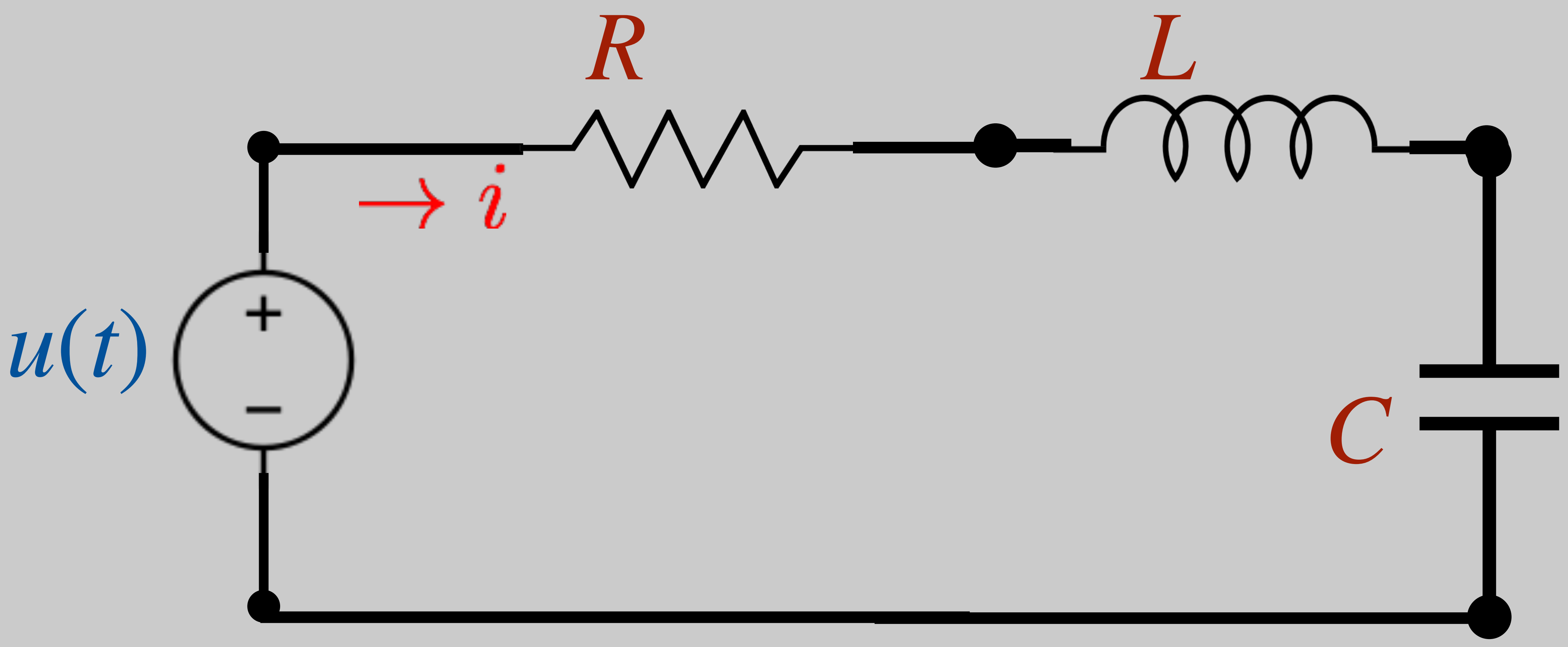
EE16B
Designing Information Devices and Systems II

Lecture 3B
RLC circuits and 2nd Order Diff EQ

Announcements

- Last time:
  - Transistors Switch RC model
  - Finish review of complex numbers
    - Euler Formula
  - Solution for sinusoidal input
- Today:
  - Second order systems
  - Intro to VDE
  - RLC circuits
  - Resonant Tank

Example 1: RLC Circuits

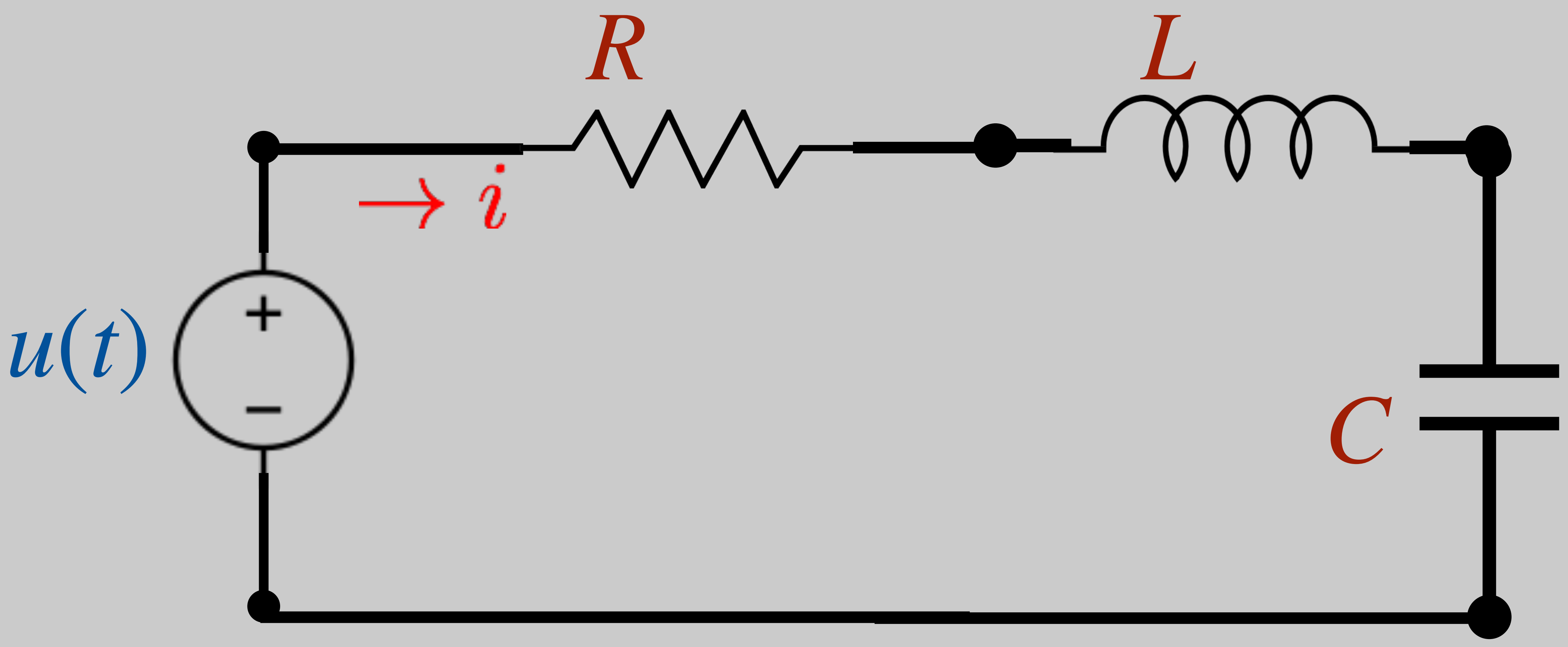


From KVL: $V_R + V_L + V_C = u$

$$L\frac{di(t)}{dt} = V_L = u - V_c - V_R = u - V_c - Ri$$

$$C\frac{dV_C(t)}{dt} = i$$

Example 1: RLC Circuits



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$$C\frac{dV_C(t)}{dt} = i$$

$$\frac{d}{dt}V_c(t) = \frac{1}{C}i(t)$$

$$\frac{d}{dt}i(t) = -\frac{1}{L}V_c(t) - \frac{R}{L}i(t) + \frac{1}{L}u(t)$$

Example 1: RLC Circuits

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Two (coupled) 1st order Diff. EQ!

Write in matrix form:

$$\begin{bmatrix} \frac{d}{dt}V_{c_1} \\ \frac{d}{dt}i \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} V_c \\ i \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u(t)$$

Example 1: RLC Circuits

$$\frac{d}{dt}V_c(t) = \frac{1}{C}i(t)$$

$$\frac{d}{dt}i(t) = -\frac{1}{L}V_c(t) - \frac{R}{L}i(t) + \frac{1}{L}u(t)$$

Two (coupled) 1st order Diff. EQ!

Write in matrix form:

$$\begin{bmatrix} \frac{d}{dt}V_{c_1} \\ \frac{d}{dt} i \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} V_c \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

$$\frac{d}{dt}\vec{x}(t) \qquad A \qquad \vec{x}(t) \qquad \vec{B}$$

VDE / State Space

$$\begin{bmatrix} \frac{d}{dt}V_{c_1} \\ \frac{d}{dt}V_{c_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} V_c \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

General Case:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

Also called state-space (more in week 6!)

Q: How to solve for $\vec{x}(t)$?

Change of Variables - diagonalization

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

Let T be an invertible matrix: $\vec{z}(t) = T\vec{x}(t) \Leftrightarrow \vec{x}(t) = T^{-1}\vec{z}(t)$

Then, $$\frac{d}{dt}\vec{z}(t) = T\frac{d}{dt}\vec{x}(t) = TA\vec{x}(t) + TBu(t)$$

$$= TAT^{-1}\vec{z}(t) + TBu(t)$$

Choose T, such that $A_{\text{new}} = TAT^{-1}$ is diagonal!

$$\frac{d}{dt}\vec{z}(t) = A_{\text{new}}\vec{z}(t) + B_{\text{new}}u(t)$$

Change of Variables - diagonalization

$$\frac{d}{dt}\vec{z}(t) = A_{\text{new}}\vec{z}(t) + B_{\text{new}}u(t)$$

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

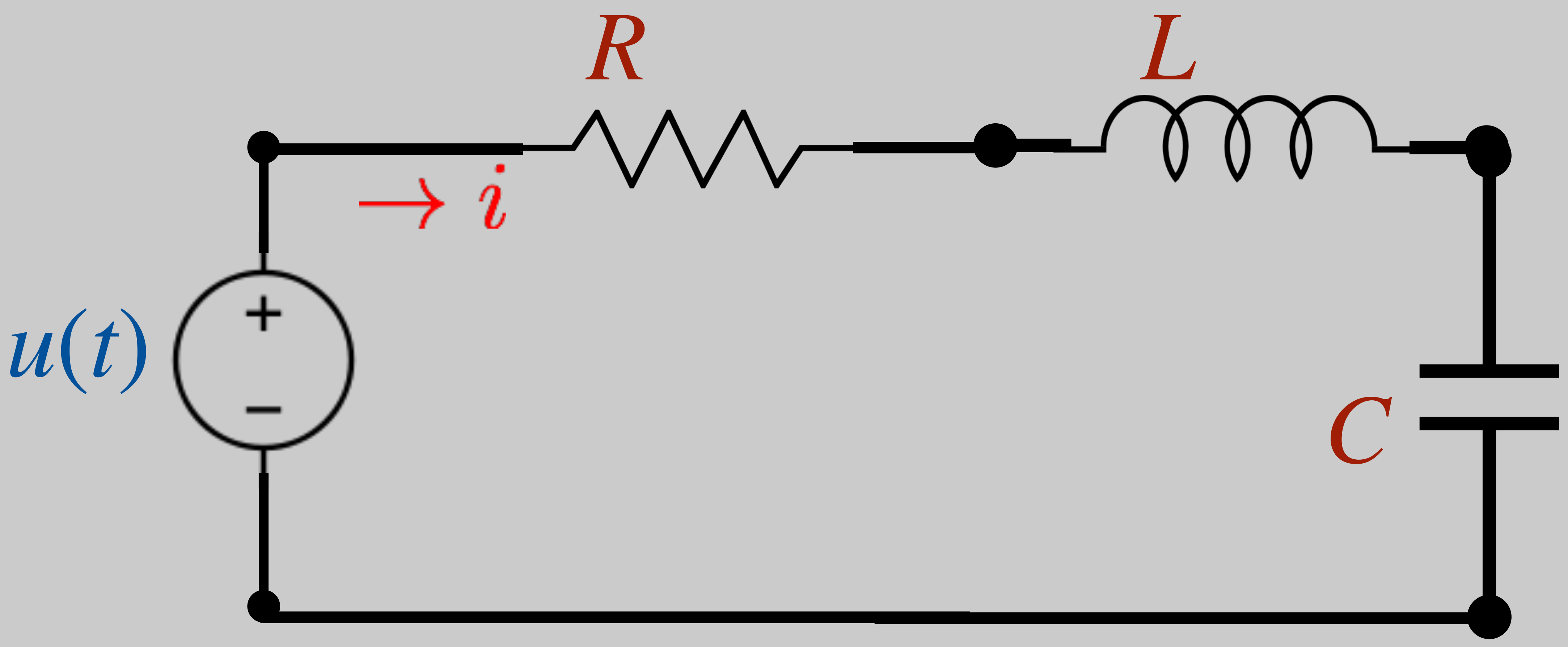
$$\overrightarrow{v(t)} = B_{\text{new}}u(t)$$

$$\frac{d}{dt}\vec{z}(t) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\vec{z}(t) + \overbrace{B_{\text{new}}u(t)}$$

$$\frac{d}{dt}z_1(t) = \lambda_1 z_1(t) + v_1(t)$$

Go back to original variable by: $\vec{x}(t) = T^{-1}\vec{z}(t)$

Back to the RLC circuit



(1) $\frac{d}{dt}V_c(t) = \frac{1}{C}i(t)$

(2) $\frac{d}{dt}i(t) = -\frac{1}{L}V_c(t) - \frac{R}{L}i(t) + \frac{1}{L}u(t)$

Time derivative of (1) Substitute (2)

$$\Rightarrow \frac{d^2}{dt^2}V_c(t) = \frac{1}{C}\frac{d}{dt}i(t) = -\frac{1}{LC}V_c(t) - \frac{R}{LC}i(t) + \frac{1}{LC}u(t)$$

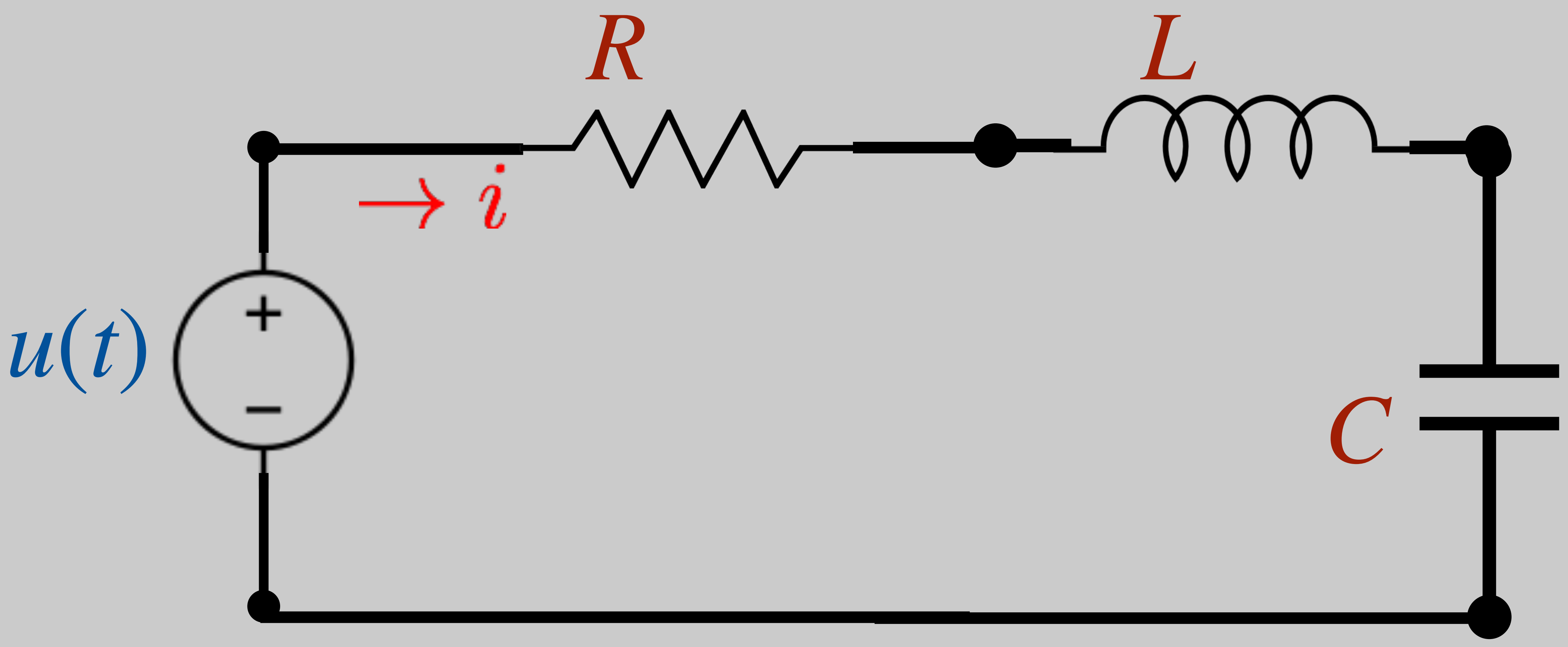
Substitute (1)

$$= -\frac{1}{LC}V_c(t) - \frac{RC}{LC}\frac{d}{dt}V_c(t) + \frac{1}{LC}u(t)$$

$$\frac{d^2}{dt^2}V_c(t) + \frac{R}{L}\frac{d}{dt}V_c(t) + \frac{1}{LC}V_c(t) = \frac{1}{LC}u(t)$$

One 2nd order Diff. EQ!

Homogeneous Solution



$$\frac{d^2}{dt^2}V_c(t) + \frac{R}{L}\frac{d}{dt}V_c(t) + \frac{1}{LC}V_c(t) = 0$$

Define:

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = 0$$

Guess: $V_c(t) = Ae^{st}$

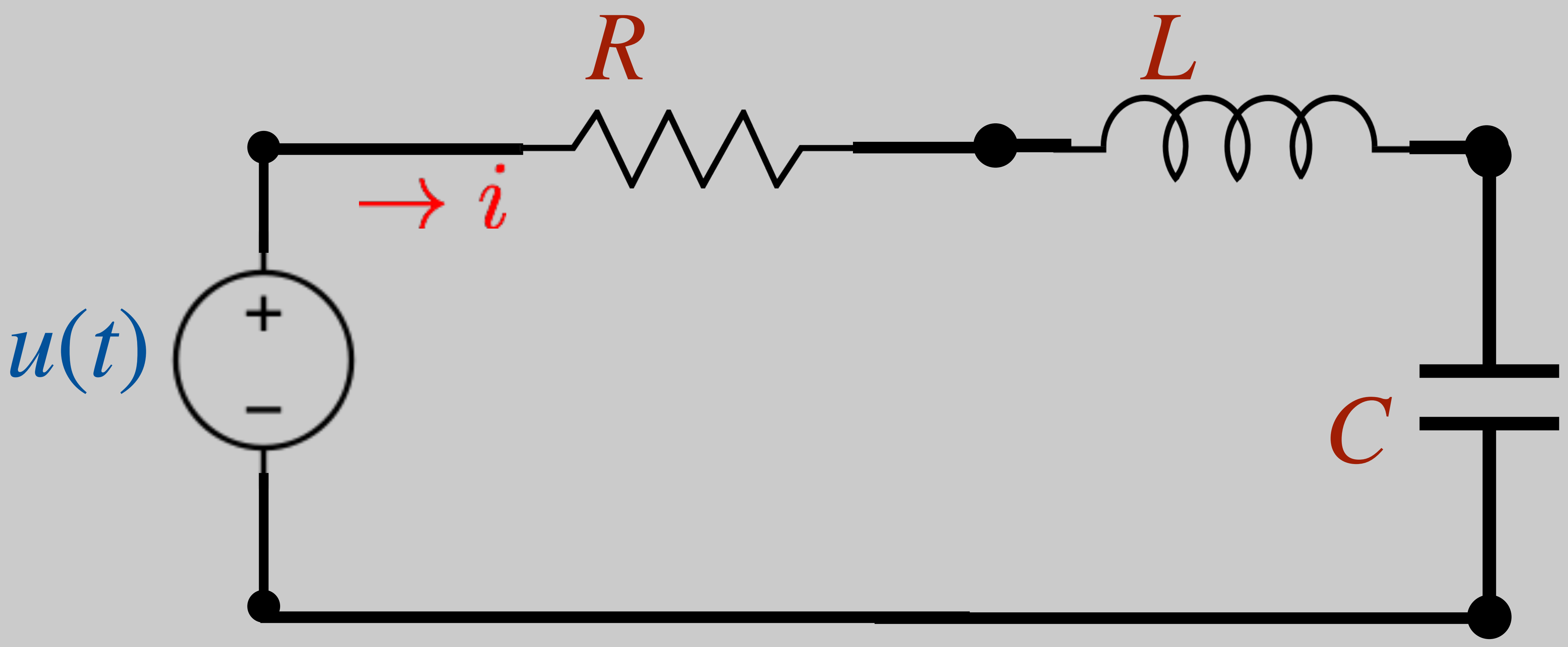
$$As^2e^{st} \quad +2\alpha Ase^{st} \quad +\omega_0^2 Ae^{st} \ = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Homogeneous Solution



$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = 0$$

$a = \dfrac{R}{2L}$ Damping coefficient (associated with decay)

$\omega_0 = \dfrac{1}{\sqrt{LC}}$ Resonance frequency (associated with oscillations)

$\zeta = \dfrac{\alpha}{\omega_0}$ Damping Ratio

$$V_c(t) = Ae^{st}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

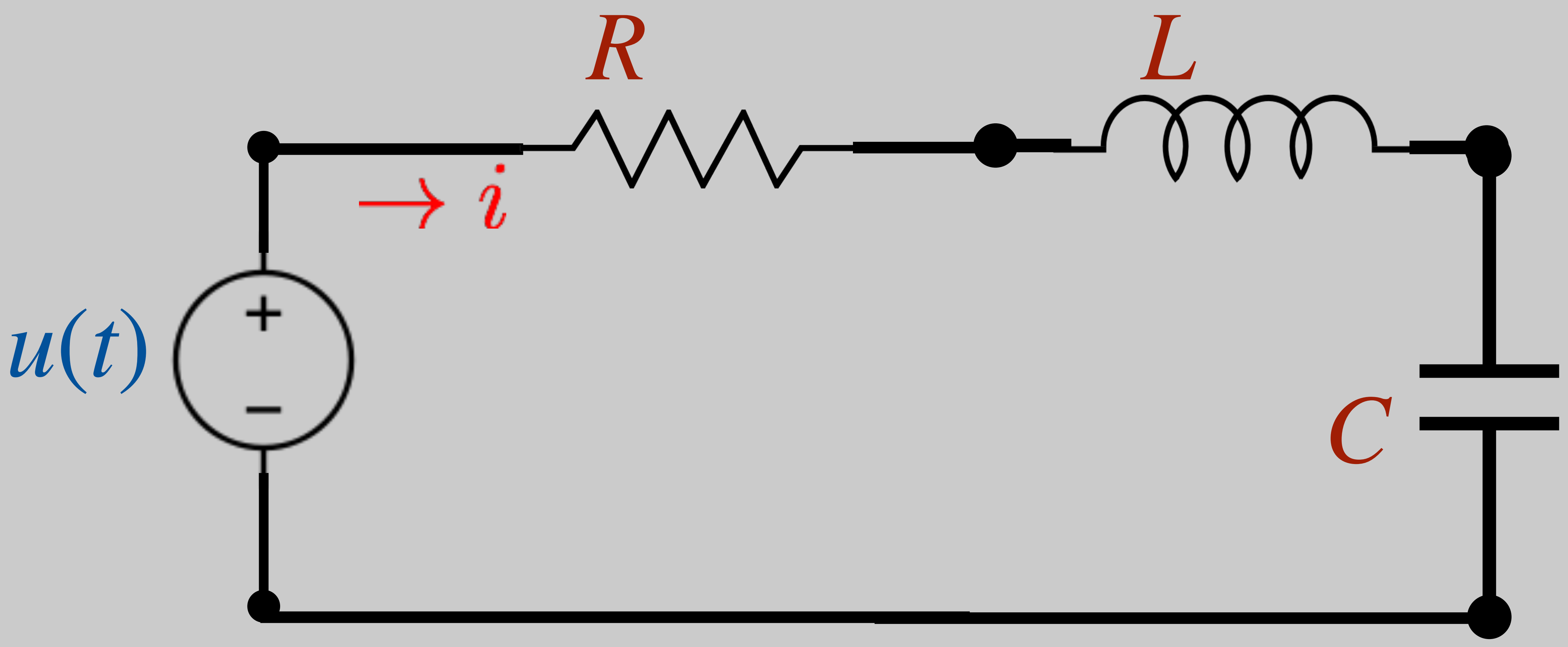
$\alpha > \omega_0, \quad \zeta > 1 \Rightarrow$ 2 real solutions $\Rightarrow$ Exponential decay

$\alpha = \omega_0, \quad \zeta = 1 \Rightarrow$ 2 same solutions $\Rightarrow$ Exponential decay

$\alpha < \omega_0, \quad \zeta < 1 \Rightarrow$ 2 complex conj. solutions $\Rightarrow$ decay+oscillations

Overdamped

$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = 0$$



$a > \omega_0$ 2 real solutions:

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$

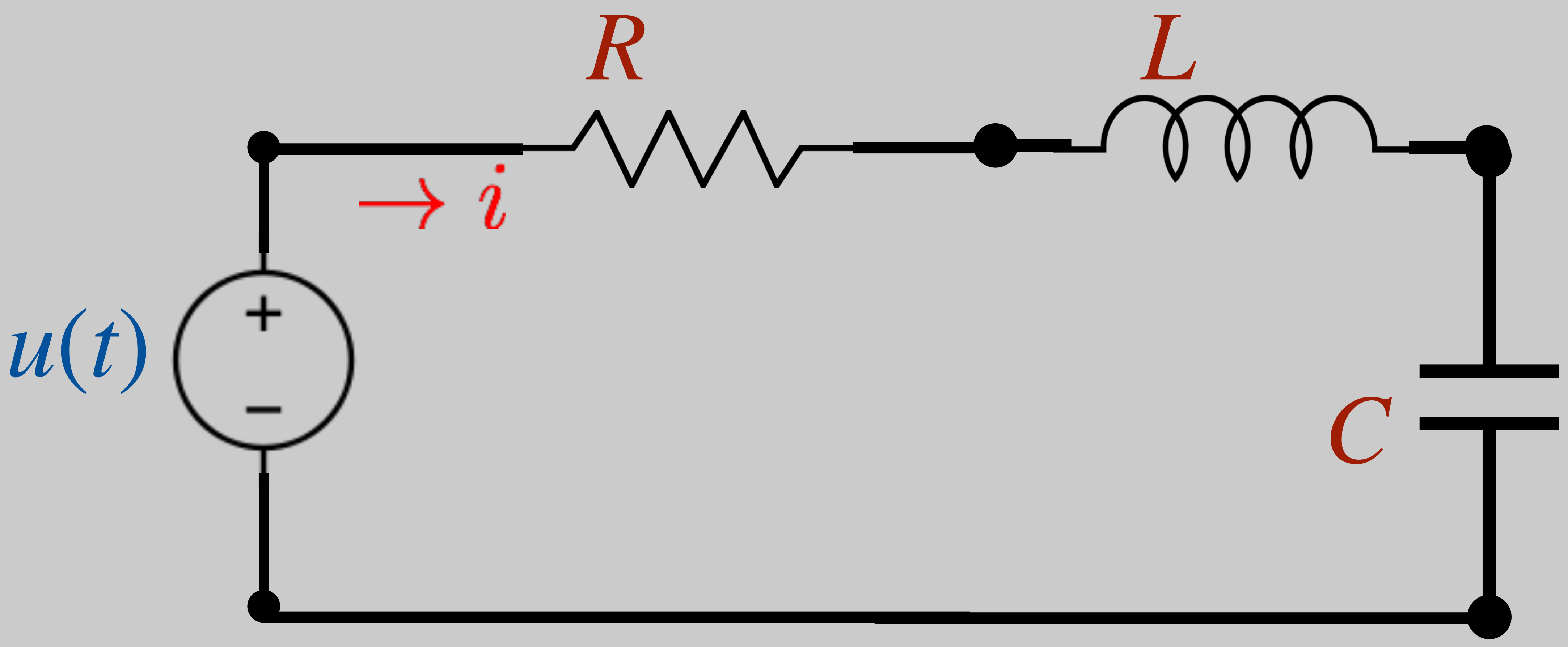
From Initial conditions:

$$u(0) = V_{DD}$$

$$V_c(0) = V_{DD} \quad \Rightarrow V_c(0) = Ae^0 + Be^0 = V_{DD} \quad \Rightarrow A + B = V_{DD}$$

$$i_L(0) = 0 \quad \Rightarrow i_L(0) = 0 \quad \Rightarrow C\frac{d}{dt}V_c(0) = 0 \quad \Rightarrow As_1 + Bs_2 = 0$$

Overdamped



$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = 0$$

$a > \omega_0$ 2 real solutions:

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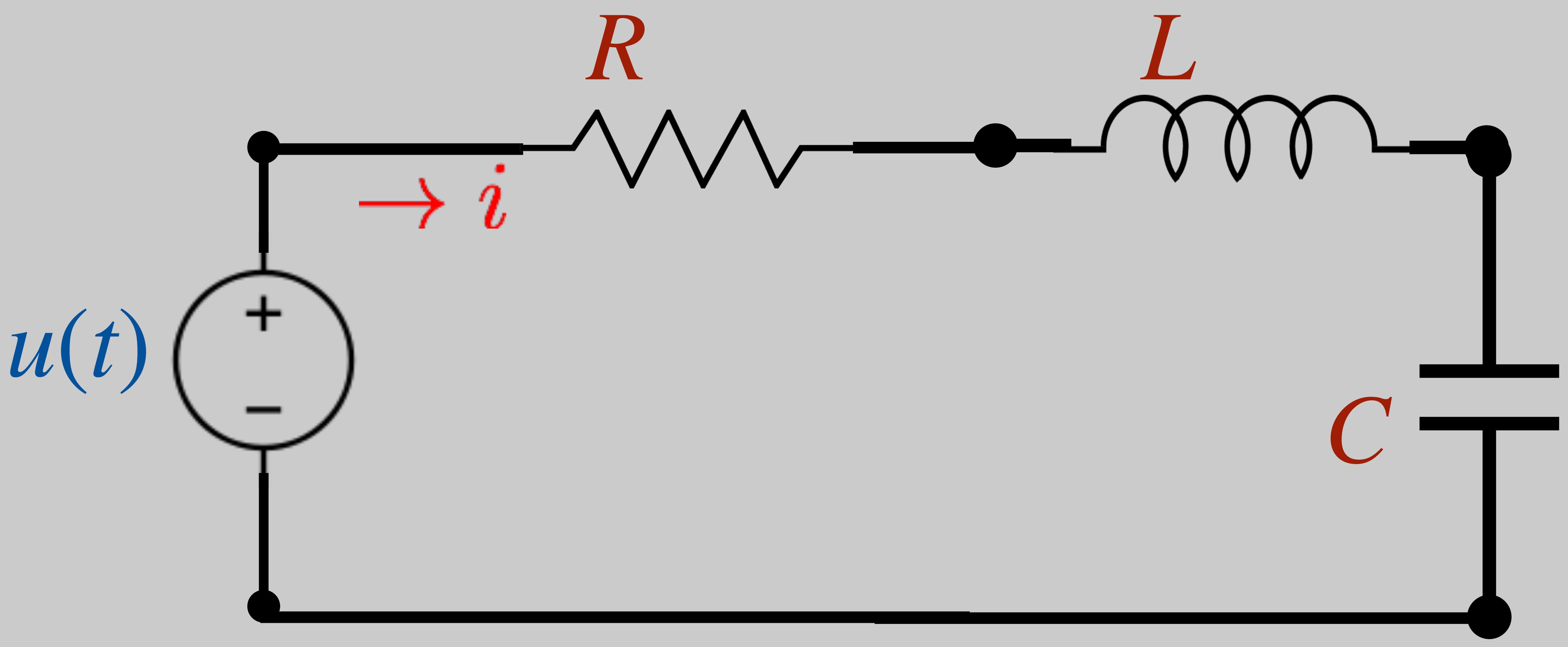
$i_L(0) = 0 \quad \Rightarrow i_L(0) = 0 \Rightarrow C\frac{d}{dt}V_c(0) = 0 \quad \Rightarrow As_1 + Bs_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ s_1 & s_2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{bmatrix} V_{DD} \\ 0 \end{bmatrix} \Rightarrow \frac{1}{s_2 - s_1}\begin{pmatrix} s_2 & -1 \\ -s_1 & 1 \end{pmatrix}\begin{pmatrix} V_{DD} \\ 0 \end{pmatrix} =$$

$$A = \frac{s_2}{s_2 - s_1}V_{DD}$$

$$B = -\frac{s_1}{s_2 - s_1}V_{DD}$$

Overdamped



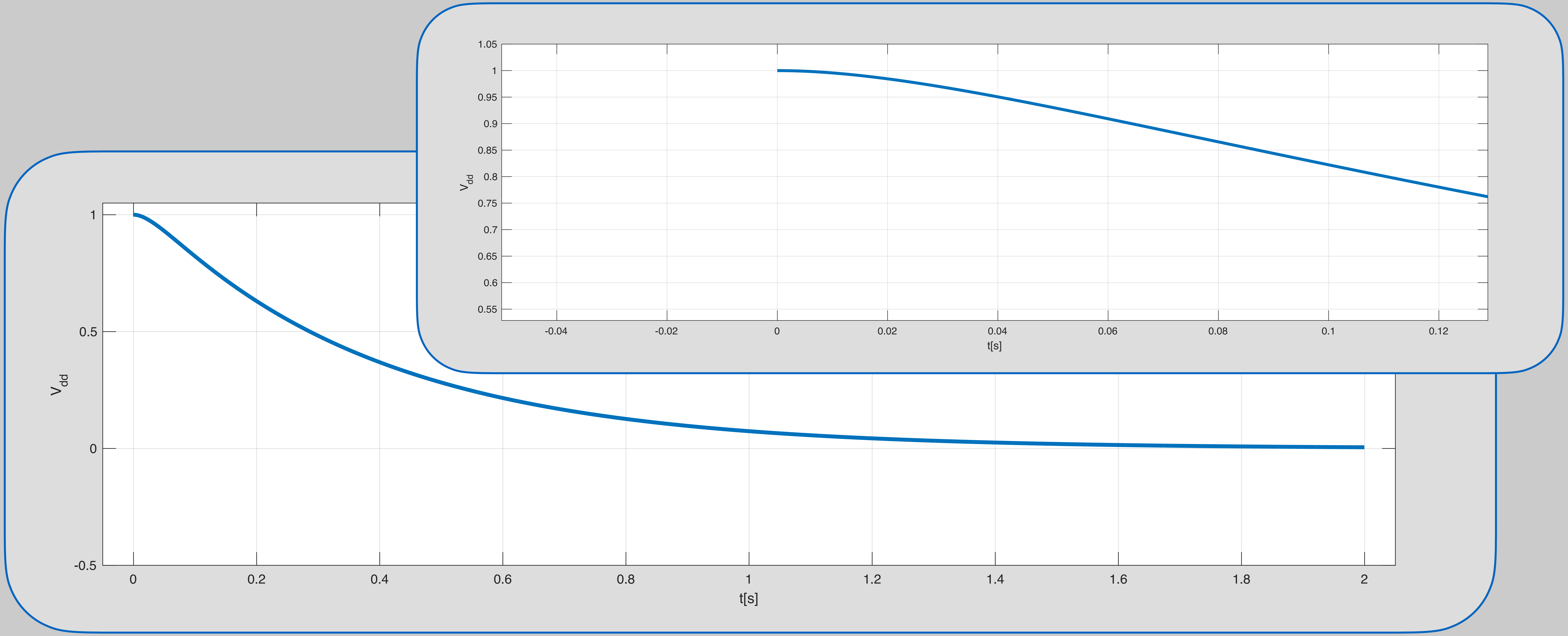
$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$$A = \frac{s_2}{s_2 - s_1} V_{DD} \qquad B = -\frac{s_1}{s_2 - s_1} V_{DD}$$

$\alpha = 20$

$\omega_0 = 10$

$\zeta = 2$

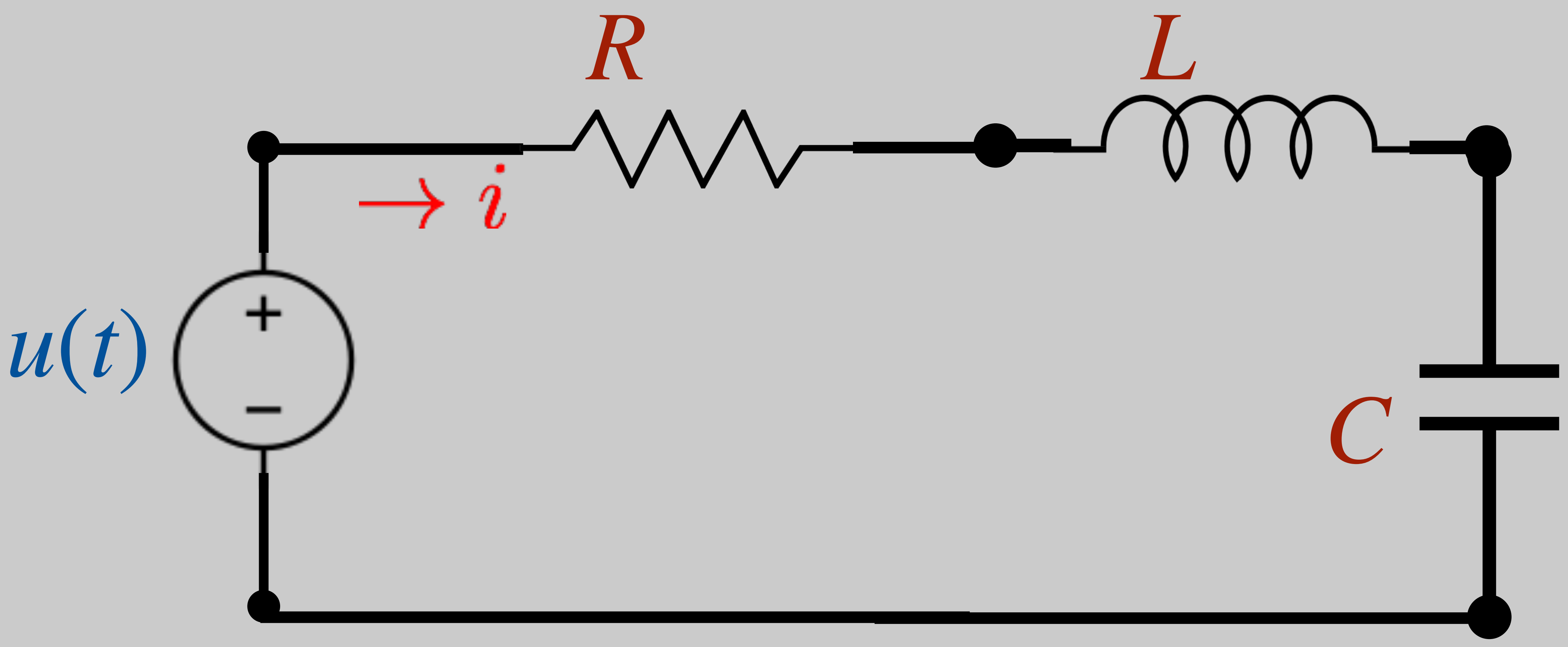


Underdamped

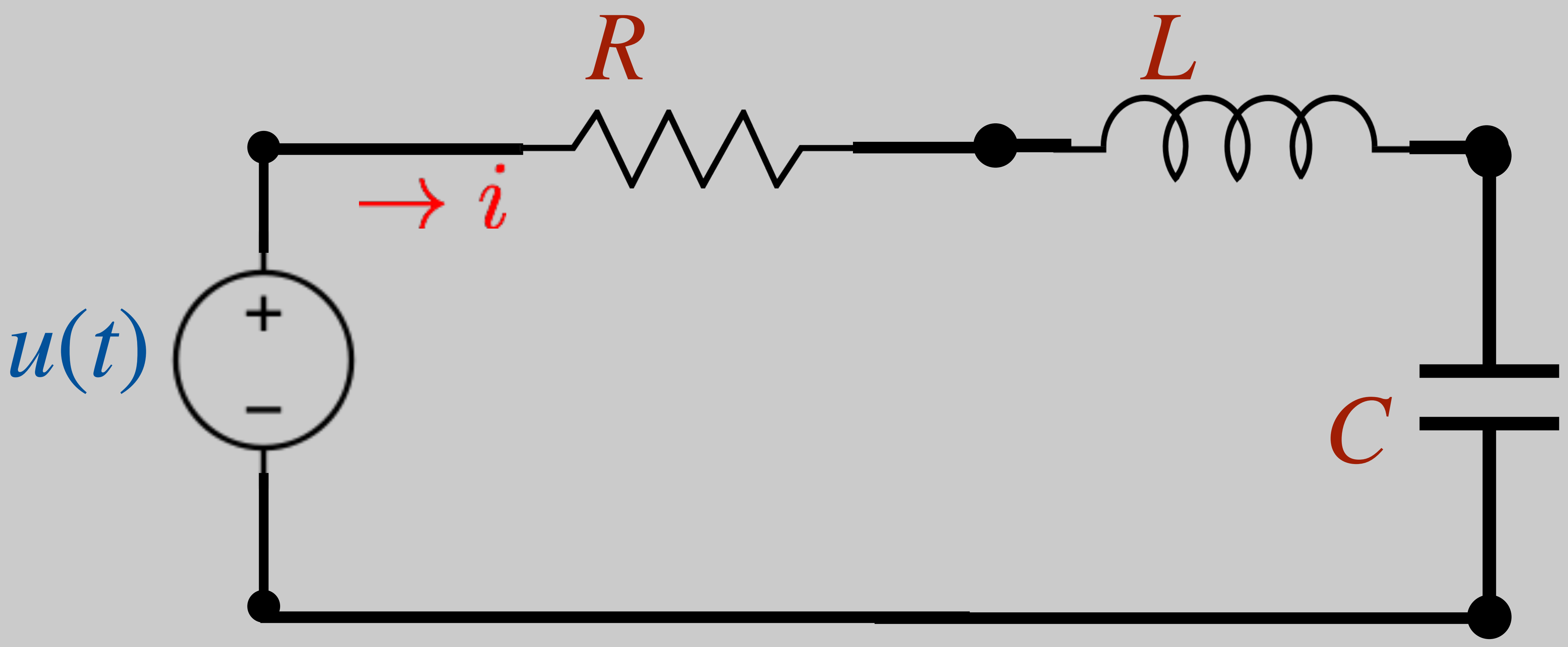
$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = 0$$

$a < \omega_0$ 2 complex conjugate solutions:

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$



Underdamped



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$a < \omega_0$ 2 complex conjugate solutions:

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$

From Initial conditions:

$u(0) = V_{DD}$

$V_c(0) = V_{DD} \quad \Rightarrow V_c(0) = Ae^0 + Be \ldots + B = V_{DD}$

$i_L(0) = 0 \quad \Rightarrow i_L(0) = 0 \quad \Rightarrow C\frac{d}{dt}V_c(0) = 0 \quad \Rightarrow \ldots$

Same as before!!!!

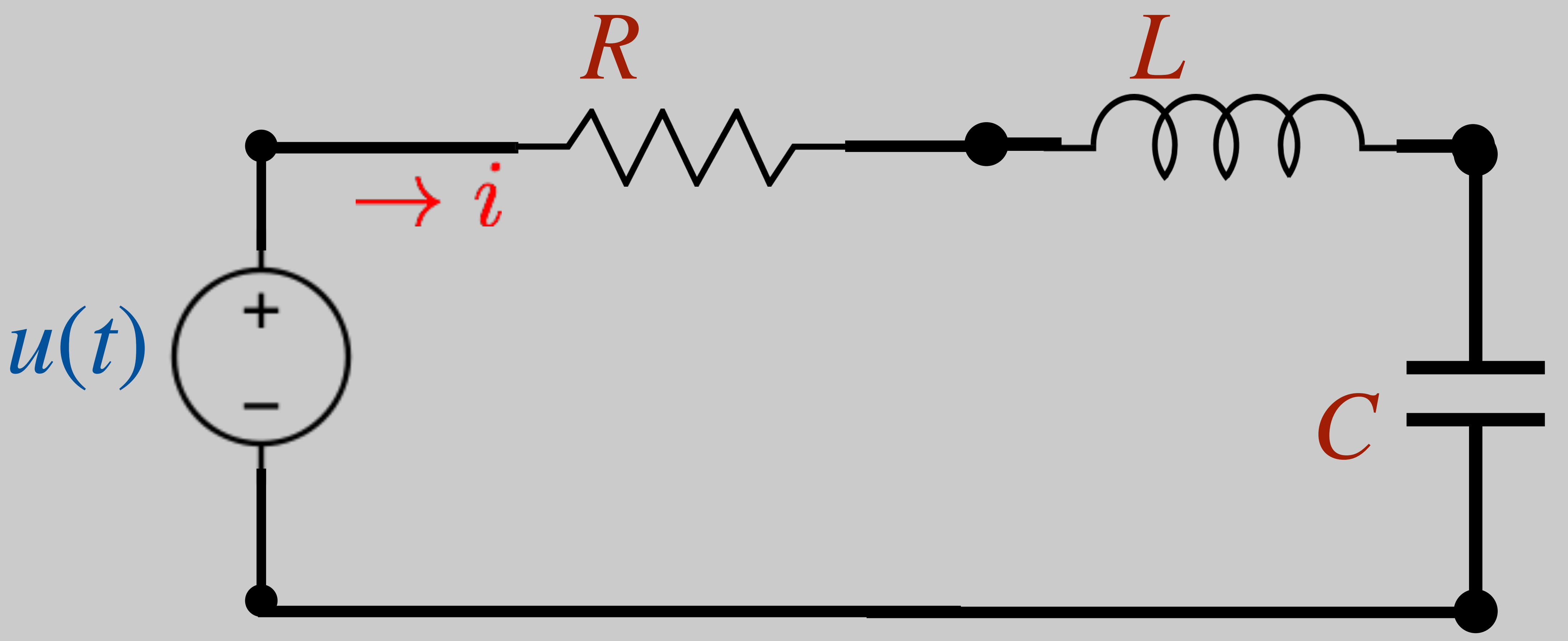
$$\begin{pmatrix} 1 & 1 \\ s_1 & s_2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{bmatrix} V_{DD} \\ 0 \end{bmatrix} \Rightarrow \frac{1}{s_2 - s_1}\begin{pmatrix} s_2 & -1 \\ -s_1 & 1 \end{pmatrix}\begin{pmatrix} V_{DD} \\ 0 \end{pmatrix} =$$

$\ldots \frac{\ldots}{\ldots s_1} V_{DD}$

$$B = -\frac{s_1}{s_2 - s_1}V_{DD}$$

~~Overdamped~~ Underdamped

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$



$$A = \frac{s_2}{s_2 - s_1} V_{DD} \qquad B = -\frac{s_1}{s_2 - s_1} V_{DD}$$

$a < \omega_0$ 2 complex conjugate solutions:

$$A = \frac{-\alpha - j\omega_n}{-2j\omega_n} V_{DD} \qquad B = -\frac{-\alpha + j\omega_n}{-2j\omega_n} V_{DD}$$

$$V_c(t) = \frac{\alpha + j\omega_n}{2j\omega_n} V_{DD} e^{(-\alpha + j\omega_n)t} + \frac{-\alpha + j\omega_n}{2j\omega_n} V_{DD} e^{(-\alpha - j\omega_n)t}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$j\omega_n = \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = -\alpha + j\omega_n$$

$$s_2 = -\alpha - j\omega_n$$

Overdamped Underdamped

$$V_c(t) = \frac{\alpha + j\omega_n}{2j\omega_n} V_{DD} e^{(-\alpha + j\omega_n)t} + \frac{-\alpha + j\omega_n}{2j\omega_n} V_{DD} e^{(-\alpha - j\omega_n)t}$$

$$= V_{DD} e^{-\alpha t} \left(\frac{e^{j\omega_n t} + e^{-j\omega_n t}}{2} \right)$$

Overdamped Underdamped

$$V_c(t) = \frac{\alpha + j\omega_n}{2j\omega_n} V_{DD} e^{(-\alpha + j\omega_n)t} + \frac{-\alpha + j\omega_n}{2j\omega_n} V_{DD} e^{(-\alpha - j\omega_n)t}$$

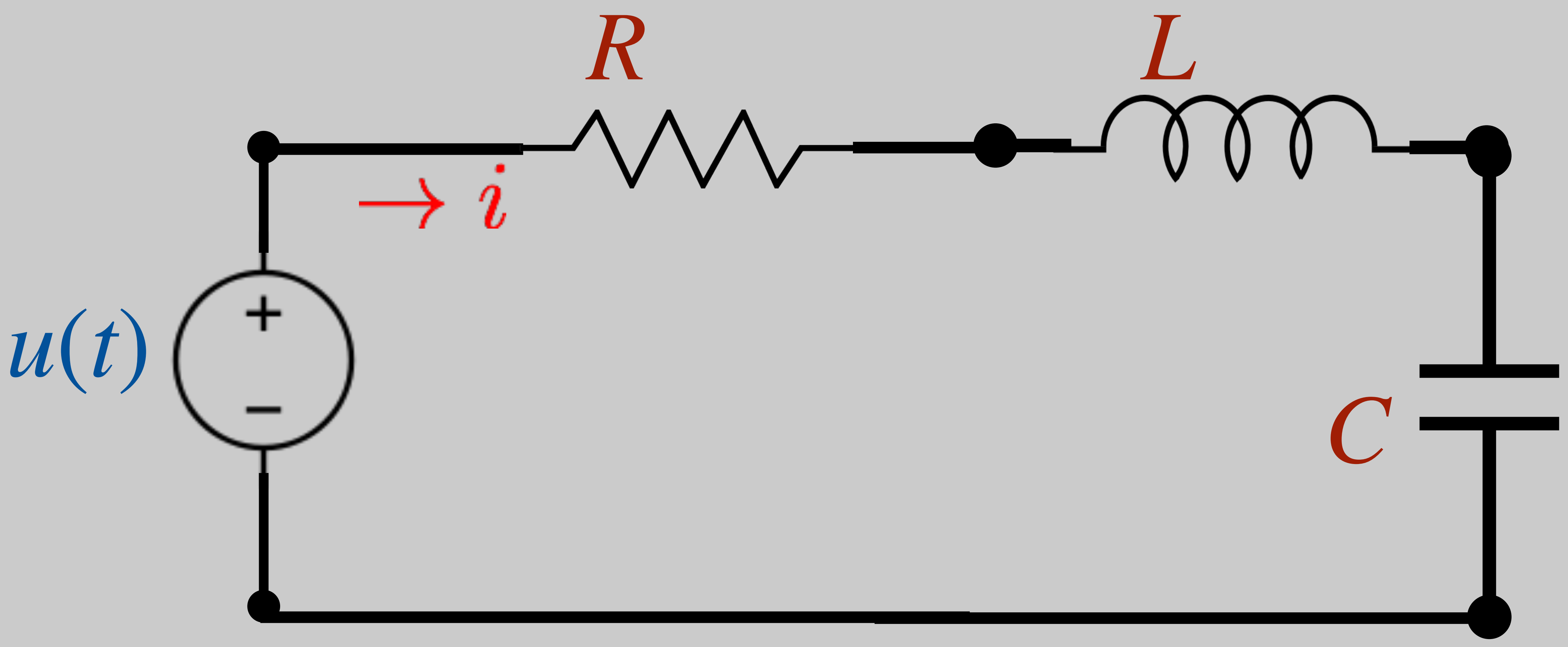
$$= V_{DD} e^{-\alpha t} \left(\frac{e^{j\omega_n t} + e^{-j\omega_n t}}{2} \right) + V_{DD} \frac{\alpha}{\omega_n} e^{-\alpha t} \left(\frac{e^{j\omega_n t} - e^{-j\omega_n t}}{2j} \right)$$

$$= V_{DD} e^{-\alpha t} \cos(\omega_n t) \; + V_{DD} \frac{\alpha}{\omega_n} e^{-\alpha t} \sin(\omega_n t)$$

$$= V_{DD} e^{-\alpha t} \left(\cos(\omega_n t) + \frac{\alpha}{\omega_n} \sin(\omega_n t) \right)$$

~~Overdamped~~ Underdamped

$$V_c = V_{DD} e^{-\alpha t} \left(\cos(\omega_n t) + \frac{\alpha}{\omega_n} \sin(\omega_n t) \right)$$



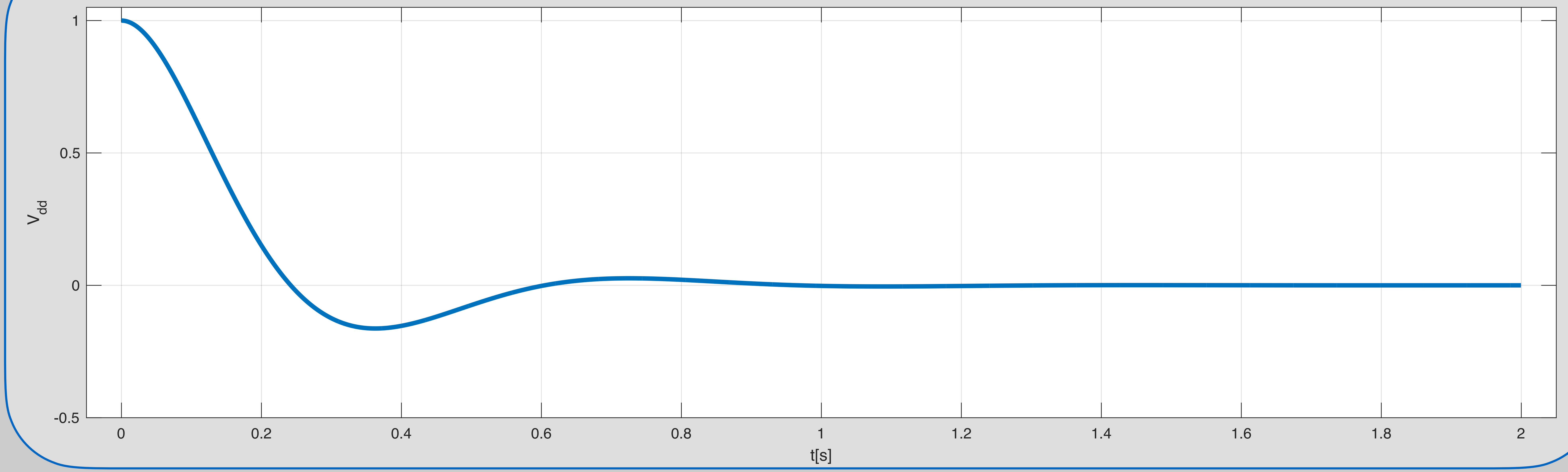
$\alpha = 5$

$\omega_0 = 10$

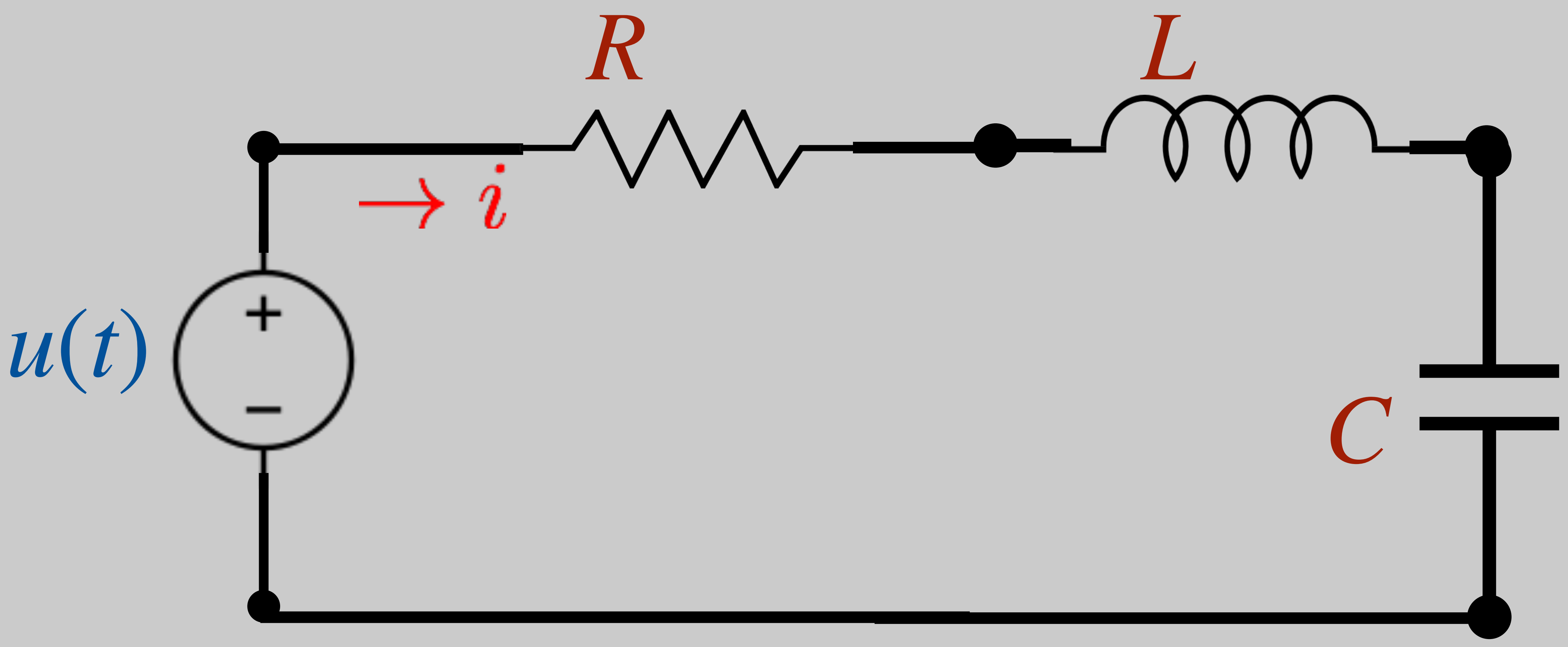
$\zeta = 0.5$

ω_n: oscillation frequency

ω_0: resonance frequency, when no damping ($\alpha = 0,\ R = 0$)



Overdamped Underdamped



$$V_c = V_{DD} e^{-\alpha t} \left(\cos(\omega_n t) + \frac{\alpha}{\omega_n} \sin(\omega_n t) \right)$$

$\alpha = 5$

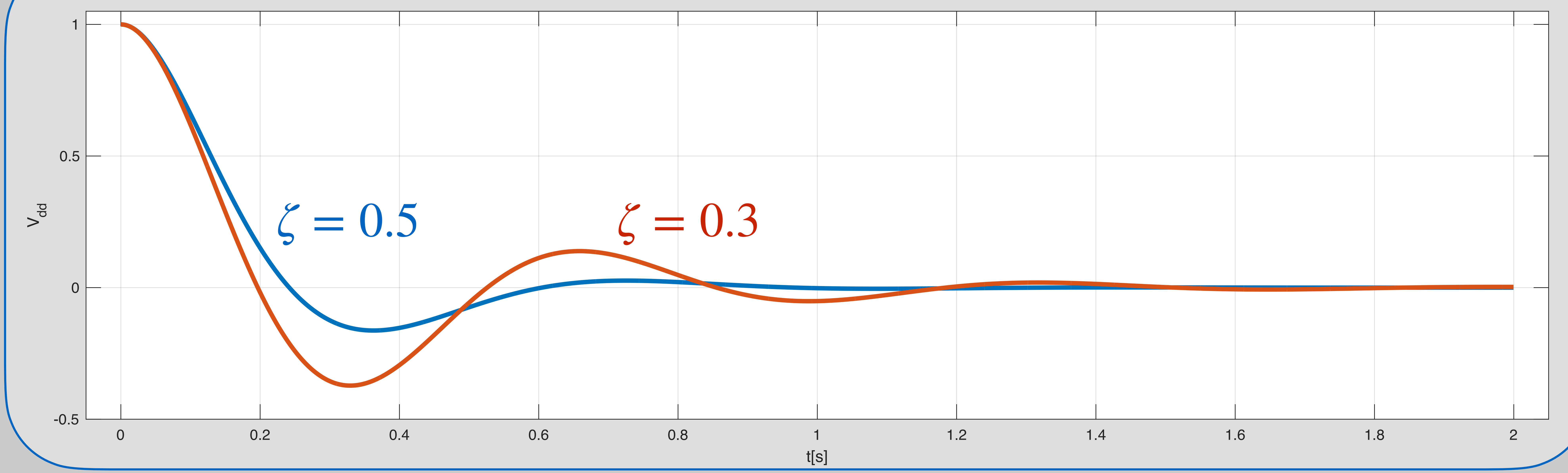
$\omega_0 = 10$

$\zeta = 0.5$

$\zeta = 0.3$

ω_n: oscillation frequency

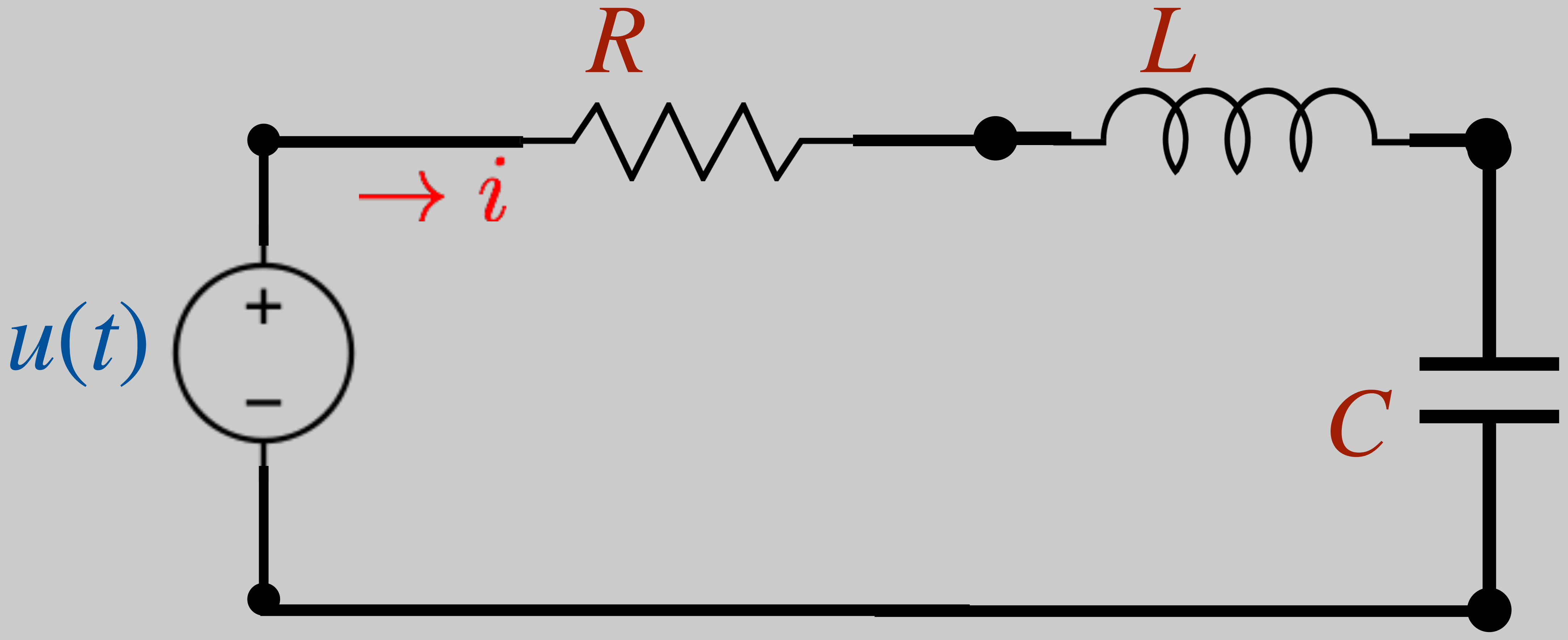
ω_0: resonance frequency, when no damping ($\alpha = 0,\ R = 0$)

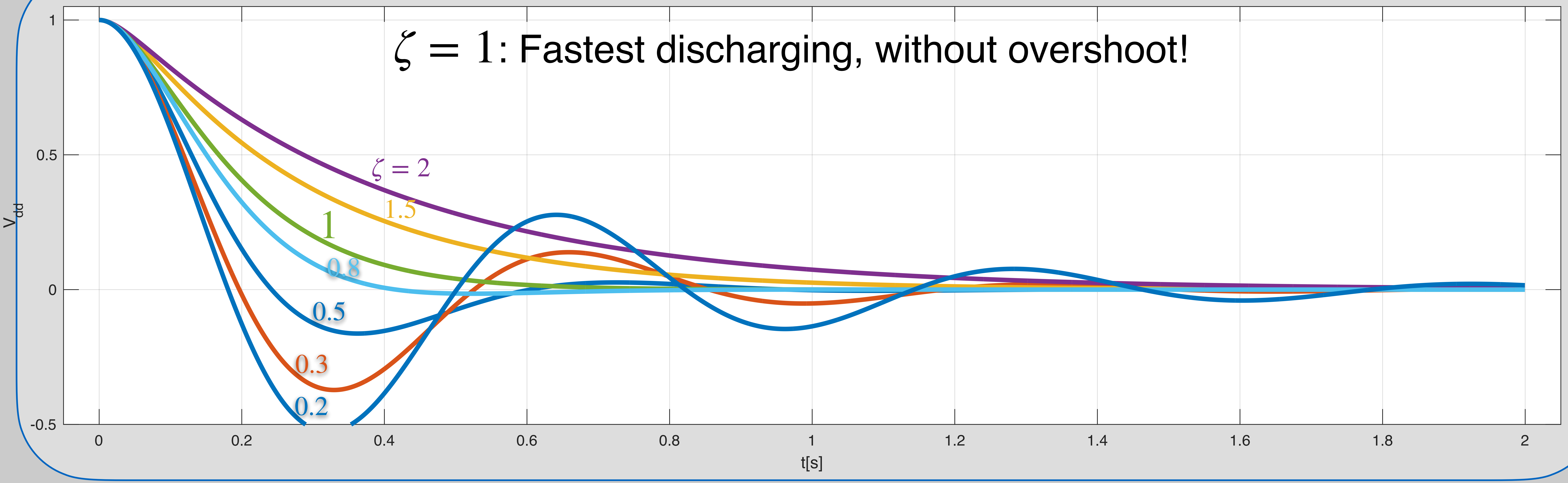


RLC Homogeneous Response

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$

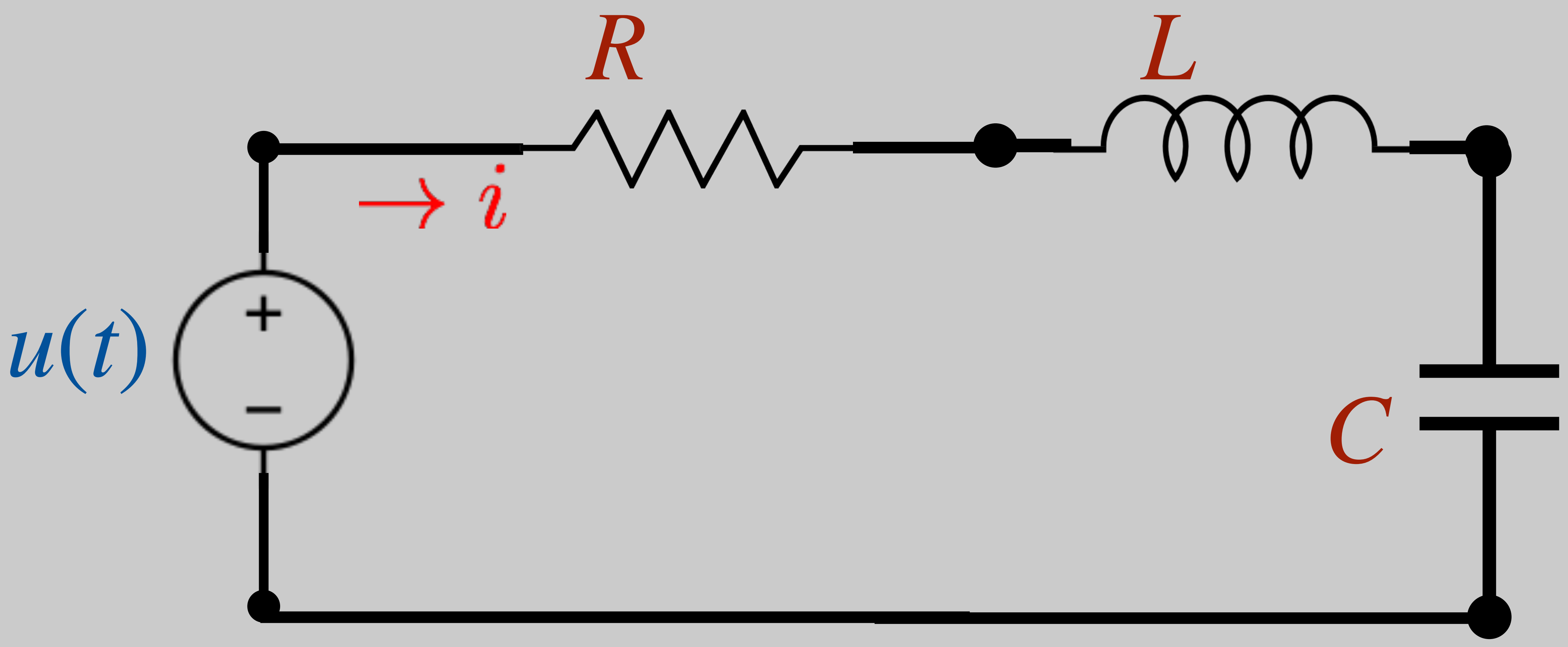
$$A = \frac{s_2}{s_2 - s_1} V_{DD} \qquad B = -\frac{s_1}{s_2 - s_1} V_{DD}$$





Critically Damped

$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = 0$$



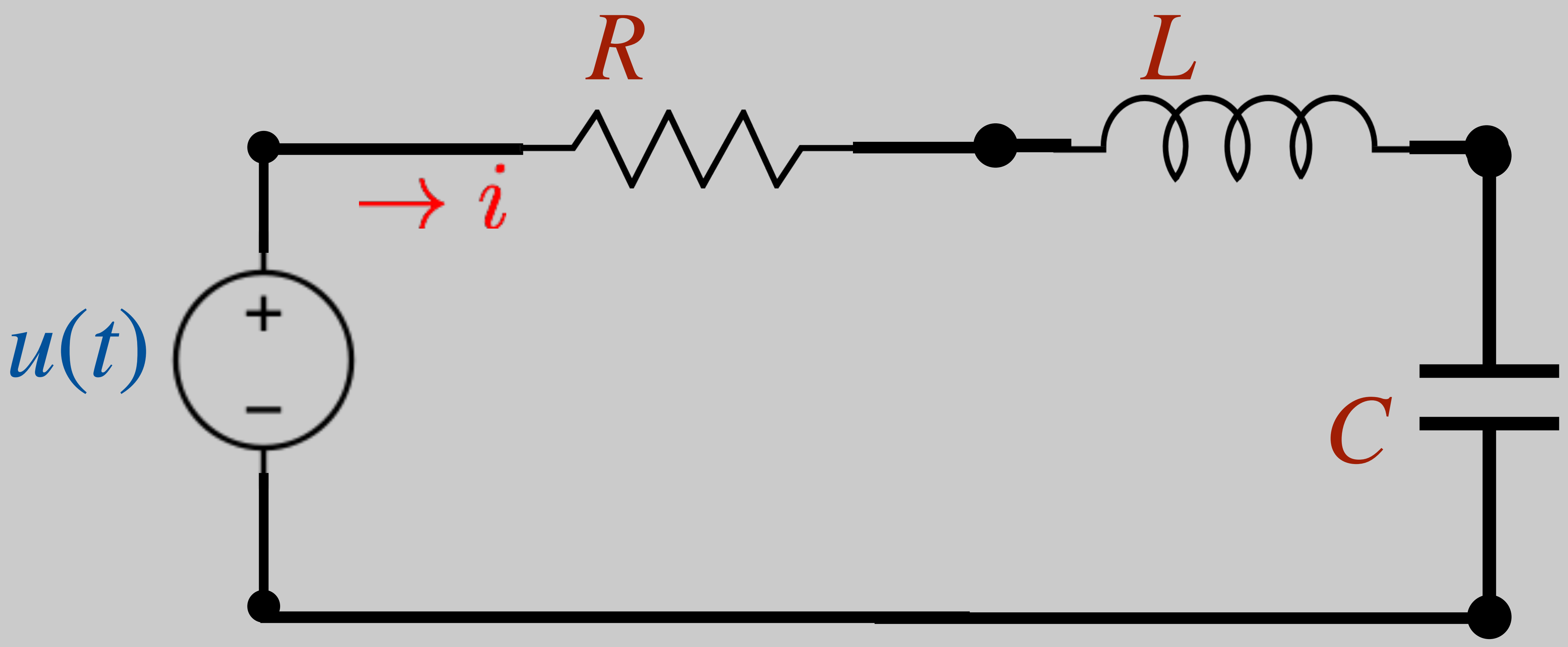
$a = \omega_0$ 2 identical solutions:

$$V_c(t) = Ae^{s_1 t} + Be^{s_1 t}$$

Problem! One initial condition….

Proceed with s_1, s_2 for now….

Critically Damped



$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = 0$$

$a = \omega_0$ 2 identical solutions:

$$V_c(t) = Ae^{s_1 t} + Be^{s_1 t}$$

Problem! One initial condition….

Proceed with s_1, s_2 for now….

From Initial conditio…

Same as before!!!!

$u(0) = V_{DD}$

$V_c(0) = V_{DD} \quad \Rightarrow V_c(0) \ldots V_{DD} \quad \Rightarrow A + B = V_{DD}$

$i_L(0) = 0 \quad \Rightarrow i_L(0) = 0 \quad \Rightarrow C\frac{d}{dt}V_c(\ldots + Bs_2 = 0$

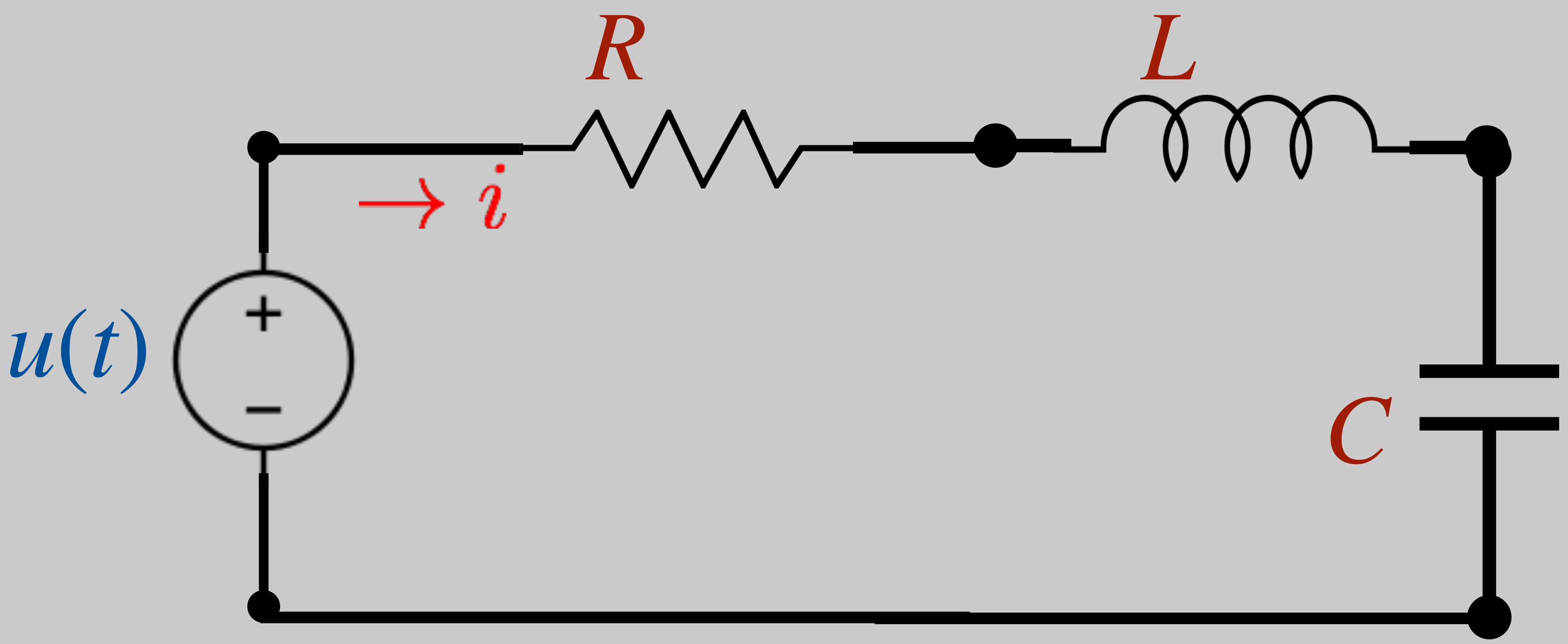
$$\begin{pmatrix} 1 & 1 \\ s_1 & s_2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{bmatrix} V_{DD} \\ 0 \end{bmatrix} \Rightarrow \frac{1}{s_2 - s_1}\begin{pmatrix} s_2 & -1 \\ -s_1 & 1 \end{pmatrix}\begin{pmatrix} \ldots \\ 0 \end{pmatrix}$$

$$A = \frac{s_2}{s_2 - s_1}V_{DD}$$

$$B = -\frac{s_1}{s_2 - s_1}V_{DD}$$

Critically Damped

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$



$$A = \frac{s_2}{s_2 - s_1} V_{DD} \qquad B = -\frac{s_1}{s_2 - s_1} V_{DD}$$

$$\lim_{s_2 \rightarrow s_1} V_c(t) = V_{DD} \frac{s_2 e^{s_1 t} - s_1 e^{s_2 t}}{s_2 - s_1}$$

L'Hopital's rule:

$$= V_{DD} \frac{e^{s_1 t} - s_1 t e^{s_2 t}}{1} \quad = V_{DD} \left(e^{s_1 t} - s_1 t e^{s_1 t} \right)$$

General solution for repeated roots:

$$V_c(t) = Ae^{s_1 t} + Bte^{s_1 t}$$

Recap:

$$\frac{d^2}{dt^2}V_c(t) + 2\alpha\frac{d}{dt}V_c(t) + \omega_0^2 V_c(t) = u(t)$$

$$a = \frac{R}{2L}$$ Damping coefficient (associated with decay)

$$\omega_0 = \frac{1}{\sqrt{LC}}$$ Resonance frequency

$$\zeta = \frac{\alpha}{\omega_0}$$ Damping Ratio

$$Q = \frac{1}{2\zeta} = \frac{\omega_0 L}{R} = \sqrt{\frac{L}{CR^2}}$$ Quality factor

$\alpha > \omega_0, \quad \zeta > 1 \Rightarrow$ Overdamped $\Rightarrow$ Exponential decays

$\alpha = \omega_0, \quad \zeta = 1 \Rightarrow$ Critically damped $\Rightarrow$ Fastest decay without oscillations

$\alpha < \omega_0, \quad \zeta < 1 \Rightarrow$ Underdamped $\Rightarrow$ decay+oscillations

$\omega_n = \sqrt{\alpha^2 - \omega^2}$ Oscillation frequency