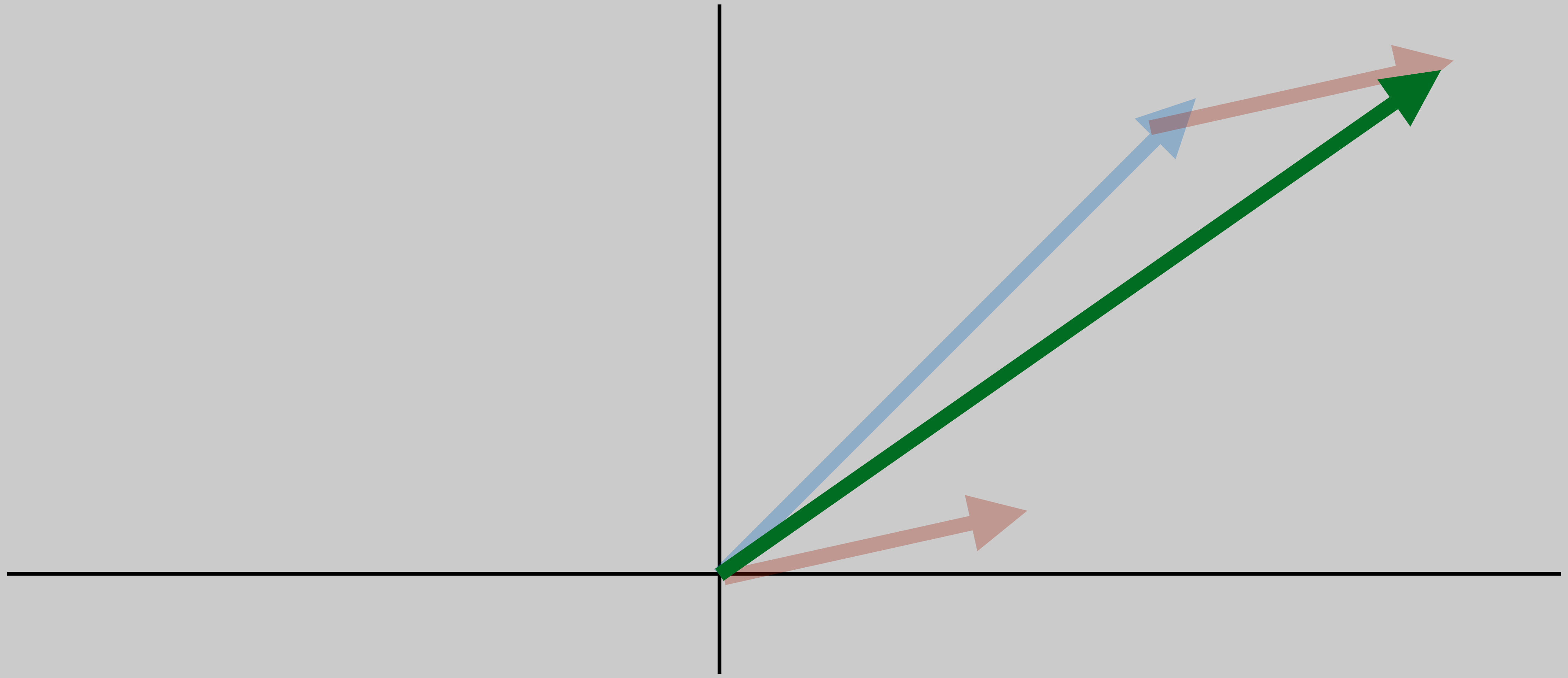
Phasor Arithmetic

$$\tilde{V} = |V| e^{j\theta}$$

$$\tilde{V}_1 + \tilde{V}_2 = \tilde{V}_1 + \tilde{V}_2 =$$



Phasor Arithmetic



$$\tilde{V} = |V| e^{j\theta}$$

$$\tilde{V}_1 + \tilde{V}_2 = \tilde{V}_1 + \tilde{V}_2 =$$

$$|V_1|\cos(\theta_1) + |V_2|\cos(\theta_2) + j\left(|V_1|\sin(\theta_1) + |V_2|\sin(\theta_2)\right)$$

$$\tilde{V}_1 \tilde{V}_2 = |V_1||V_2| e^{j(\theta_1+\theta_2)}$$

$$j\tilde{V} = \tilde{V} e^{j\frac{\pi}{2}}$$

$$\frac{\tilde{V}_1}{\tilde{V}_2} = \frac{|V_1|}{|V_2|} e^{j(\theta_1-\theta_2)}$$

$$\frac{\tilde{V}}{j} = \tilde{V} e^{-j\frac{\pi}{2}}$$

Impedance in AC circuits

Complex valued "resistance"

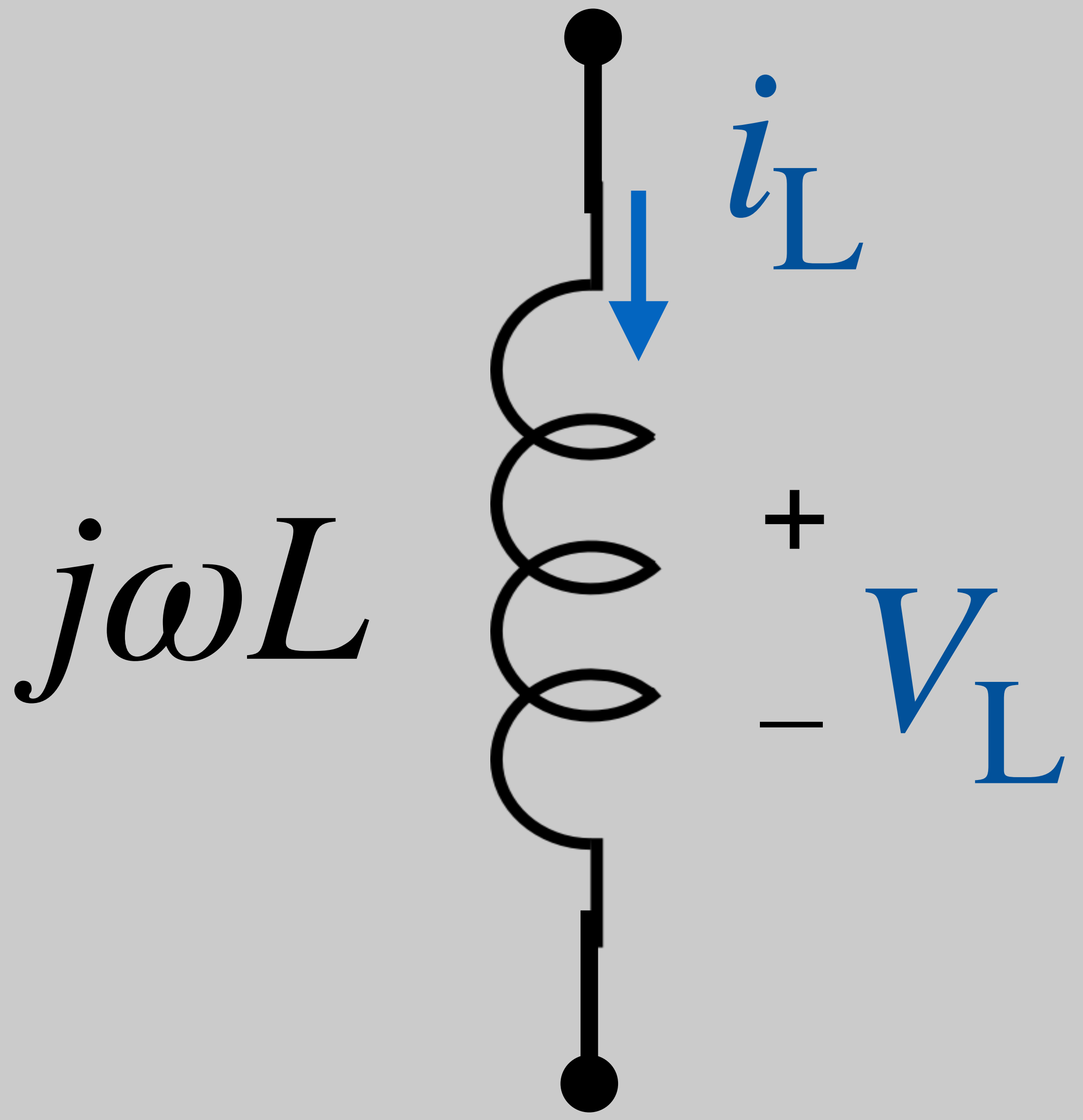
$$Z = R + jX$$

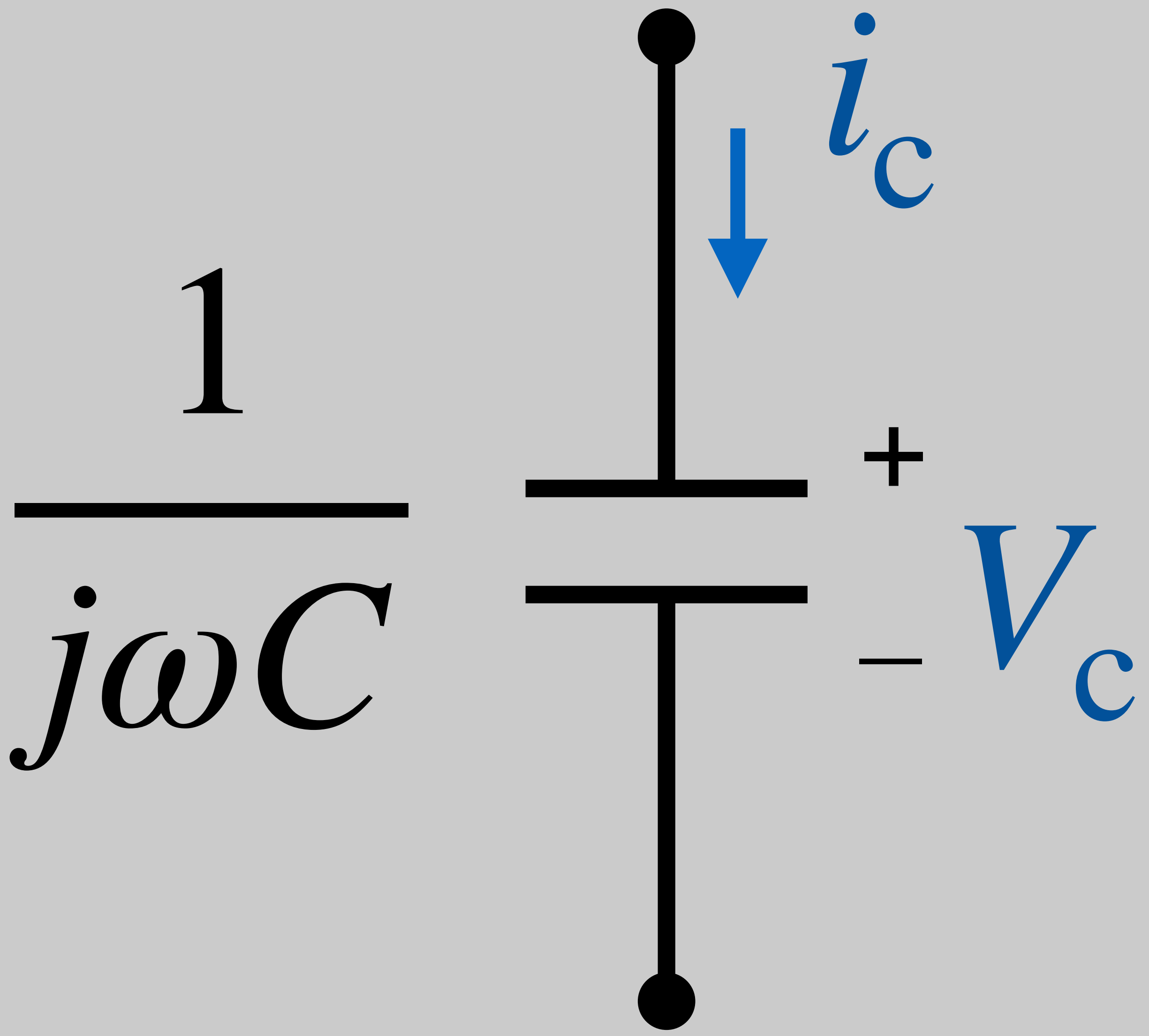
Impedance — Resistance — Reactance

$$Z^{-1} = Y = G + jB$$

Conductance — Susceptance

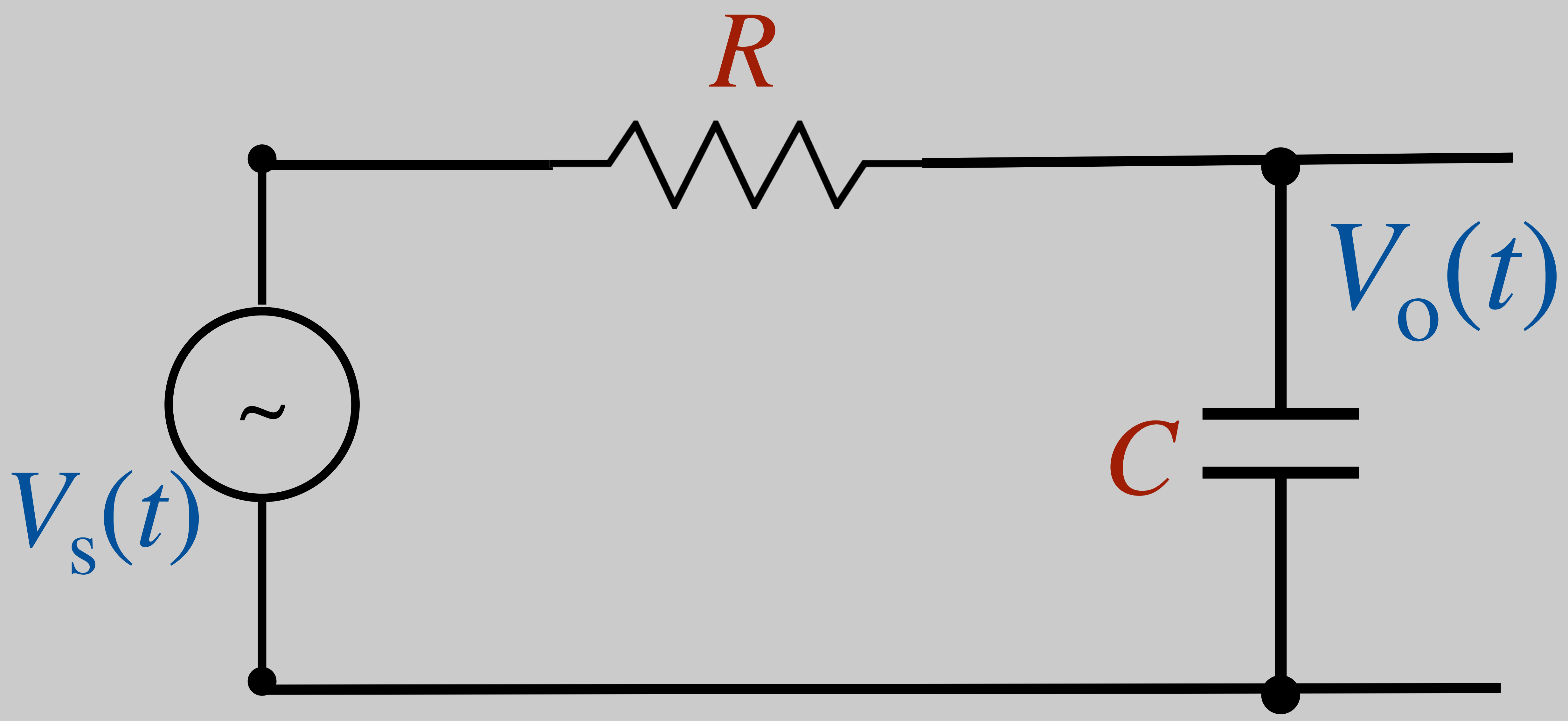
New Ohm's law: $\tilde{V} = \tilde{I}Z$





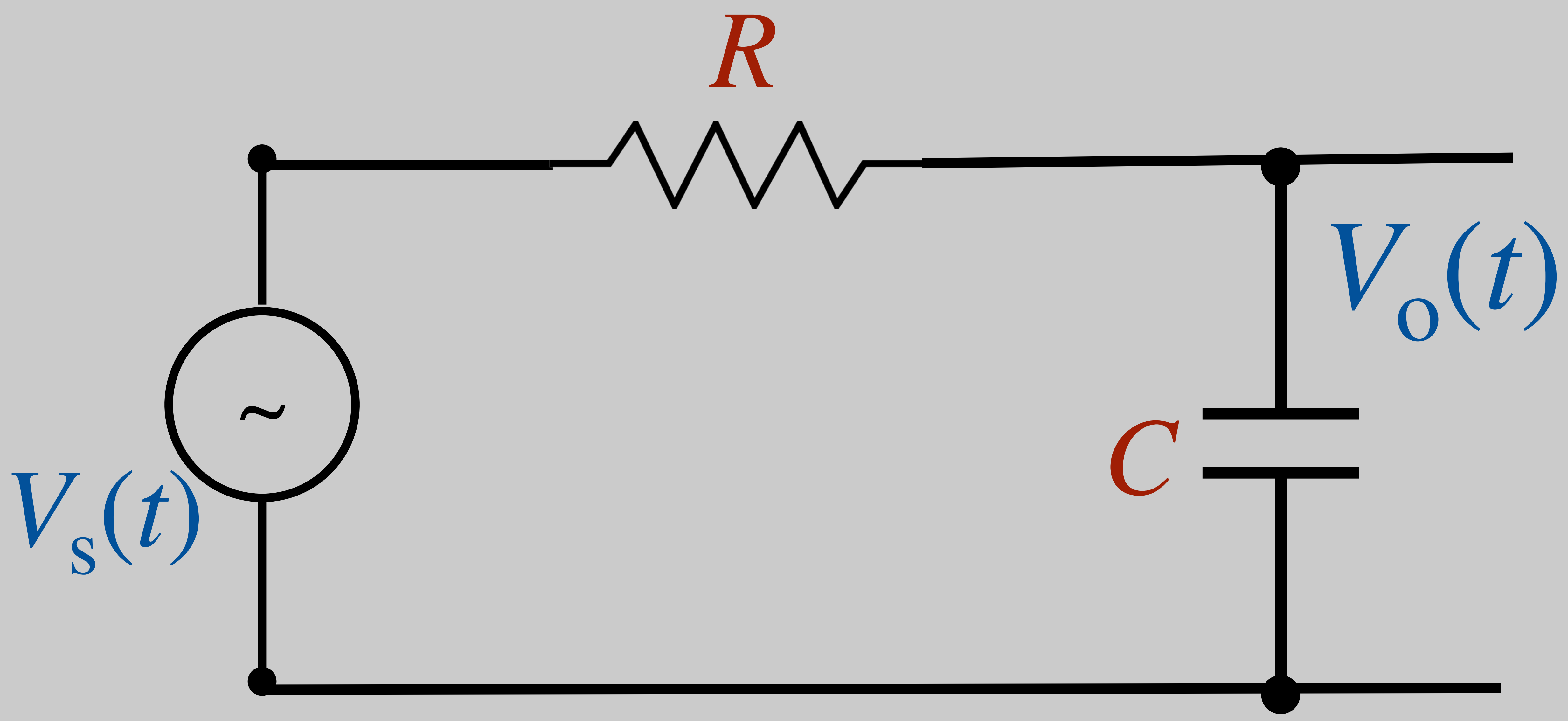
Example 1: AC Response of an RC

$$V_s(t) = V_{DD}\cos(\omega t)$$

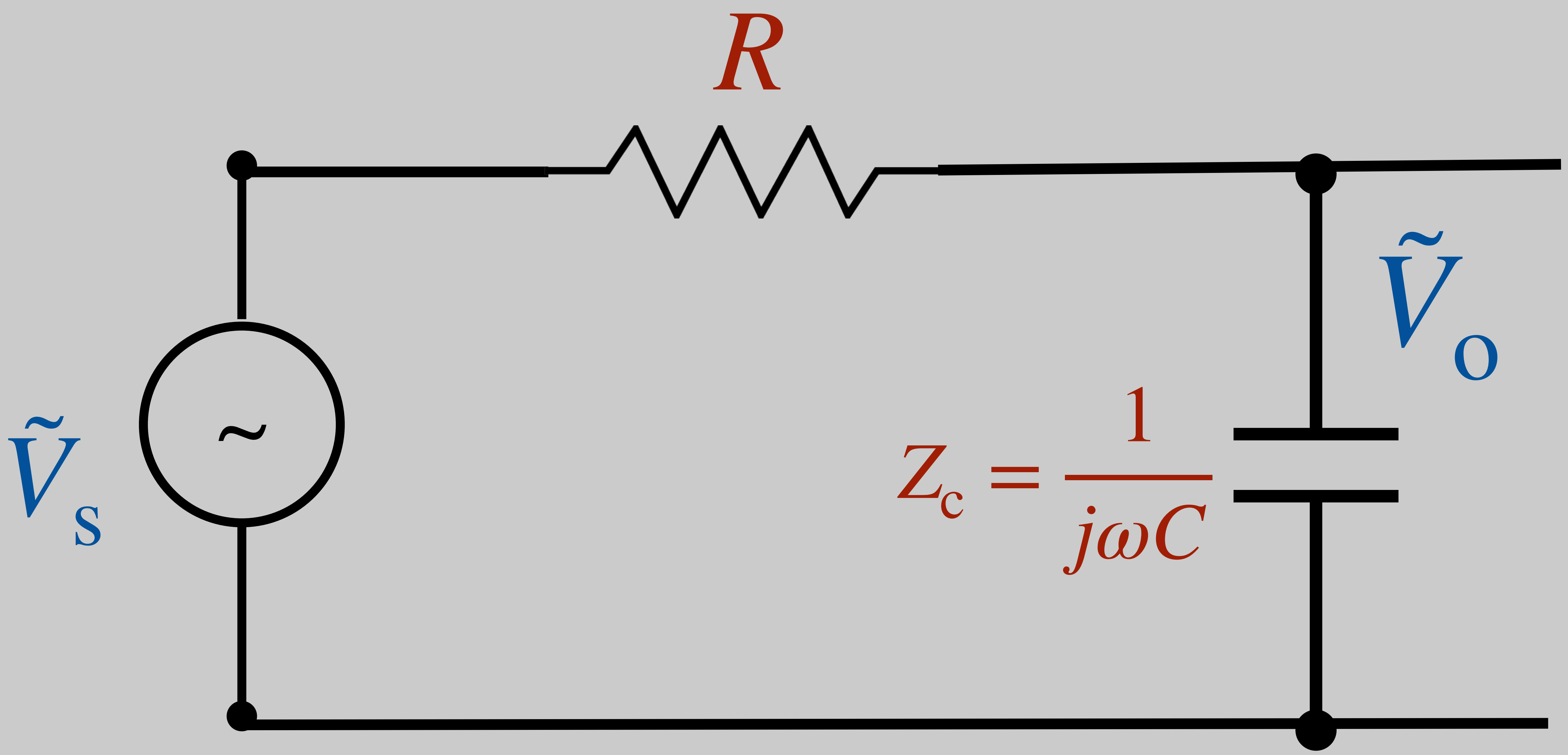


Example 1: AC Response of an RC

$$V_s(t) = V_{DD}\cos(\omega t)$$



AC responses using phasors:



Voltage divider:

$$\tilde{V}_o = \frac{Z_c}{R + Z_c}\tilde{V}_s$$

$$\tilde{V}_o = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}\tilde{V}_s$$

$$\tilde{V}_o = \frac{1}{j\omega RC + 1}\tilde{V}_s$$

Example 1: AC Response of an RC

$$\tilde{V}_o = \frac{1}{j\omega RC + 1}\tilde{V}_s$$

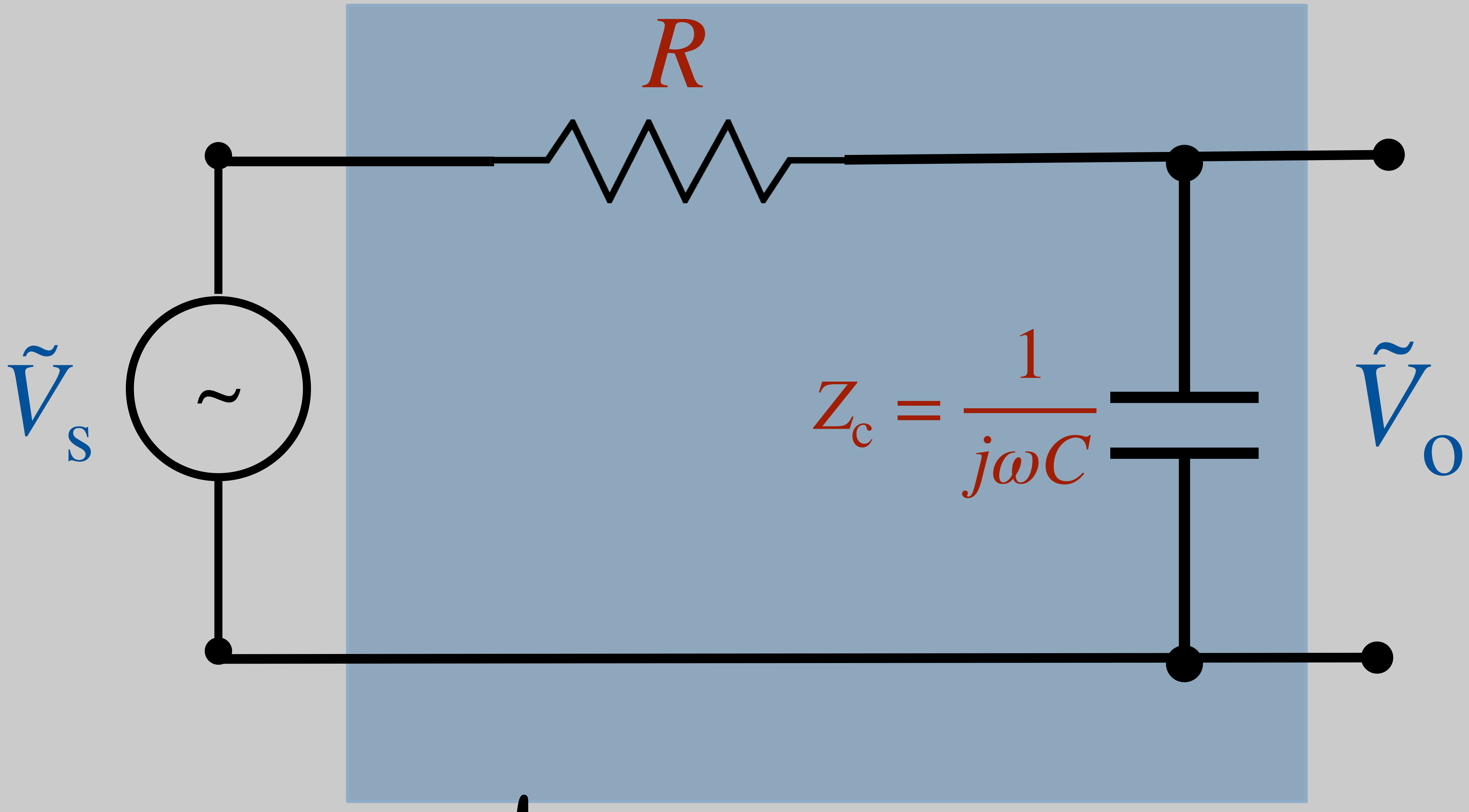
$$H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_s} = \frac{1}{j\omega RC + 1}$$

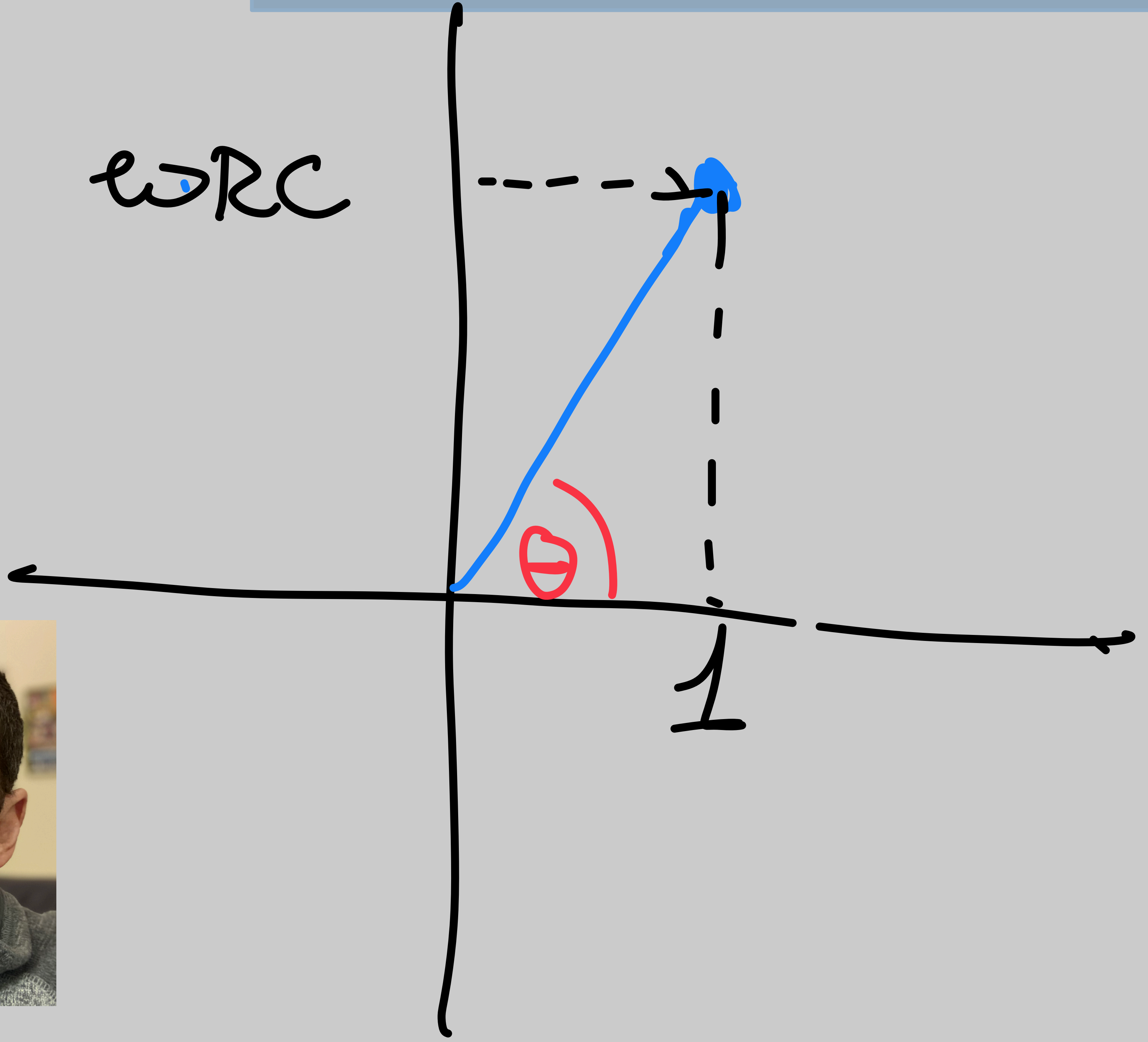
magnitude Resp:

$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2(RC)^2 + 1}}$$

phase Resp:

$$\angle H(j\omega) = -\,\text{atan}(\omega RC)$$





Example 1: AC Response of an RC

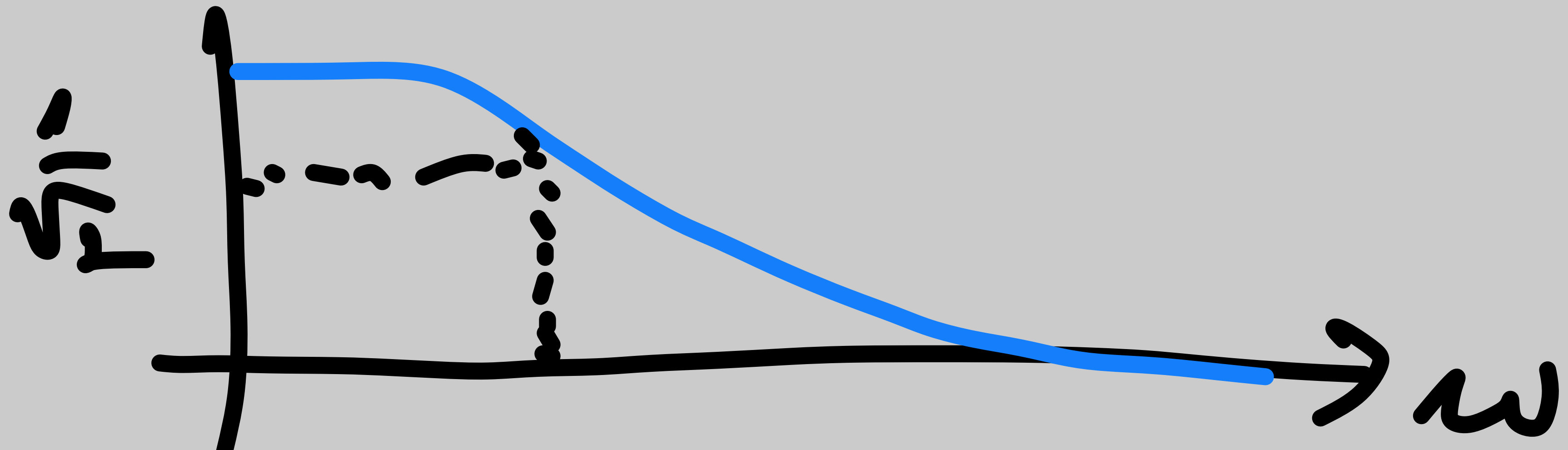
$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2(RC)^2 + 1}} \qquad \angle H(j\omega) = -\operatorname{atan}(\omega RC)$$

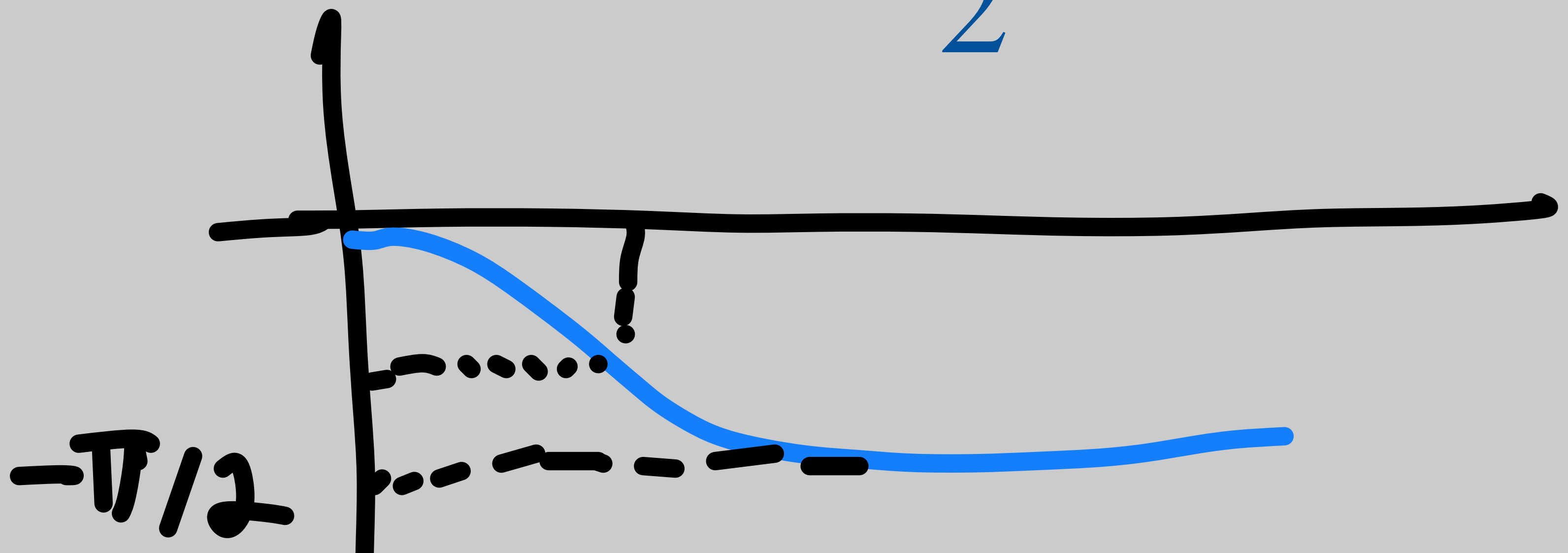
$\omega = 0$ $\qquad |H(j\omega)| = 1$ $\qquad \angle H(j\omega) = 0$

$\omega = \frac{1}{RC}$ $\qquad |H(j\omega)| = \frac{1}{\sqrt{2}}$ $\qquad \angle H(j\omega) = -\frac{\pi}{4} = -45^\circ$

$\omega \gg \frac{1}{RC}$ $\qquad |H(j\omega)| \approx \frac{1}{\omega(RC)}$

$\omega \to \infty$ $\qquad |H(j\omega)| \to 0$ $\qquad \angle H(j\omega) = -\frac{\pi}{2} = -90^\circ$



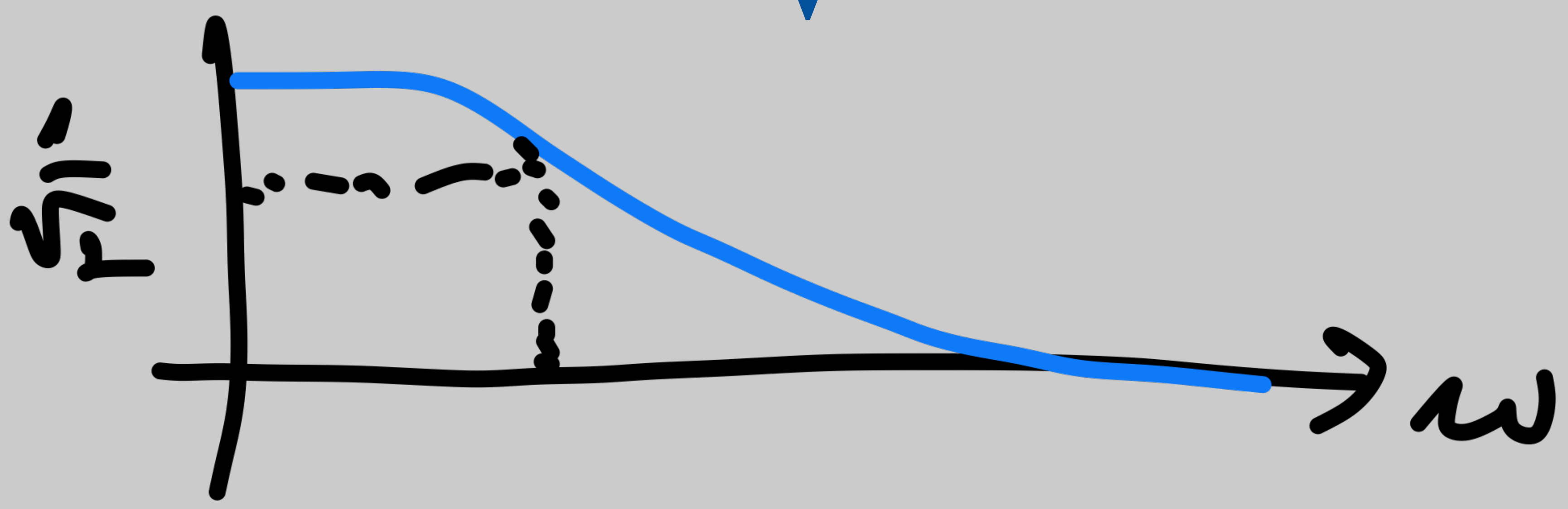


Example 1: AC Response of an RC

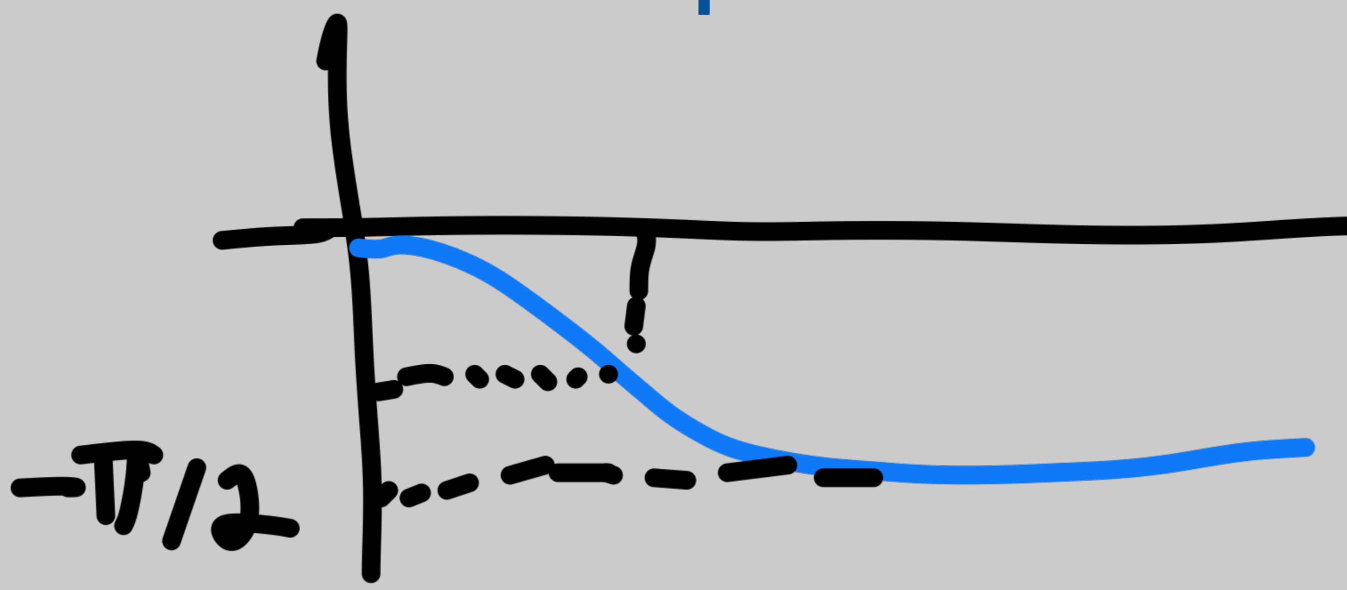
$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2(RC)^2 + 1}} \qquad \angle H(j\omega) = -\operatorname{atan}(\omega RC)$$

$$\omega = \frac{1}{RC} = 2\pi 100$$

$$|H(j\omega)| = \frac{1}{\sqrt{2}}$$



$$\angle H(j\omega) = -\frac{\pi}{4} = -45^\circ$$



Input:

$$V_s(t) = V_{DD}\cos(2\pi \cdot 100t)$$

Output:

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