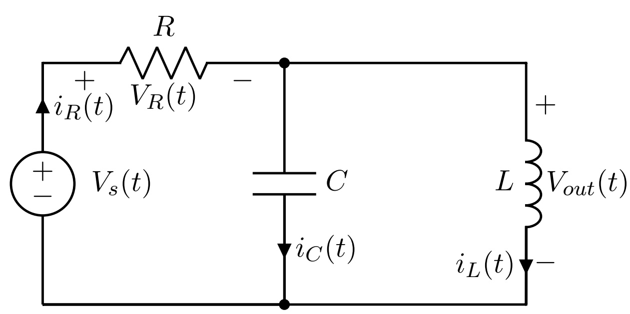
EECS 16B Designing Information Devices and Systems II
Spring 2021 UC Berkeley **Homework 15**

This homework is optional and we are releasing it with solutions. These problems are selected since they might be useful for reviewing for the final.

1. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{out}(t)$.



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2}\cos\left(100t - \frac{\pi}{4}\right)$$
$$R = 5\,\Omega$$
$$L = 50\,\text{mH}$$
$$C = 2\,\text{mF}$$

(a) **Give the amplitude V_0, oscillation frequency ω, and phase ϕ of the input voltage V_s.**

Solution: A sinusoid takes the form $v(t) = V_0 cos(\omega t + \phi)$. Given $V_s(t)$, we find:

$$V_0 = 10\sqrt{2}\text{V}$$
$$\omega = 100\text{rad/sec}$$
$$\phi = -\frac{\pi}{4}\text{rad}$$

(b) Transform the circuit into the phasor domain. **What are the impedances of the resistor, capacitor, and inductor? What is the phasor $\widetilde{V}_S$ of the input voltage $V_s(t)$?**

Solution:

$$\begin{aligned}
Z_L &= j\omega L \\
Z_C &= \frac{1}{j\omega C} \\
Z_R &= R \\
\widetilde{V}_s &= \frac{|V_s|}{2} e^{j\angle V_s} = 5\sqrt{2}e^{-j\frac{\pi}{4}}
\end{aligned}$$

(c) Use the circuit equations to **solve for $\widetilde{V}_{out}$**, the phasor representing the output voltage.

Solution:

We have

$$\begin{aligned}
i_R &= \frac{\widetilde{V}_S - \widetilde{V}_{out}}{R} \\
i_L &= \frac{\widetilde{V}_{out}}{j\omega L} \\
i_C &= \widetilde{V}_{out} \cdot j\omega C
\end{aligned}$$

Rewriting the current relation in terms of voltage phasors gives:

$$\begin{aligned}
\frac{\widetilde{V}_S - \widetilde{V}_{out}}{R} &= \frac{\widetilde{V}_{out}}{j\omega L} + \widetilde{V}_{out} \cdot j\omega C \\
\frac{\widetilde{V}_S}{R} &= \widetilde{V}_{out}\left(\frac{1}{j\omega L} + j\omega C + \frac{1}{R}\right) \\
\frac{\widetilde{V}_S}{R} &= \widetilde{V}_{out}\left(\frac{R + (j\omega)^2 RLC + j\omega L}{j\omega RL}\right)
\end{aligned}$$

Solving for $\widetilde{V}_{out}$:

$$\widetilde{V}_{out} = \widetilde{V}_S\left(\frac{j\omega L}{R - \omega^2 RLC + j\omega L}\right)$$

Plugging in for the values of R, L, C and ω:

$$\begin{aligned}
\widetilde{V}_{out} &= \widetilde{V}_S\left(\frac{j \cdot 100 \cdot 50 \times 10^{-3}}{5 - 100^2 \cdot 5 \cdot 50 \times 10^{-3} \cdot 2 \times 10^{-3} + j100 \cdot 50 \times 10^{-3}}\right) \\
&= \widetilde{V}_S\left(\frac{j5}{5 - 5 + j5}\right) \\
&= \widetilde{V}_S\left(\frac{j5}{j5}\right) \\
\widetilde{V}_{out} &= \widetilde{V}_S
\end{aligned}$$

We found that $\widetilde{V}_{out} = \widetilde{V}_S$ because this circuit is in resonance; i.e., the capacitor and inductor have the exact values that cause current and voltage to endlessly oscillate between them at this frequency. If we chose a different value for ω with these same component values, the circuit would not be in resonance and $\widetilde{V}_{out}$ and $\widetilde{V}_S$ would no longer be equal.

Alternative Solution:
The equivalent impedance of the parallel combination of inductor and capacitor is given by

$$\begin{aligned}
Z_{eq} &= \frac{Z_L Z_C}{Z_L + Z_C} \\
&= \frac{L/C}{j\omega L + \frac{1}{j\omega C}} \\
&= \frac{L/C}{j\left(\omega L - \frac{1}{\omega C}\right)} \\
&= \frac{(50 \times 10^{-3})/(2 \times 10^{-3})}{j\left(100 \cdot 50 \times 10^{-3} - \frac{1}{100 \cdot 2 \times 10^{-3}}\right)} \\
&= \frac{25}{j(5-5)} = \frac{25}{j0} = \infty
\end{aligned}$$

Hence the parallel inductor and capacitor combine to form infinite impedance at the resonant frequency ω. Then the circuit becomes 'open', so the resistor R is floating and hence has no voltage drop, i.e.

$$\widetilde{V}_R = 0 \implies \widetilde{V}_{out} = \widetilde{V}_S$$

(d) **Convert the phasor $\tilde{V}_{out}$ back to get the time-domain signal $V_{out}(t)$.**
Solution: Since $\widetilde{V}_{out} = \widetilde{V}_S$,

$$v_{out}(t) = 10\sqrt{2}\cos\left(100t - \frac{\pi}{4}\right)$$

2. Latch

The circuit below is a type of latch, which is one of the fundamental components of memory in many digital systems. The latch is a bistable circuit, which means that there are two possible stable states: one representing a stored '1' bit and the other a stored '0' bit.

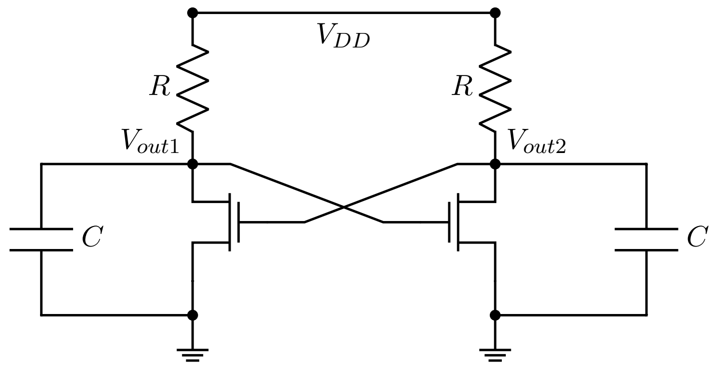


Figure 1: Simplified latch: the gate capacitances have been pulled out explicitly.

(a) To get a basic understanding of the stable operating points for the latch, consider the following simplified circuit using the pure switch model for MOSFETs (and a threshold voltage of $\frac{V_{DD}}{2}$).

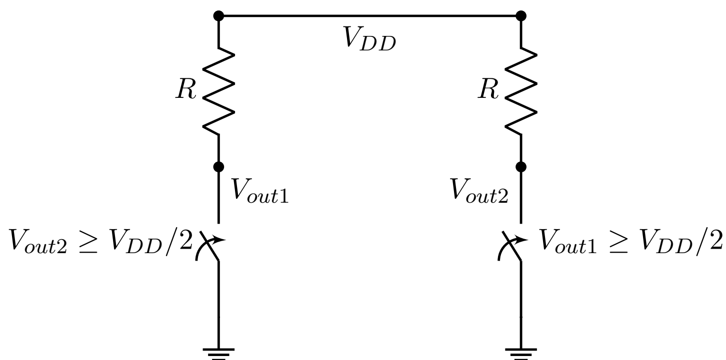


Figure 2: Pure switch model for a latch

First assume that $V_{out1} = 0$. What is V_{out2}? Are the left and right switches open or closed?

	Open or V_{DD}	**Closed or 0**
Left Switch	◯	◯
Right Switch	◯	◯
V_{out2}	◯	◯

Solution: If $V_{out1} = 0$, then the switch on the right is open because that transistor is clearly off. (You can look at the switch model or think about the threshold of $V_{DD}/2$, either way.) At that point, there can be no current flowing through the resistor above it, and $V_{out2} = V_{DD}$. Now, we can look at the switch to the left. It is clearly closed since that transistor is clearly on.

So, the first set of answers should be:

Notice how in this, this is self-fulfilling. Having the left transistor closed means that there is current in the corresponding resistor and V_{out1} is indeed pulled down to 0 since the transistor is being modeled with no output resistance. So the initial assumption is self-consistent.

	Open or V_{DD}	**Closed or 0**
Left Switch	○	●
Right Switch	●	○
V_{out2}	●	○

	Open or V_{DD}	**Closed or 0**
Left Switch	○	○
Right Switch	○	○
V_{out2}	○	○

Suppose that $V_{out1} = V_{DD}$. What is V_{out2}? Are the left and right switches open or closed?

Solution: If $V_{out1} = V_{DD}$, then the switch on the right is closed because that transistor is clearly on. (You can look at the switch model or think about the threshold of $V_{DD}/2$, either way.) At that point, there is current flowing through the resistor above it, and $V_{out2} = 0$. This is because we are assuming that there is no resistance in our pure switch model of the transistor. Now, we can look at the switch to the left. It is clearly open since that transistor is clearly off because V_{out2} is low.

So, the second set of answers should be:

	Open or V_{DD}	**Closed or 0**
Left Switch	●	○
Right Switch	○	●
V_{out2}	○	●

Notice how in this case also, this is self-fulfilling. Having the left transistor open means that there is no current in the corresponding resistor and V_{out1} is indeed V_{DD} for consistency.

(b) To get an understanding of latch dynamics, we will now break it down into smaller pieces. Below is one half of the latch circuit.

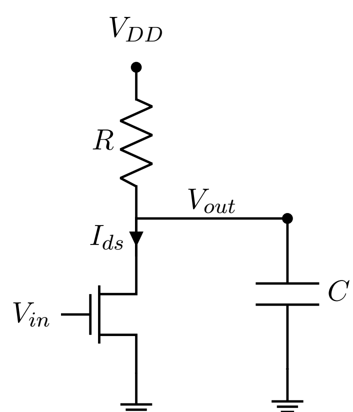


Figure 3: Latch half-circuit

Write a differential equation for the voltage V_{out} in terms of the drain to source current I_{ds}. Treat I_{ds} as some specified input signal and treat the transistor as a current source connected to ground. (i.e. There is no dependence on V_{in} in this part. In this part, treat the I_{ds} as a constant that you are given.)

Solution: Using KCL at the output node, let's equate the currents in (from the resistor) to the currents out (to the capacitor and through the transistor).

$$\frac{V_{DD} - V_{out}}{R} = I_{ds} + C\frac{d}{dt}(V_{out})$$

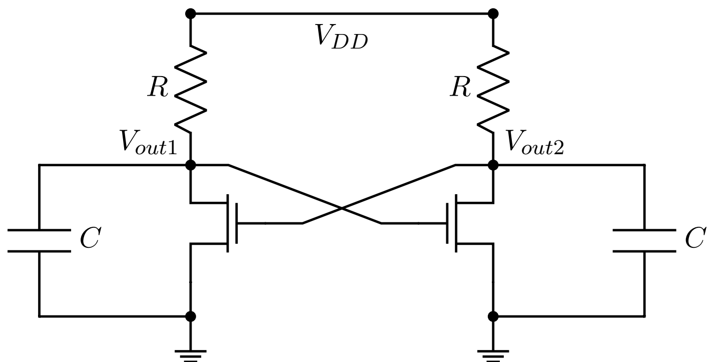
Rearranging terms, we get:

$$\frac{dV_{out}}{dt} = \frac{1}{C}(\frac{V_{DD}}{R} - \frac{V_{out}}{R} - I_{ds})$$

Notice that this can be made into a more familiar form as:

$$\frac{d}{dt}V_{out}(t) = \frac{-1}{RC}V_{out}(t) + \frac{V_{DD}}{RC} - \frac{I_{ds}}{C}$$

Here, we see that $\frac{V_{DD}}{RC} - \frac{I_{ds}}{C}$ is playing the role of the "input" in our standard view of a first-order linear scalar differential equation.



(c)

For this circuit, we care about more detailed analog characteristics of the MOSFETs, so we will model their behavior more accurately as current sources that are controlled by their gate voltages V_{in} with the following equation:

$$I_{ds} = g(V_{in})$$

Where $g(V_{in})$ is a some nonlinear function. **Using this I_{ds} expression together with the result from the previous part, write down a system of differential equations for V_{out1} and V_{out2} in vector form:**

$$\frac{d}{dt}\begin{bmatrix} V_{out1}(t) \\ V_{out2}(t) \end{bmatrix} = \vec{f}\left(\begin{bmatrix} V_{out1}(t) \\ V_{out2}(t) \end{bmatrix}\right).$$

(Hint: Notice that the latch above can be constructed by taking two of the circuit in Figure 3, and connecting the V_{out} of one to the gate V_{in} of the other and vice-versa.)

Solution:

We observe that V_{in} for the half-circuit on the left is V_{out2}. That means that the I_{ds} for the differential equation corresponding to V_{out1} is given by $g(V_{out2})$ and similarly, the I_{ds} for the differential equation corresponding to V_{out2} is given by $g(V_{out1})$.

Writing those two equations with that substitution gives:

$$\frac{d}{dt}V_{out1}(t) = \frac{-1}{RC}V_{out1}(t) + \frac{V_{DD}}{RC} - \frac{g(V_{out2}(t))}{C}$$

$$\frac{d}{dt}V_{out2}(t) = \frac{-1}{RC}V_{out2}(t) + \frac{V_{DD}}{RC} - \frac{g(V_{out1}(t))}{C}$$

Putting this into vector form yields:

$$\frac{d}{dt}\begin{bmatrix}V_{out1}(t)\\V_{out2}(t)\end{bmatrix}=\vec{f}\left(\begin{bmatrix}V_{out1}(t)\\V_{out2}(t)\end{bmatrix}\right)=\begin{bmatrix}\frac{-1}{RC}V_{out1}(t)+\frac{V_{DD}}{RC}-\frac{g(V_{out2}(t))}{C}\\\frac{-1}{RC}V_{out2}(t)+\frac{V_{DD}}{RC}-\frac{g(V_{out1}(t))}{C}\end{bmatrix}.$$

(d) For the rest of this problem, assume that your analysis yields the following system of nonlinear differential equations:

$$\begin{bmatrix}\frac{dV_{out1}}{dt}\\\frac{dV_{out2}}{dt}\end{bmatrix}=\begin{bmatrix}1-V_{out_1}-g(V_{out2})\\1-V_{out_2}-g(V_{out1})\end{bmatrix}$$

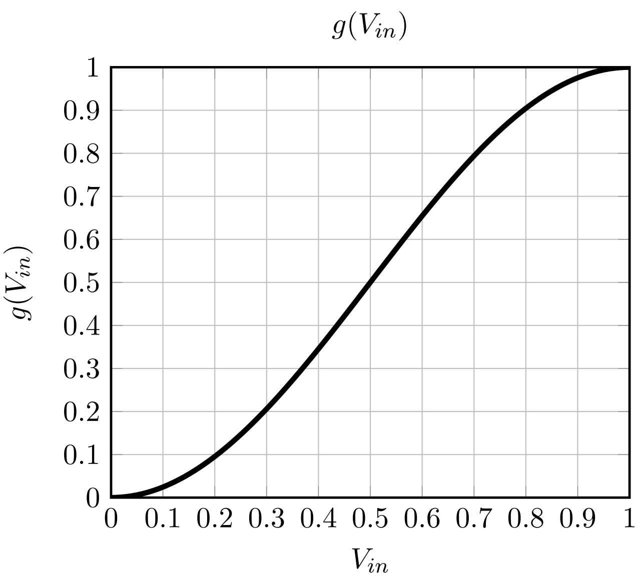
Suppose that you put this latch into an ideal circuit simulator, and measure $g(V_{in})$ and $\frac{dg}{dt}(V_{in})$. The results from these measurements are shown in the graphs below. From your simulations, you also can see that for the following initial conditions, the latch voltages do not change over time:

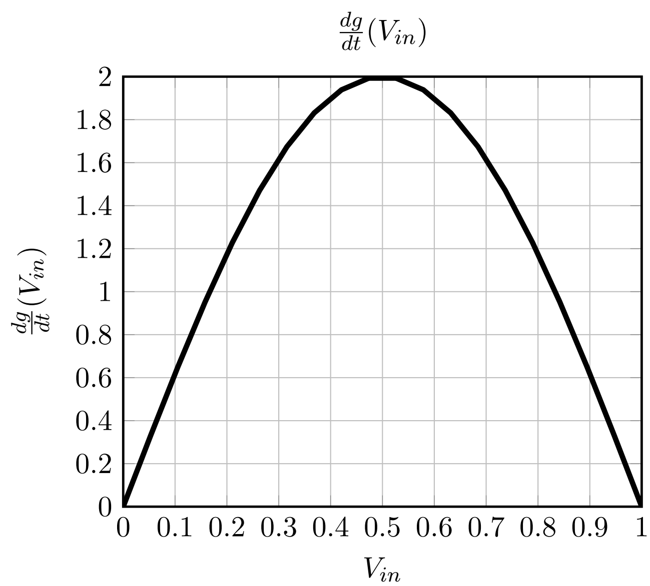
$$\begin{bmatrix}V^*_{out1}\\V^*_{out2}\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0.5\\0.5\end{bmatrix}$$

Use the graphs below to linearize the differential equations around the three operating points.

Write a linearized system of differential equations around each of those operating points $\begin{bmatrix}V^*_{out1}\\V^*_{out2}\end{bmatrix}$**.**

For which of the provided $\begin{bmatrix}V^*_{out1}\\V^*_{out2}\end{bmatrix}$ **points is the latch locally stable? For which of the provided points is the latch locally unstable? Why?**





Solution: To linearize these differential equations, we need to take the derivative of $\vec{f}$ with respect to $\begin{bmatrix}V_{out1}\\V_{out2}\end{bmatrix}$. This will give a matrix. The derivative of the first component of $\vec{f}$ with respect to V_{out1} is just -1 since that dependence is linear. Similarly for the derivative of the second component of $\vec{f}$

with respect to V_{out2} being -1. The off-diagonal derivatives are clearly $-g'(V_{out2})$ and $-g'(V_{out1})$ respectively, where $g'(V_{in}) = \frac{dg(V_{in})}{dV_{in}}$. This means that we get:

$$\vec{f}\left(\begin{bmatrix} V_{out1} \\ V_{out2} \end{bmatrix}\right) = \vec{f}\left(\begin{bmatrix} V_{out1}^* + \delta V_1 \\ V_{out2}^* + \delta V_2 \end{bmatrix}\right) \tag{1}$$

$$= \vec{f}\left(\begin{bmatrix} V_{out1}^* \\ V_{out2}^* \end{bmatrix}\right) + \frac{d}{d\vec{v}} f(\vec{v})|_{\vec{v}=\begin{bmatrix} V_{out1}^* \\ V_{out2}^* \end{bmatrix}} \begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix} + \vec{w} \tag{2}$$

$$= \begin{bmatrix} 1 - V_{out1}^* - g(V_{out2}^*) \\ 1 - V_{out2}^* - g(V_{out1}^*) \end{bmatrix} + \begin{bmatrix} -1 & -g'(V_{out2}^*) \\ -g'(V_{out1}^*) & -1 \end{bmatrix} \begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix} + \vec{w} \tag{3}$$

where the disturbance term $\vec{w}$ captures all the approximation errors that are coming from linearization. That is what lets us write equalities above instead of $\approx$.

Using this in the differential equation, since the operating point is not changing with time, we get:

$$\frac{d}{dt}\begin{bmatrix} V_{out1}(t) \\ V_{out2}(t) \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) = \begin{bmatrix} -1 & -g'(V_{out2}^*) \\ -g'(V_{out1}^*) & -1 \end{bmatrix} \begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) + \begin{bmatrix} 1 - V_{out1}^* - g(V_{out2}^*) \\ 1 - V_{out2}^* - g(V_{out1}^*) \end{bmatrix} + \vec{w}(t)$$

.

With the general form established, we need to actually write this out for the neighborhoods of the provided operating points.

For $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we know from the provided graphs that $g(0) = 0$, $g(1) = 1$, and $g'(0) = 0, g'(1) = 0$. Here, we view $\begin{bmatrix} \delta V_1(t) \\ \delta V_2(t) \end{bmatrix}$ as the local deviation from the given operating point of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. In other words, $\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} \delta V_1(t) \\ \delta V_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Substituting everything in, we get:

$$\frac{d}{dt}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) = \begin{bmatrix} 1 - 1 - 0 \\ 1 - 0 - 1 \end{bmatrix} + \begin{bmatrix} -1 & -0 \\ -0 & -1 \end{bmatrix} \begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) + \vec{w}(t) \tag{4}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) + \vec{w}(t). \tag{5}$$

This clearly has two eigenvalues of -1 and so is locally stable.

For $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we know from the provided graphs that $g(0) = 0$, $g(1) = 1$, and $g'(0) = 0, g'(1) = 0$. Here, we view $\begin{bmatrix} \delta V_1(t) \\ \delta V_2(t) \end{bmatrix}$ as the local deviation from the given operating point of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In other words,

$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} \delta V_1(t) \\ \delta V_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Substituting everything in, we get:

$$\frac{d}{dt}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) = \begin{bmatrix} 1-1-0 \\ 1-0-1 \end{bmatrix} + \begin{bmatrix} -1 & -0 \\ -0 & -1 \end{bmatrix}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) + \vec{w}(t) \tag{6}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) + \vec{w}(t). \tag{7}$$

This also clearly has two eigenvalues of -1 and so is locally stable.

So both of these operating points for the latch are stable. Small disturbances will be rejected by the local dynamics and the memory will regenerate its value. This is important in digital circuits.

For $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, we know from the provided graphs that $g(0.5) = 0.5$, $g(0.5) = 0.5$, and $g'(0.5) = 2$. Here, we view $\begin{bmatrix} \delta V_1(t) \\ \delta V_2(t) \end{bmatrix}$ as the local deviation from the given operating point of $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$. In other words, $\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} \delta V_1(t) \\ \delta V_2(t) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$. Substituting everything in, we get:

$$\frac{d}{dt}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) = \begin{bmatrix} 1-0.5-0.5 \\ 1-0.5-0.5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) + \vec{w}(t) \tag{8}$$

$$= \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} \delta V_1 \\ \delta V_2 \end{bmatrix}(t) + \vec{w}(t). \tag{9}$$

Notice how the constant terms went away (which makes sense since this is an operating point) but now we need to calculate some eigenvalues to understand if this is locally stable or not.

$$\det\left(\lambda I - \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}\right) = (\lambda+1)^2 - 4 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1).$$

This has one stable eigenvalue of -3 and an unstable eigenvalue of $+1$. This means that this operating point is unstable overall.

You will learn in more advanced digital circuits courses that this instability is actually desirable for this particular operating point because that is what lets us more effectively use this circuit as a latch. To set the value of the latch, we just have to drive it past the unstable point and then the unstable point will push the state to the region of attraction of the desired stable operating point, which will then hold that value. If the local gain g' at the middle operating point was too low, then this would also be a stable operating point and the circuit could end up remembering a useless intermediate voltage that is not cleanly interpretable as a binary 0 or binary 1.

3. DFT

Consider the DFT matrix B as defined below

$$B = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{4}} & e^{j2\frac{2\pi}{4}} & e^{j3\frac{2\pi}{4}} \\ 1 & e^{j2\frac{2\pi}{4}} & e^{j4\frac{2\pi}{4}} & e^{j6\frac{2\pi}{4}} \\ 1 & e^{j3\frac{2\pi}{4}} & e^{j6\frac{2\pi}{4}} & e^{j9\frac{2\pi}{4}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}.$$

(a) The DFT coefficients $\vec{F}$ are related to a vector of samples $\vec{f}$ by the relationship $\vec{f} = B\vec{F}$. In other words, $\vec{F}$ represents $\vec{f}$ in the basis given by the columns of B. Similarly for the DFT coefficients $\vec{G}$ and a vector of samples $\vec{g}$ — they too satisfy the relationship $\vec{g} = B\vec{G}$.
What are the DFT coefficients $\vec{H}$ for $\vec{h} = \alpha\vec{f} + \beta\vec{g}$ in terms of $\vec{F}$ and $\vec{G}$. Here, α and β are constant real numbers.

Solution: The DFT coefficients of $\vec{H}$ are

$$\vec{H} = B^{-1}\vec{h} = B^{-1}(\alpha\vec{f} + \beta\vec{g}) = \alpha \cdot B^{-1}\vec{f} + \beta \cdot B^{-1}\vec{g} = \alpha\vec{F} + \beta\vec{G}.$$

(b) **Explicitly find the DFT coefficients $\vec{F}$ of the vector $\vec{f} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$** i.e. we want $\vec{F}$ so that $\vec{f} = B\vec{F}$.

Solution:

$$\vec{F} = B^{-1}\vec{f} = B^{*}\vec{f} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -j \\ 0 \\ j \end{bmatrix}.$$

(c) **Show that if $\vec{f}$ is a real vector, then:**

- **$F[0]$ is always real and so is $F[2]$.**
- **$F[1] = \overline{F}[3]$.**

(HINT: What do you know about B^{-1}?)

Solution: Let

$$\vec{f} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

where a, b, c, d are real numbers. Then,

$$\vec{F} = B^{-1}\vec{f} = B^{*}\vec{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+b+c+d \\ a-bj-c+dj \\ a-b+c-d \\ a+bj-c-dj \end{bmatrix}.$$

Hence, we see that $F[0]$ and $F[2]$ are always real since a, b, c, d are real, while $F[1] = \overline{F}[3]$ since everything multiplied by j in one is multiplied by $-j$ in the other and vice versa.

This is a direct demonstration of the desired result.

Alternatively, you could have noticed that if we consider $B = [\vec{b}_0, \vec{b}_1, \vec{b}_2, \vec{b}_3]$, then $\vec{b}_0$ and $\vec{b}_2$ are real while $\vec{b}_1$ and $\vec{b}_3$ are complex conjugates by inspection. The formula $\vec{F} = B^*\vec{f}$ means that $F[i] = \vec{b}_i^*\vec{f}$. Now, $\vec{b}_0$ and $\vec{b}_2$ are real and that immediately gives us what we want since the conjugate transpose of a real vector is just the transpose. Similarly, $F[1] = \overline{F[3]}$ since $\overline{F[3]} = \overline{\vec{b}_3^*\vec{f}} = \overline{\vec{b}_3^*}\vec{f} = \vec{b}_1^*\vec{f} = F[1]$.

Either approach works.

4. DFT and Circuit Filters

You have been introduced to low-pass and high-pass filter circuits that pass some range of input signal frequencies while attenuating other ranges of signal frequency. You have also seen how we can break signals down and view the frequency components of sampled signals using the DFT. In this problem, we will see how we can combine these two bases of knowledge. Throughout this problem, if we have an N-dimensional vector $\vec{x}$, its DFT coefficients are given by the vector $\vec{X} = F_N\vec{x}$ where the DFT transformation matrix is

$$F_N = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}3} & \cdots & e^{-j\frac{2\pi}{N}(N-1)1} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & e^{-j\frac{2\pi}{N}6} & \cdots & e^{-j\frac{2\pi}{N}(N-1)2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}3(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}$$

and the inverse is $F_N^{-1} = F_N^*$.

(a) If you sample every Δ seconds and you take N samples, the 0^{th} DFT coefficient $\vec{X}[0]$ corresponds to the DC (or constant) term. The 1^{st} DFT coefficient $\vec{X}[1]$ corresponds to the fundamental frequency $f_0 = \frac{1}{N\Delta}$.

Say you have a signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$. You take $N = 9$ samples of the function every $\Delta = 1$ second; i.e. at $t = \{0, 1, 2, \ldots, 8\}$, forming a 9 element vector of samples $\vec{v}_{in}$. **What are the DFT coefficients $\vec{V}_{in}$ of the sampled signal $\vec{v}_{in}$?**

Solution:

We can rewrite

$$v_{in}(t) = \frac{1}{2}\left(e^{-j\frac{2\pi}{3}t} + e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{9}t} + e^{j\frac{2\pi}{9}t}\right).$$

If we define the k^{th} row of the DFT matrix F_9 (with $N = 9$) to be

$$\vec{u}_k^T = \frac{1}{\sqrt{N}}\begin{bmatrix} e^{-j\frac{2\pi}{9}(k)(0)} & e^{-j\frac{2\pi}{9}(k)(1)} & \cdots & e^{-j\frac{2\pi}{9}(k)(8)} \end{bmatrix},$$

we can write the sampled version of $v_{in}(t)$ in vector notation as the column vector

$$\vec{v}_{in} = \frac{\sqrt{N}}{2}\left(\vec{u}_3 + \overline{\vec{u}_3} + \vec{u}_1 + \overline{\vec{u}_1}\right).$$

One property of the rows of the DFT matrix is that $\vec{u}_k = \overline{\vec{u}_{N-k}}$, where N is the number of samples. Thus we can rewrite

$$\vec{v}_{in} = \frac{\sqrt{N}}{2}\left(\vec{u}_3 + \vec{u}_6 + \vec{u}_1 + \vec{u}_8\right).$$

Since the rows of the DFT matrix are also orthogonal and have norm $\|\vec{u}_k\| = 1$, the inner product $\vec{u}_k^*\vec{u}_k = 1$ and $\vec{u}_{i,\ i\neq k}^*\vec{u}_k = 0$. Therefore when we calculate

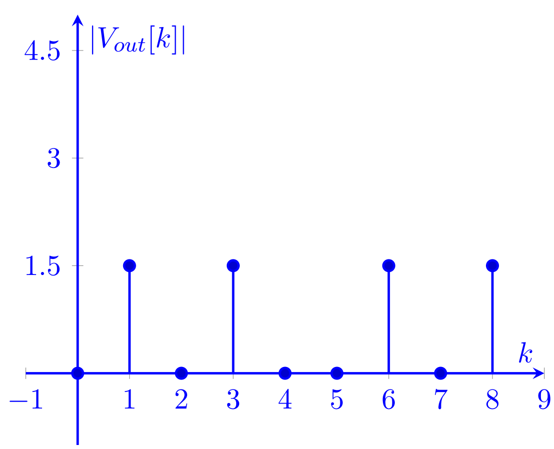
$$\vec{V}_{in} = F_9 \vec{v}_{in} = \frac{\sqrt{9}}{2} \begin{bmatrix} \vec{u}_0^T \\ \vec{u}_1^T \\ \vec{u}_2^T \\ \vec{u}_3^T \\ \vec{u}_4^T \\ \vec{u}_5^T \\ \vec{u}_6^T \\ \vec{u}_7^T \\ \vec{u}_8^T \end{bmatrix} (\vec{u}_3 + \vec{u}_6 + \vec{u}_1 + \vec{u}_8) = \frac{3}{2} \begin{bmatrix} \vec{u}_0^* \\ \vec{u}_8^* \\ \vec{u}_7^* \\ \vec{u}_6^* \\ \vec{u}_5^* \\ \vec{u}_4^* \\ \vec{u}_3^* \\ \vec{u}_2^* \\ \vec{u}_1^* \end{bmatrix} (\vec{u}_3 + \vec{u}_6 + \vec{u}_1 + \vec{u}_8) = \frac{3}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The coefficients are given by

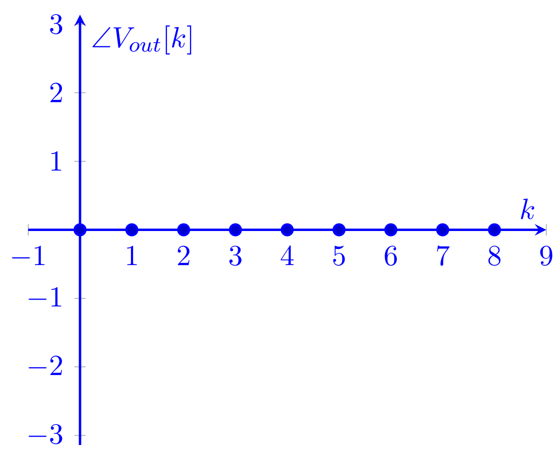
$$V_{out}[k] = \begin{cases} \frac{3}{2} & k = 1, 3, 6, 8 \\ 0 & k = 0, 2, 4, 5, 7 \end{cases} \tag{10}$$

and can be plotted

Frequency Domain Magnitude



Frequency Domain Phase



(b) You are given the circuit below.

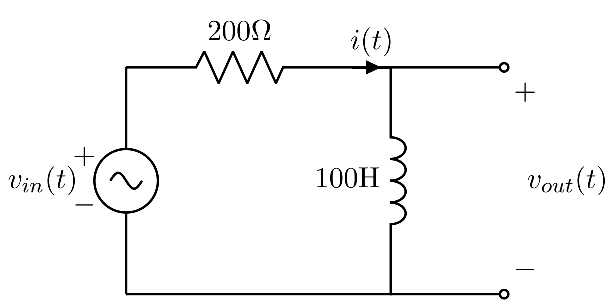
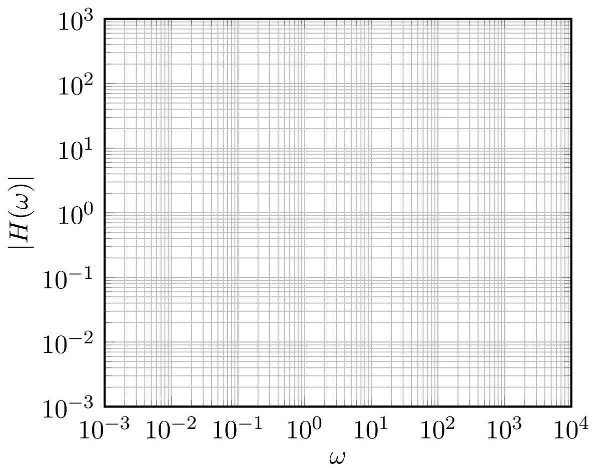


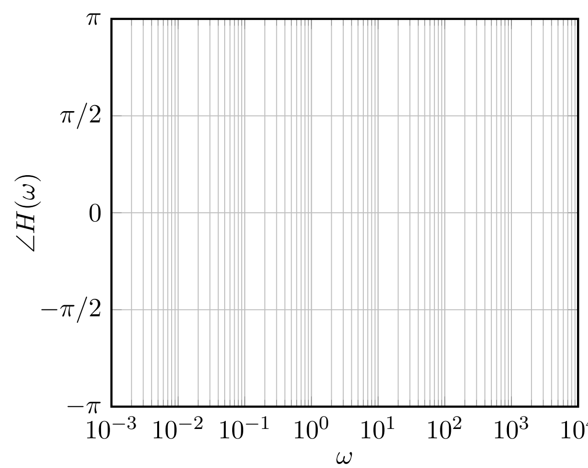
Figure 4: Filter circuit

Is this a high-pass or low-pass filter? What is its cutoff angular frequency, ω_c? Sketch the piecewise-linear approximations of the magnitude and phase Bode plots of the transfer function $H(\omega) = \frac{\widetilde{V}_{out}(\omega)}{\widetilde{V}_{in}(\omega)}$ below.

Log-log plot of transfer function magnitude



Semi-log plot of transfer function phase

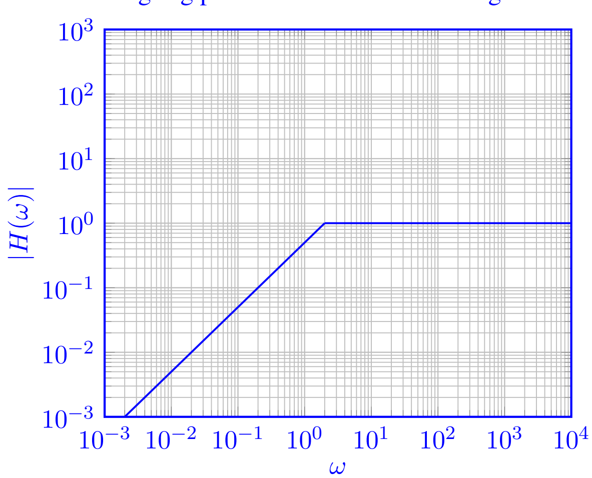


Solution: Since the inductor will behave as a closed circuit for low-frequency and DC signals and as an open circuit for high-frequency signals, this is a high-pass filter.
From KVL, we know:

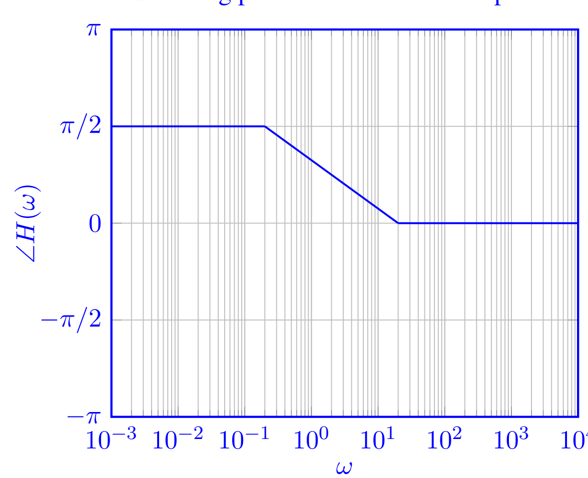
$$\widetilde{V}_{in} = \widetilde{I}R + \widetilde{I}j\omega L$$
$$\widetilde{V}_{out} = \widetilde{I}j\omega L$$
$$H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{\widetilde{I}j\omega L}{\widetilde{I}R + \widetilde{I}j\omega L} = \frac{j\omega L}{R + j\omega L} = \frac{j\omega/(\frac{R}{L})}{1 + j\omega/(\frac{R}{L})}$$

The cut-off frequency of the filter is $\omega_c = \frac{R}{L} = 2$ rad/sec $= 2 \times 10^0$ rad/sec.
We can draw the straight-line Bode plots by considering the behavior of the circuit for $\omega \to 0$ and $\omega \to \infty$. For $\omega \to 0$, $H(\omega \to 0) \approx \frac{j0}{1}$, giving $|H(\omega \to 0)| \approx 0$ and $\angle H(\omega \to 0) \approx \frac{\pi}{2}$ rad. For $\omega \to \infty$, $H(\omega \to \infty) \approx \frac{j\infty}{j\infty} \approx \frac{\infty}{\infty}$, giving $|H(\omega \to \infty)| \approx 1$ and $\angle H(\omega \to \infty) \approx 0$ rad.
The magnitude plot's "corner" occurs at ω_c. The angle plot's two "corners" occur at $0.1\omega_c$ and $10\omega_c$. The plots are shown below.

Log-log plot of transfer function magnitude



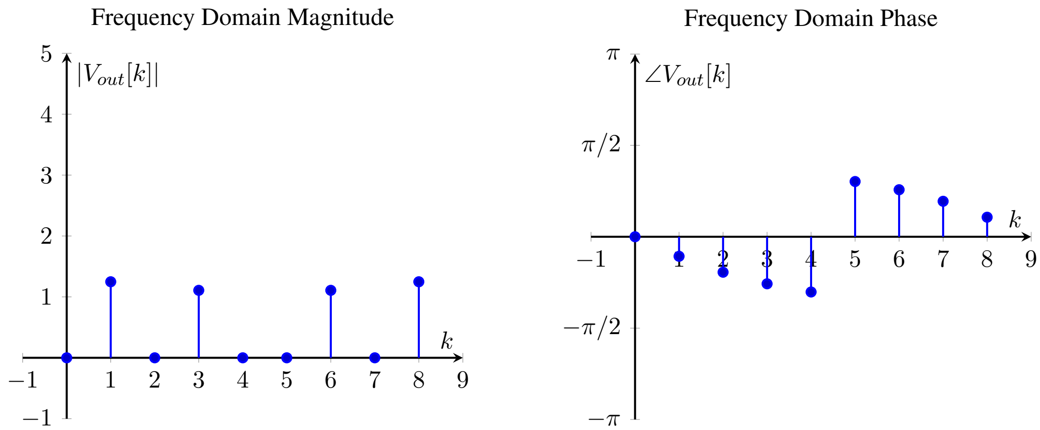
Semi-log plot of transfer function phase



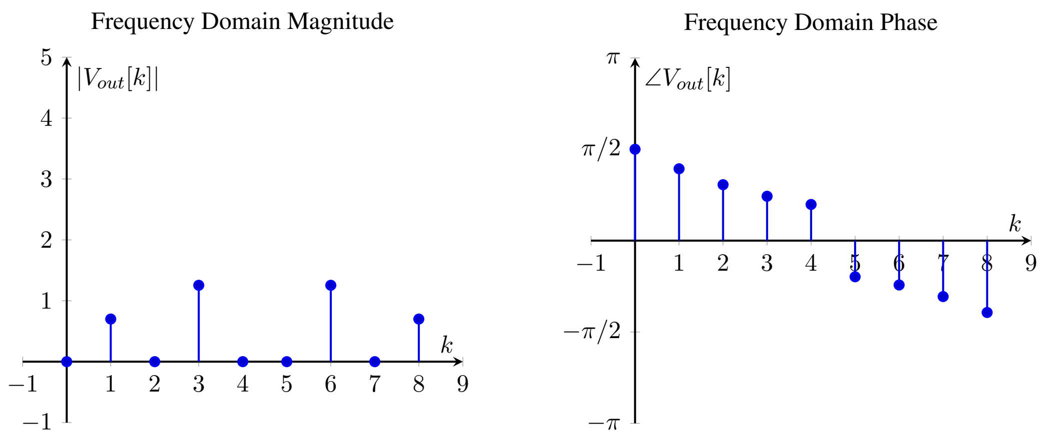
(c) The signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$ is input into the circuit in Figure 4, giving output signal $v_{out}(t)$. You take $N = 9$ samples of the function $v_{out}(t)$ every $\Delta = 1$ seconds; i.e. at $t =$

$\{0, 1, 2, \ldots, 8\}$, forming a 9 element vector of samples $\vec{v}_{out}$. We have given you several possible plots below that may represent the DFT coefficients $\vec{V}_{out}$ of the sampled signal $\vec{v}_{out}$. **For each of the four candidate solutions, circle the statement which is true. Provide a one-sentence explanation for your choice in the box provided.** *Reminder:* $\omega = 2\pi f$.

(HINT: Exactly one of the candidate solutions below is correct. Consequently, no precise numerical calculations are required to get full credit.)

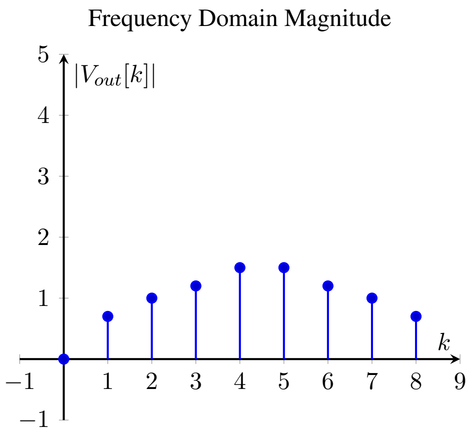


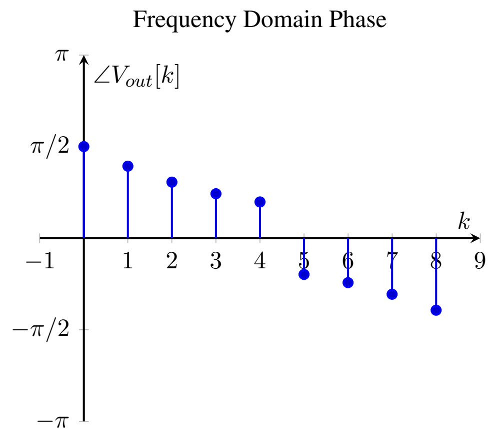
Solution: Incorrect. This is a low-pass filtered signal (not a high-pass filtered signal, as this circuit would produce).



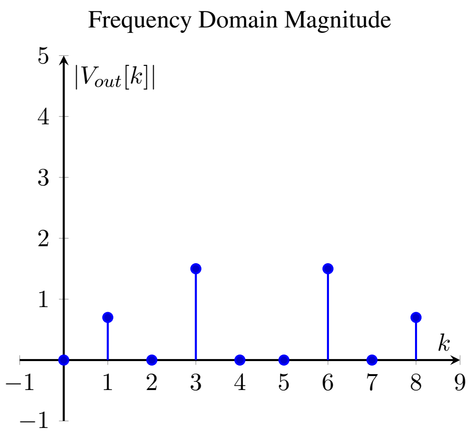
Solution: Correct. This is a high-pass filtered signal with the correct nonzero frequency components and correct phase. The signal is also conjugate-symmetric, as we would expect from a real signal.

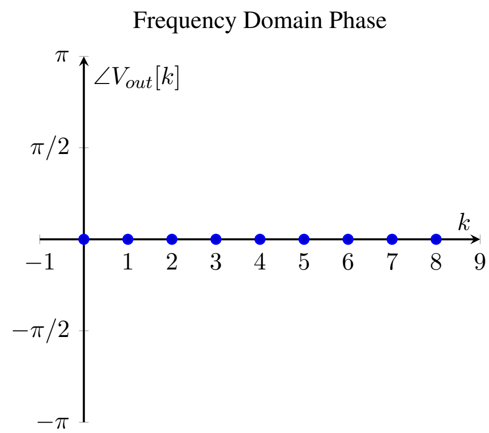
Notice that here, we have phases that are not zero in the plot for magnitudes that are zero. There is nothing wrong with that since a zero magnitude corresponds to zero, no matter what the phase is.





Solution: Incorrect. Frequency components that were zero in the input will not increase in magnitude by being filtered.





Solution: Incorrect. The high-pass filtered magnitude is correct, but there is no phase change from the input, which is incorrect.

5. Adapting Proofs to the Complex Case

At many points in the course, we have made assumptions that various matrices or eigenvalues are real while discussing various theorems. If you have noticed, this has always happened in contexts where we have invoked orthonormality during the proof or statement of the result. Now that you understand the idea of orthonormality for complex vectors, and how to adapt Gram-Schmitt to complex vectors, you can go back and remove those restrictions. This problem asks you to do exactly that.

Unlike many of the problems that we have given you in 16A and 16B, this problem has far less hand-holding — there aren't multiple parts breaking down each step for you. Fortunately, you have the existing proofs in your notes to work based on. So this problem has a secondary function to help you solidify your understanding of these earlier concepts ahead of the final exam.

There is one concept that you will need beyond the idea of what orthogonality means for complex vectors as well as the idea of conjugate-transposes of vectors and matrices. The analogy of a real symmetric matrix S that satisfies $S = S^\top$ is what is called a Hermitian matrix H that satisfies $H = H^*$ where $H^* = \overline{H^\top}$ is the conjugate-transpose of H.

(a) The upper-triangularization theorem for all (potentially complex) square matrices A says that there exists an orthonormal (possibly complex) basis V so that V^*AV is upper-triangular.

Adapt the proof from the real case with assumed real eigenvalues to prove this theorem.

Feel free to assume that any square matrix has an (potentially complex) eigenvalue/eigenvector pair. You don't need to prove this. But you can make no other assumptions.

(HINT: Use the exact same argument as before, just use conjugate-transposes instead of transposes.)

Solution:

The proof proceeds by induction over the size of matrices. The result is trivialy true for 1-dimensional matrices since every 1-dimensional matrix is already diagonal, and hence upper-triangular.

We now assume (for purposes of induction) that the statement is true for all $n-1$-dimensional matrices. This means that given any $n-1$-dimensional matrix M, we can find an orthonormal basis V_{n-1} with $n-1$ vectors so that $V_{n-1}^* MV$ is upper-triangular.

Given an n-dimensional matrix A, we find a possibly complex eigenvalue/eigenvector pair $\lambda, \vec{v}$ with $\|\vec{v}\| = 1$ so that $\lambda\vec{v} = A\vec{v}$. We take the eigenvector $\vec{v}$ and extend it to an orthonormal potentially complex basis for all of n-dimensional complex space $U = [\vec{v}, R = \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_{n-1}]]$. This can be done using Gram-Schmitt, which discussion showed works just as well for complex vectors. Then we change coordinates to this orthonormal basis by computing

$$U^*AU = [\vec{v}, R]^* A[\vec{v}, R] \tag{11}$$

$$= \begin{bmatrix} \vec{v}^* \\ R^* \end{bmatrix} [A\vec{v}, AR] \tag{12}$$

$$= \begin{bmatrix} \vec{v}^* \\ R^* \end{bmatrix} [\lambda\vec{v}, AR] \tag{13}$$

$$= \begin{bmatrix} \lambda\vec{v}^*\vec{v} & \vec{v}^*AR \\ \lambda R^*\vec{v} & R^*AR \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} \lambda & \vec{v}^*AR \\ \vec{0} & R^*AR \end{bmatrix} \tag{15}$$

where we simply use the orthogonality of R and $\vec{v}$ to get the structure of the first column.

It is clear we want to recurse into the R^*AR block.

So we use the inductive hypothesis on $M = R^*AR$ to get a basis V_{n-1} such that $V_{n-1}^* M V_{n-1} = V_{n-1}^* R^* A R V_{n-1} = (RV_{n-1})^* A (RV_{n-1})$ is upper-triangular.

Now, define $V = [\vec{v}, RV_{n-1}]$ and compute:

$$
\begin{aligned}
V^*AV &= [\vec{v}, RV_{n-1}]^* A [\vec{v}, RV_{n-1}] && (16)\\
&= \begin{bmatrix} \vec{v}^* \\ V_{n-1}^* R^* \end{bmatrix} [A\vec{v}, ARV_{n-1}] && (17)\\
&= \begin{bmatrix} \vec{v}^* \\ V_{n-1}^* R^* \end{bmatrix} [\lambda\vec{v}, ARV_{n-1}] && (18)\\
&= \begin{bmatrix} \lambda & \vec{v}^* ARV_{n-1} \\ \lambda V_{n-1} R^* \vec{v} & V_{n-1}^* R^* ARV_{n-1} \end{bmatrix} && (19)\\
&= \begin{bmatrix} \lambda & \vec{v}^* ARV_{n-1} \\ V_{n-1}\vec{0} & V_{n-1}^* R^* ARV_{n-1} \end{bmatrix} && (20)\\
&= \begin{bmatrix} \lambda & \vec{v}^* ARV_{n-1} \\ \vec{0} & (RV_{n-1})^* A (RV_{n-1}) \end{bmatrix} && (21)
\end{aligned}
$$

which is upper triangular since by the inductive-hypothesis, $(RV_{n-1})^* A (RV_{n-1})$ is upper-triangular. All that remains is to show that V is an orthonormal matrix, which can be seen by computing

$$
\begin{aligned}
V^*V &= [\vec{v}, RV_{n-1}]^* [\vec{v}, RV_{n-1}] && (22)\\
&= \begin{bmatrix} \vec{v}^* \\ V_{n-1}^* R^* \end{bmatrix} [\vec{v}, RV_{n-1}] && (23)\\
&= \begin{bmatrix} \vec{v}^*\vec{v} & \vec{v}^* RV_{n-1} \\ V_{n-1}^* R^* \vec{v} & V_{n-1}^* R^* RV_{n-1} \end{bmatrix} && (24)\\
&= \begin{bmatrix} 1 & \vec{0}^* V_{n-1} \\ V_{n-1}^* \vec{0} & V_{n-1}^* (R^*R) V_{n-1} \end{bmatrix} && (25)\\
&= \begin{bmatrix} 1 & \vec{0}^* \\ \vec{0} & V_{n-1}^* V_{n-1} \end{bmatrix} && (26)\\
&= \begin{bmatrix} 1 & \vec{0}^* \\ \vec{0} & I \end{bmatrix} && (27)\\
&= I && (28)
\end{aligned}
$$

which completes the proof. By induction, all complex matrices can be upper-triangularized by an orthonormal change of coordinates.

Congratulations, once you have completed this part you essentially can solve all systems of linear differential equations based on what you know, and you can also complete the proof that having all the eigenvalues being stable implies BIBO stability.

(b) The spectral theorem for Hermitian matrices says that a Hermitian matrix has all real eigenvalues and an orthonormal set of eigenvectors.

Adapt the proof from the real symmetric case to prove this theorem.

(HINT: As before, you should just leverage upper-triangularization and use the fact that $(ABC)^* = C^*B^*A^*$. There is a reason that this part comes after the first part.)

Solution: By upper-triangularization, a Hermitian matrix H can be upper-triangularized so that an upper-triangular matrix $U = V^*HV$ where V is orthonormal. But $U^* = (V^*HV)^* = V^*H^*(V^*)^* = V^*H^*V = V^*HV = U$ since H is Hermitian. This means that the off-diagonal elements of U must be zero since conjugate transposing would make the upper-triangular elements become the conjugates of the lower triangular elements, which would be equal to zero since U is upper-triangular. Furthermore, along the diagonal, $U[i][i] = \overline{U}[i][i]$ implies that all these elements must be real. Consequently, U must be a real diagonal matrix. Call $U = \Lambda$ to emphasize this fact.

Now we know that $\Lambda = V^*HV$ which implies that $H = V\Lambda V^*$. Consider the i-th column vector $\vec{v}_i$ of V. Computing $H\vec{v}_i = V\Lambda V^*\vec{v}_i = V\Lambda\vec{e}_i = V\lambda_i\vec{e}_i = \lambda_i\vec{v}_i$ where we use $\vec{e}_i$ to denote the i-th standard basis vector — i.e. the i-th column of the identity matrix — and λ_i is the i-th diagonal entry in Λ which we know from above must be a real number.

This means that all the columns of V are eigenvectors corresponding to real eigenvalues, establishing the theorem we want to prove.

(c) The SVD for complex matrices says that any rectangular (potentially complex) matrix A can be written as $A = \sum_i \vec{u}_i\sigma_i\vec{w}_i^*$ where σ_i are real positive numbers and the collection $\{\vec{u}_i\}$ are orthonormal (but potentially complex) as well as $\{\vec{w}_i\}$.

Adapt the derivation of the SVD from the real case to prove this theorem.

Feel free to assume that A is wide. (Since you can just conjugate-transpose everything to get a tall matrix to become wide.)

(HINT: Analogously to before, you're going to have to show that the matrix A^*A is Hermitian and that it has non-negative eigenvalues. Use the previous part. There is a reason that this part comes after the previous parts.)

Solution:

Consider $H = A^*A$. Clearly $H^* = (A^*A)^* = A^*(A^*)^* = A^*A = H$ and so H is Hermetian. From the previous part, it can be written $H = V\Lambda V^*$ where V are the eigenvalues and λ_i are the real eigenvalues. Notice that $\vec{v}_i^*H\vec{v}_i = \vec{v}_i^*\lambda_i\vec{v}_i = \lambda_i$ but also that $\vec{v}_i^*H\vec{v}_i = \vec{v}_i^*A^*A\vec{v}_i = (A\vec{v}_i)^*(A\vec{v}_i) = \|A\vec{v}_i\|^2 \geq 0$. This means that $\lambda_i \geq 0$ and so we can just ask for them to be sorted in descending order, since we are free to rearrange eigenvalues to whatever order we want as long as we sort the corresponding eigenvectors to maintain the correspondance.

Let $\sigma_i = \sqrt{\lambda_i}$ for all the nonzero λ_i. Because $\lambda_i > 0$, this means that $\sigma_i > 0$ as well. Suppose that there are r of them. Clearly, for all the others $\vec{v}_k$ for $k > r$, we know by the fact that we sorted the non-negative eigenvalues in descending order that $H\vec{v}_k = \vec{0}$. But this means that $\vec{v}_k^*H\vec{v}_k = 0$ and hence $\|(A\vec{v}_k)\|^2 = (A\vec{v}_k)^*(A\vec{v}_k) = 0$. This means that $A\vec{v}_k = \vec{0}$ as well since the only vector with zero length is the zero vector. That means that all these $\vec{v}_k$ lie in the nullspace of A.

Anyway, for the rest, define $\vec{u}_i = \frac{1}{\sigma_i}A\vec{v}_i$ for $i = 1, \ldots, r$. Computing the norm $\|\vec{u}_i\| = \sqrt{\vec{u}_i^*\vec{u}_i} = \sqrt{\frac{\vec{v}_i^*A^*A\vec{v}_i}{\sigma_i^2}} = \sqrt{\frac{\vec{v}_i^*H\vec{v}_i}{\lambda_i}} = \sqrt{\frac{\lambda_i\vec{v}_i^*\vec{v}_i}{\lambda_i}} = 1$ so they're normalized. Meanwhile, the inner products $\vec{u}_k^*\vec{u}_i = \frac{\vec{v}_k^*H\vec{v}_i}{\sigma_k\sigma_i} = \frac{\lambda_i\vec{v}_k^*\vec{v}_i}{\sigma_k\sigma_i} = 0$ so these are also orthogonal and hence orthonormal.

Defining $\vec{w}_i = \vec{v}_i$ we now have two sets of orthonormal vectors.

All that remains is to show that $A = \sum_{i=1}^r \vec{u}_i\sigma_i\vec{w}_i^*$. To do this, consider any arbitrary vector $\vec{x}$. Because the V is an orthonormal basis by the previous part, we know that there exist coefficients so that $\vec{x} = \sum_j \alpha_j\vec{v}_j$.

Now we just compute:

$$A\vec{x} = A\sum_j \alpha_j \vec{v}_j \tag{29}$$
$$= (\sum_{j=1}^{r} \alpha_j A\vec{v}_j) + (\sum_{j>r} \alpha_j A\vec{v}_j) \tag{30}$$
$$= (\sum_{j=1}^{r} \alpha_j A\vec{v}_j) + \vec{0} \tag{31}$$
$$= \sum_{j=1}^{r} \alpha_j \sigma_j \vec{u}_j \tag{32}$$

where the last line follows from the definition of $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$. Now, compare what happened above with

$$(\sum_{i=1}^{r} \vec{u}_i \sigma_i \vec{w}_i^*)\vec{x} = (\sum_{i=1}^{r} \vec{u}_i \sigma_i \vec{w}_i^*) \sum_j \alpha_j \vec{v}_j \tag{33}$$
$$= \sum_{i=1}^{r} \vec{u}_i \sigma_i \sum_j \alpha_j \vec{w}_i^* \vec{v}_j \tag{34}$$
$$= \sum_{i=1}^{r} \vec{u}_i \sigma_i \alpha_i \tag{35}$$

where the second sum over j vanished because $\vec{w}_i^* \vec{v}_j = \vec{v}_i^* \vec{v}_j$ and the V are orthonormal. This means that only the $i = j$ term survives in that sum.

Because the two are the same no matter what $\vec{x}$ is, the two matrices must be the same — since a matrix is defined by what it does to vectors. If two matrices always do the same thing to every vector, they must be the same matrix.

This establishes the SVD in the context of complex matrices.

In all of these proofs, the proof was identical to the real case except we had to use complex conjugate transposes instead of transposes.

6. Rank 1 Decomposition

In this problem, we will decompose a few images into linear combinations of rank 1 matrices. Remember that outer product of two vectors $\vec{s}\vec{g}^T$ gives a rank 1 matrix. It has rank 1 because clearly, the column span is one-dimensional — multiples of $\vec{s}$ only — and the row span is also one dimensional — multiples of $\vec{g}^T$ only.

(a) Consider a standard 8×8 chessboard shown in Figure 5. Assume that black colors represent -1 and that white colors represent 1.

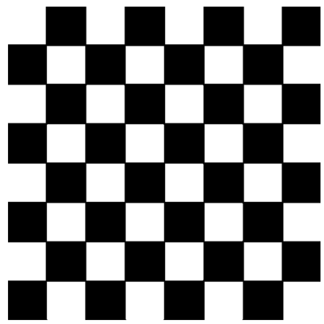


Figure 5: 8×8 chessboard.

Hence, that the chessboard is given by the following 8×8 matrix C_1:

$$C_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Express C_1 as a linear combination of outer products. *Hint: In order to determine how many rank 1 matrices you need to combine to represent the matrix, find the rank of the matrix you are trying to represent.*

Solution:

The matrix C_1 only has rank 1, since column vectors 1, 3, 5, and 7 are the same, and column vectors 2, 4, 6, and 8 are multiples of the other columns. This means that we can express C_1 by multiplying the first column vector $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^T$ by the multiples required to generate the other columns, which are $1, -1, 1, \ldots, -1$. As a result, we get the following outer product form:

$$C_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}^T$$

(b) For the same chessboard shown in Figure 5, now assume that black colors represent 0 and that white colors represent 1.

Hence, the chessboard is given by the following 8×8 matrix C_2:

$$C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Express C_2 as a linear combination of outer products.

Solution:

The chessboard is now a rank 2 image, so we need to decompose it.

There are multiple valid solutions. This is just one of them. Give yourself full credit for any valid solution.

We can look at the new matrix as a linear combination of the matrix from part (a) and a new matrix. Hence, we can write it as:

$$\begin{aligned} C_2 &= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{2}\left(\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \right) \\ &= \frac{1}{2}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}^T + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \end{aligned}$$

(c) Now consider the Swiss flag shown in Figure 6. Assume that red colors represent 0 and that white colors represent 1.

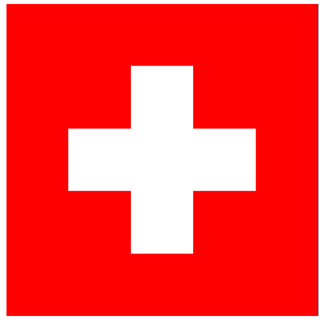


Figure 6: Swiss flag.

Assume that the Swiss flag is given by the following 5×5 matrix S:

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Furthermore, we know that the Swiss flag can be viewed as a superposition of the following pairs of images:

Figure 7: Pairs of images - Option 1

Figure 8: Pairs of images - Option 2

Express the S in two different ways: i) as a linear combination of the outer products inspired by the Option 1 images and ii) as a linear combination of outer products inspired by the Option 2 images.

Solution: Based on the given images, we can decompose the Swiss flag into the following rank-1 matrices.

Option 1:

$$S = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T$$

Option 2:

$$S = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T$$

Note here that the there does not necessarily exist a unique decomposition for an image.

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