EECS 16B Designing Information Devices and Systems II
Spring 2021 UC Berkeley

Homework 9

This homework is due on Sunday, March 21, 2021, at 11:00PM. Self-grades and HW Resubmission are due on Tuesday, March 30, 2021, at 11:00PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 8 and Note 9

   (a) What is the matrix test for controllability of a general linear discrete time system $\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u[t]$?

   Solution: We construct the controllability matrix $\mathcal{C} = \begin{bmatrix} b & Ab & A^2b & \dots & A^{n-1}b \end{bmatrix}$. If $rank(\mathcal{C}) = n$, meaning it is full rank, then the system is controllable.

   Just as a reminder, this means that given any initial state, we can construct a sequence of inputs that lead us to any goal state in n timesteps.

2. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}[t+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t].$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[t+1] = A\vec{x}[t] + Bu[t]$.

(a) **Is this system controllable?**

Solution: We calculate

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Observe that the $\mathcal{C}$ matrix has linearly independent columns (we can't get the 1 as a multiple of the 0) and hence our system is controllable.

(b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

Solution: We have to calculate the eigenvalues of matrix A. Thus,

$$\begin{aligned} \det(\lambda I - A) &= 0 \\ \det \begin{bmatrix} \lambda + 2 & -2 \\ 2 & \lambda - 3 \end{bmatrix} &= 0 \\ \lambda^2 - \lambda - 2 &= 0 \\ \lambda_1 = 2, \lambda_2 &= -1 \end{aligned}$$

Since the magnitude of eigenvalue λ_1 is greater than 1, and the magnitude of $\lambda_2 = 1$, the discrete-time system is unstable.

(c) Suppose we use state feedback of the form $u[t] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[t]$

Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

Solution: The closed loop system using state feedback has the form

$$\vec{x}[t+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[t] = \left(\begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[t]$$

Thus, the closed loop system has the form

$$\vec{x}[t+1] = \underbrace{\begin{bmatrix} -2 + f_1 & 2 + f_2 \\ -2 + f_1 & 3 + f_2 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}[t]$$

Finding the characteristic polynomial of the above system, we have

$$\det\left(\lambda I - \begin{bmatrix} -2+f_1 & 2+f_2 \\ -2+f_1 & 3+f_2 \end{bmatrix}\right) = (\lambda+2-f_1)(\lambda-3-f_2) - (-2-f_2)(2-f_1)$$
$$= \lambda^2 - f_1\lambda - f_2\lambda - \lambda + f_1f_2 - 6 - 2f_2 + 3f_1 - (-4 + f_1f_2 + 2f_1 - 2f_2)$$
$$= \lambda^2 - (1+f_1+f_2)\lambda + f_1 - 2$$

However, we want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. That means we want

$$\lambda^2 - (1+f_1+f_2)\lambda + f_1 - 2 = \left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right)$$

or equivalently:

$$\lambda^2 - (1+f_1+f_2)\lambda + f_1 - 2 = \lambda^2 - \frac{1}{4}$$

Equating the coefficients of the different powers of λ on both sides of the equation, we get,

$$1 + f_1 + f_2 = 0$$
$$f_1 - 2 = -\frac{1}{4}$$

The above system of equations gives us $f_1 = \frac{7}{4}, f_2 = -\frac{11}{4}$.

(d) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix A and B from (a) for the system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] + w[t]$$

into the Jupyter notebook "**eigenvalue_placement.ipynb**" and use the random input $w[t]$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}[t]$ evolves over time for the given A. **What do you see happening to the norm of the state?**

Solution: See Jupyter notebook "**eigenvalue_placement_sol.ipynb**" for solution. The norm of $\vec{x}(t)$ increases with time for the given A. This is because the matrix A has eigenvalues with magnitude greater than one as we discussed in (b) and thus the state keeps growing at each time step.

(e) Add the feedback computed in part (c) to the system in the notebook and **explain how the norm of the state changes.**

Solution: The eigenvalues of the closed loop system are at $\frac{1}{2}$ and -$\frac{1}{2}$. Thus, the norm of the state variable is now bounded with time. Check the solution in the Jupyter notebook.

(f) **[OPTIONAL]** Now suppose we've got a different system described by the controlled scalar difference equation $z[t+1] = z[t] + 2z[t-1] + u[t]$. To convert this second-order difference equation to a two-dimensional discrete time system, we will let $\vec{y}[t] = \begin{bmatrix} z[t-1] \\ z[t] \end{bmatrix}$ **Write down the system representation for $\vec{y}$ in the following matrix form:**

$$\vec{y}(t+1) = A_y\vec{y}(t) + B_y u(t). \tag{1}$$

Please specify what the matrix A_y and the vector B_y.

Solution: From the problem, we have $z[t+1] = 2z[t-1] + z[t] + u[t]$, which will become the second row of our system. We can then write the equation in matrix form as

$$\vec{y}[t+1] = \begin{bmatrix} z[t] \\ z[t+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} z[t-1] \\ z[t] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$$

where $A_y = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(g) **[OPTIONAL]** We will now show how the initial system for $\vec{x}[t]$ can be converted to the system for $\vec{y}[t]$ using a change of basis. Suppose we change coordinates with the transformation $\vec{y}[t] = P\vec{x}[t]$. **Write down the state-transition matrices of $\vec{y}[t]$ in terms of the state transition matrices of $\vec{x}[t]$, i.e., express A_y and B_y in terms of A, B, and P.** Additionally, confirm that for $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, the resulting state space representation of $\vec{y}[t]$ is the same as in the previous part (i.e. we get the same A_y, B_y).

Solution: As we know from before, $\vec{x}[t+1] = A\vec{x}[t] + Bu[t]$. Then,

$$\begin{aligned} \vec{y}[t+1] &= P\vec{x}[t+1] \\ &= P(A\vec{x}[t] + Bu[t]) \\ &= PA\vec{x}[t] + PBu[t] \\ &= PAP^{-1}\vec{y}[t] + PBu[t] \end{aligned}$$

Thus,

$$\begin{aligned} A_y &= PAP^{-1} \\ B_y &= PB \end{aligned}$$

We confirm that,

$$\begin{aligned} PAP^{-1} &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = A_y \end{aligned}$$

(Note: the above is not a typo. The inverse of this particular P matrix is really itself.)

We also confirm that

$$PB = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B_y$$

(h) **[OPTIONAL]** For the $\vec{y}$ system from part (f), **design a feedback gain matrix** $\begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix}$ **to place the closed-loop eigenvalues at** $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$**. Confirm that** $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P$.

Solution: Solving for the new feedback matrix: The closed loop system using state feedback has the form

$$\vec{y}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{y}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t] = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{y}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} \vec{y}[t] \right) = \left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} \right) \vec{y}[t]$$

Thus, the closed loop system has the form

$$\vec{y}[t+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 2+\bar{f}_1 & 1+\bar{f}_2 \end{bmatrix}}_{A_{\text{cl}}} \vec{y}[t]$$

Thus, finding the eigenvalues of the above system we have

$$\det(\lambda I - \begin{bmatrix} 0 & 1 \\ 2+\bar{f}_1 & 1+\bar{f}_2 \end{bmatrix}) = 0 \Rightarrow \lambda^2 - (1+\bar{f}_2)\lambda - (2+\bar{f}_1) = 0$$

However, we want to place the eigenvalue at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Thus, this means that

$$\lambda^2 - (1+\bar{f}_2)\lambda - \bar{f}_1 - 2 = \left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right)$$

$$\lambda^2 - (1+\bar{f}_2)\lambda - \bar{f}_1 - 2 = \lambda^2 - \frac{1}{4}$$

Equating the co-efficients of λ on both sides, we get

$$\begin{aligned} 1 + \bar{f}_2 &= 0 \\ -\bar{f}_1 - 2 &= -\frac{1}{4} \end{aligned}$$

The above system of equations gives us $\bar{f}_1 = \frac{-7}{4}, \bar{f}_2 = -1$
Matrix multiplication shows that

$$\begin{bmatrix} \frac{7}{4} & -\frac{11}{4} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P = \begin{bmatrix} -\frac{7}{4} & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

.

3. Tracking a Desired Trajectory in Continuous Time

The treatment in 16B so far has treated closed-loop control as being about holding a system steady at some desired operating point, by placing the eigenvalues of the state transition matrix. This control used the actual current state to apply a control signal designed to bring the eigenvalues in the region of stability. Meanwhile, the idea of controllability itself was more general and allowed us to make an open-loop trajectory that went pretty much anywhere. This problem is about combining these two ideas together to make feedback control more practical — how we can get a system to more-or-less closely follow a desired trajectory, even though it might not start exactly where we wanted to start and in principle could be affected by small disturbances throughout.

In this question, we will also see that everything that you have learned to do closed-loop control in discrete-time can also be used to do closed-loop control in continuous time.

Consider the specific 2-dimensional system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + \vec{w}(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + \vec{w}(t) \tag{2}$$

where $u(t)$ is a scalar valued continuous control input and $\vec{w}(t)$ is a bounded disturbance (noise).

(a) **Would the given system be controllable if we viewed the parameters A and $\vec{b}$ as the parameters of a discrete-time system, i.e. $\vec{x}_d[t+1] = A\vec{x}_d[t] + \vec{b}u_d[t]$?**

Solution: By substituting the matrix A and $\vec{b}$ into the controllability matrix, we have:

$$\mathcal{C}_2 = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \text{ (full rank)}$$

Since $\mathcal{C}_2$ is full rank, the system would be controllable if it were a discrete-time system.

It turns out (although we will not prove so in this course), being controllable also means that we can navigate the system state to any desired state by choosing an input trajectory $u(t)$ in continuous-time as well. Showing this is a bit more subtle than it is for the discrete-time case, which is why we don't do it. But as this problem shows, we certainly can see that controllability allows us to place the closed-loop eigenvalues wherever we want. This is the same in discrete-time and continuous-time.

(b) In an ideal noiseless scenario, the desired control signal $u^*(t)$ makes the system follow the desired trajectory $\vec{x^*}(t)$ that satisfies the following dynamics:

$$\frac{d}{dt}\vec{x^*}(t) = A\vec{x^*}(t) + \vec{b}u^*(t) \tag{3}$$

The presence of the bounded noise term $\vec{w}(t)$ makes the actual state $\vec{x}(t)$ deviate from the desired $\vec{x^*}(t)$ and follow (2) instead. In the following subparts, we will analyze how we can adjust the desired control signal $u^*(t)$ in (3) to the control input $u(t)$ in (2) so that the deviation in the state caused by $\vec{w}(t)$ remains bounded.

Represent the state as $\vec{x}(t) = \vec{x^*}(t) + \vec{v}(t)$ and $u(t) = u^*(t) + u_v(t)$. Using (2) and (3), we can represent the evolution of the trajectory deviation $\vec{v}(t)$ as a function of the control deviation $u_v(t)$ and the bounded disturbance $\vec{w}(t)$ as:

$$\frac{d}{dt}\vec{v}(t) = A_v\vec{v}(t) + \vec{b}_v u_v(t) + \vec{w}(t) \tag{4}$$

What are A_v and $\vec{b}_v$ in terms of the original system parameters A and $\vec{b}$? *(HINT: Write out equation (2) in terms of $\vec{x^*}(t)$, $\vec{v}(t)$, $u^*(t)$ and $u_v(t)$.)*

Solution: Using the change of variables $\vec{x}(t) = \vec{x^*}(t) + \vec{v}(t)$ and $u(t) = u^*(t) + u_v(t)$ in (2), we get

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + w(t)$$
$$\implies \frac{d}{dt}\vec{x^*}(t) + \frac{d}{dt}\vec{v}(t) = A\vec{x^*}(t) + A\vec{v}(t) + \vec{b}u^*(t) + \vec{b}u_v(t) + \vec{w}(t)$$
$$\implies \frac{d}{dt}\vec{v}(t) = A\vec{v}(t) + \vec{b}u_v(t) + \vec{w}(t) + \left(A\vec{x^*}(t) + \vec{b}u^*(t) - \frac{d}{dt}\vec{x^*}(t)\right)$$

Using (3) we know that the last term in parenthesis in zero, so

$$\frac{d}{dt}\vec{v}(t) = A\vec{v}(t) + \vec{b}u_v(t) + \vec{w}(t)$$

By pattern matching with (4), we can see that $A_v = A$, $\vec{b}_v = \vec{b}$.

Note that this implies the disturbance $\vec{w}(t)$ is entirely something that must be dealt with in the $\vec{v}$ dynamics. It doesn't affect the desired trajectory at all.

(c) **Are the dynamics that you found for $\vec{v}(t)$ in part (b) stable? Based on this, in the presence of bounded disturbance $\vec{w}(t)$, will $\vec{x}(t)$ in (2) follow the desired trajectory $\vec{x^*}(t)$ closely if we just apply the control $u(t) = u^*(t)$ to the original system in (2), i.e. $u_v(t) = 0$?**

(HINT: Use the numerical values of A and $\vec{b}$ from (2) in the solution from part (b) to determine stability of $\vec{v}(t)$.)

Solution: The key is to study the stability of $\vec{v}(t)$ when $u_v(t) = 0$ as indicated in the problem:

$$\frac{d}{dt}\vec{v}(t) = A\vec{v}(t) + \vec{w}(t)$$

Recall that the condition for stability in the continuous-time case is that the real part of the eigenvalues of the state transition matrix A must be less then zero.

$$\mathbb{R}(\lambda_A) < 0$$

If any of the eigenvalues of A has a real part that is not strictly negative, then over time the state deviation $\vec{v}(t)$ can grow without bound in response to a disturbance.

Note that since A is an upper-triangular matrix, its eigenvalues lie on the diagonal, namely, 2 and 2. In this case, since they clearly have real parts greater than zero, $\vec{v}(t)$ will follow a growing exponential trajectory in the form of e^{2t}. Hence we can see that the system is vulnerable to any bounded disturbance $\vec{w}(t)$, and we will not end up following the intended trajectory $\vec{x^*}(t)$.

Now, we want to apply state feedback control to the system using $u_v(t)$ to get it to more or less follow the desired trajectory $\vec{x^*}(t)$.

(d) **[OPTIONAL]** For the $\vec{v}(t), u_v(t)$ system, **apply feedback control by choosing $u_v(t)$ as a function of $\vec{v}(t)$ that would place both the eigenvalues of the closed-loop $\vec{v}(t)$ system at -10.** *(HINT: $u_v(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \vec{v}(t)$. Find f_0 and f_1.)*

Solution:

We can assume that the input $u_v(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \vec{v}(t)$, which is a linear function of the current state $\vec{v}(t)$. With the new input, the system equation for $\vec{v}(t)$ is given by:

$$\frac{d}{dt}\vec{v}(t) = A_v\vec{v}(t) + \vec{b}\begin{bmatrix} f_0 & f_1 \end{bmatrix}\vec{v}(t) + \vec{w}(t)$$
$$\implies \frac{d}{dt}\vec{v}(t) = \begin{bmatrix} 2+f_0 & 1+f_1 \\ f_0 & 2+f_1 \end{bmatrix}\vec{v}(t) + \vec{w}(t)$$

where we denote $A_{cl} = \begin{bmatrix} 2+f_0 & 1+f_1 \\ f_0 & 2+f_1 \end{bmatrix}$ as the state matrix for the closed loop system. The characteristic polynomial for finding the eigenvalues of A_{cl} is given by:

$$\det(\lambda I - A_{cl}) = \begin{bmatrix} \lambda - 2 - f_0 & -1 - f_1 \\ -f_0 & \lambda - 2 - f_1 \end{bmatrix}$$
$$= \lambda^2 - (4 + f_0 + f_1)\lambda + f_0 + 2f_1 + 4$$

To set the eigenvalues to be where we want, we set this equal to $(\lambda+10)(\lambda+10) = \lambda^2 + 20\lambda + 100$. By comparing the coefficients, we have:

$$-(4 + f_0 + f_1) = 20$$
$$f_0 + 2f_1 + 4 = 100$$

Solving the above system of equations, we can find $f_0 = -144$, $f_1 = 120$. Therefore, we can design the state-feedback $u_v(t) = \begin{bmatrix} -144 & 120 \end{bmatrix}\vec{v}(t)$ which will place both the eigenvalues of the closed loop system at -10.

Why did we pick -10? So that it would be stable and aggressively reject disturbances.

(e) **[OPTIONAL]** Based on what you did in the previous parts, and given access to the desired trajectory $\vec{x}^*(t)$, the desired control $u^*(t)$, and the actual measurement of the state $\vec{x}(t)$, **come up with a way to do feedback control that will keep the trajectory staying close to the desired trajectory no matter what the small bounded disturbance $\vec{w}(t)$ does.** *(HINT: Express the control input $u(t)$ in terms of $u^*(t)$, $\vec{x}^*(t)$, and $\vec{x}(t)$.)*

Solution:

From the previous parts, we have successfully found a feedback control law $u_v(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix}\vec{v}(t)$ such that the closed-loop system for $\vec{v}(t)$ is stable as long as the disturbances are bounded. As a result, by changing variables $\vec{x}(t) = \vec{x}^*(t) + \vec{v}(t)$ and $u(t) = u^*(t) + u_v(t)$ that we performed in (b), we can infer that the state $\vec{x}(t)$ will stay close to the desired trajectory $\vec{x}^*(t)$ no matter what the small bounded disturbance $\vec{w}(t)$ does.

Explicitly $u(t) = u^*(t) + u_v(t) = u^*(t) + \begin{bmatrix} -144 & 120 \end{bmatrix}(\vec{x}(t) - \vec{x}^*(t))$ is the continuous control input that we would invoke to achieve this.

This lets us figuratively have our cake and eat it too! We can use the desired/nominal system dynamics from (3) to plan, and by using closed-loop feedback we can make sure that we mostly follow our plan even in the face of disturbances.

4. [OPTIONAL] Group Re-assignment Survey

How are your study groups working out? We hope they have been helpful so far. If you feel things are not going as well as you hoped and you would prefer to be assigned to a new group, please fill out the following form:

Group Re-assignment Survey - Google Form

5. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

Contributors:

- Ashwin Vangipuram.
- Varun Mishra.
- Anant Sahai.
- Ming Jin.
- Ayan Biswas.