1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 10 and Note 11.

(a) How would you use feedback control to choose the closed-loop eigenvalues of a closed-loop discrete-time system?

Solution:

We let \( \vec{u}[i] = F \vec{x}[i] \) so \( \vec{x}[i+1] = A \vec{x}[i] + B \vec{u}[i] = A \vec{x}[i] + BF \vec{x}[i] = (A + BF) \vec{x}[i] \). Then, we calculate the determinant of \( \lambda I - (A + BF) \) to get the characteristic polynomial as a symbolic function of the entries of \( F \). Meanwhile, we calculate the target characteristic polynomial by taking our desired eigenvalues and computing \( \prod \lambda \) as a polynomial. By matching the coefficients of \( \lambda^k \), we get a system of equations where the unknowns are the entries of \( F \). Solving that system of equations gives us the entries of the \( F \) matrix that makes our closed-loop dynamics \( \vec{A}_{cl} = A + BF \) have the desired eigenvalues \( \{ \lambda_i \} \).

The same thing can be done with continuous-time systems as well. Now, \( \vec{u}(t) = F \vec{x}(t) \) so that \( \dot{\vec{x}} = \vec{x} + B \vec{u} = \vec{x} + BF \vec{x} = (A + BF) \vec{x} \).

In both cases, the closed-loop dynamics becomes \( \vec{A}_{cl} = A + BF \) and its eigenvalues can be changed by setting \( F \).

(b) What is the matrix test for controllability of a general linear discrete-time system \( \vec{x}[i+1] = A \vec{x}[i] + B \vec{u}[i] \) with a scalar input \( \vec{u}[i] \)?

Solution: We construct the controllability matrix \( C = \begin{bmatrix} \vec{b} & A \vec{b} & A^2 \vec{b} & \ldots & A^{n-1} \vec{b} \end{bmatrix} \). If \( \text{rank}(C) = n \), meaning it is full rank, then the system is controllable.

Just as a reminder, this means that given any initial state, we can construct a sequence of inputs that lead us to any goal state in \( n \) timesteps. Why? Because

\[
\vec{x}[n] = A^n \vec{x}[0] + \sum_{k=0}^{n-1} A^{n-1-k} B \vec{u}[k] = A^n \vec{x}[0] + C \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[1] \\ u[0] \end{bmatrix} .
\]

If \( \text{rank}(C) = n \), this matrix is invertible and the previous equation can always be solved for the vector of \( u[i] \) given an initial condition and a desired state \( \vec{x}[n] \).

(c) If \( \vec{b} \) above were an eigenvector of \( A \), why would this imply that the system is not controllable if the dimension of \( \vec{x} \) is larger than 1?
Solution: Because the matrix

\[
C = \begin{bmatrix}
\vec{b} & A\vec{b} & A^2\vec{b} & \ldots & A^{n-1}\vec{b}
\end{bmatrix}
\]

(1)

\[
= \begin{bmatrix}
\vec{b} & \lambda\vec{b} & \lambda^2\vec{b} & \ldots & \lambda^{n-1}\vec{b}
\end{bmatrix}
\]

(2)

since \(A\vec{b} = \lambda\vec{b}\) if \(\vec{b}\) is an eigenvector of \(A\). Since all the columns of \(C\) are multiples of each other, the rank is just 1 which is less than \(n > 1\). So the system is not controllable. We can only control the system in the direction of \(\vec{b}\).
2. Stability Criterion

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

![Complex plane divided into regions.](image)

**Figure 1:** Complex plane divided into regions.

(a) Consider the continuous-time system \[ \frac{dx(t)}{dt} = \lambda x(t) + v(t) \]
and the discrete-time system \[ y[i + 1] = \lambda y[i] + w[i] \]. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

**In which regions can the eigenvalue $\lambda$ be for the system to be stable? Fill out the table below to indicate stable regions.** Assume that the eigenvalue $\lambda$ does not fall directly on the boundary between two regions.

<table>
<thead>
<tr>
<th>Continuous Time System $x(t)$</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete Time System $y[i]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Solution:** For the continuous time system to be stable, we need the real part of $\lambda$ to be less than zero. Hence, C, D, G, H satisfy this condition.

On the other hand, for the discrete time system to be stable, we need the norm of $\lambda$ to be less than one. Hence, A, B, C, D satisfy this condition.
3. BIBO Stability

(a) Consider the circuit below with \( R = 1\, \Omega \), \( C = 0.5\, F \).

\[
\begin{array}{c}
\text{\( u(t) \)} \\
\text{\( + \)} \\
\text{\( R \)} \\
\text{\( C \)} \\
\text{\( x(t) \)} \\
\text{\( - \)}
\end{array}
\]

We know the circuit can be modeled by the differential equation

\[
\frac{d}{dt} x(t) = -2x(t) + 2u(t)
\]  

(3)

Show that this system is BIBO stable meaning \( x(t) \) remains bounded for all time if the input \( u(t) \) is bounded. Equivalently, assume \( |u(t)| < \epsilon, \forall t \geq 0 \) and \( |x(0)| < \epsilon \) and use these to show \( |x(t)| < M, \forall t \geq 0 \) for some positive constant \( M \). Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

HINT: You may want to write the expression for \( x(t) \) in terms of \( u(t) \) and \( x(0) \) and then take the norms of both sides to show a bound on \( |x(t)| \). Remember that norm in 1D is absolute value. Some helpful formulas are \(|ab| = |a||b|\), the triangle inequality \(|a + b| \leq |a| + |b|\), and the integral version of the triangle inequality \( \int_a^b |f(\tau)| \, d\tau \leq \int_a^b |f(\tau)| \, d\tau \), which just extends the standard triangle inequality to an infinite sum of terms.

Solution:

For the physical system, we can intuitively see that the voltage on the capacitor can never exceed the voltage from the voltage source, and is thus bounded.

We can also try to understand this as a differential equation and see why it must be bounded mathematically. We know that the solution to the scalar differential equation is given by

\[
x(t) = e^{-2t}x(0) + \int_0^t e^{-2(t-\tau)} 2u(\tau) \, d\tau.
\]

(4)

Then we can try to bound \( x(t) \) for \( t \geq 0 \). We first use the triangle inequality \((a + b) \leq |a| + |b|\) to get

\[
|x(t)| = \left| e^{-2t}x(0) + \int_0^t e^{-2(t-\tau)} 2u(\tau) \, d\tau \right|
\]

(5)

\[
|x(t)| \leq \left| e^{-2t}x(0) \right| + \left| \int_0^t e^{-2(t-\tau)} 2u(\tau) \, d\tau \right|
\]

(6)

We then use the property that the integral of absolute value will always be greater than the absolute value of the integral (equation (6) to (7)), and that an exponential is always positive (equation (7) to (8)):

\[
|x(t)| \leq \left| e^{-2t}x(0) \right| + \int_0^t \left| e^{-2(t-\tau)} 2u(\tau) \right| \, d\tau
\]

(7)

\[
= e^{-2t}|x(0)| + \int_0^t e^{-2(t-\tau)} 2|u(\tau)| \, d\tau
\]

(8)
Finally, plugging in our bounds for $|u(\tau)|$ and $|x(0)|$ and doing the integral:

$$|x(t)| \leq e^{-2t}e + \int_0^t e^{-2(t-\tau)}2e d\tau$$

$$= e^{-2t}e + 2e\int_0^t e^{2\tau} d\tau$$

$$= e^{-2t}e + 2e\left(\frac{e^{2t} - 1}{2}\right)$$

$$= e^{-2t}e + e\left(1 - e^{-2t}\right)$$

$$= e, \forall t \geq 0$$

So we see that our state’s magnitude is bounded for all time. Note that the negative exponent of the exponential is what makes this system stay bounded.

(b) Consider a continuous-time scalar differential equation with known solution

$$\frac{dx}{dt}(t) = ax(t) + bu(t) \quad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau) d\tau.$$  \hfill (14)

Show that if the system has $\Re\{a\} > 0$, then the system is BIBO unstable, so there exists a bounded input $(|u(t)| \leq \epsilon)$ that can result in an unbounded output. Assume $x(0) = 0$.

Solution: We are given that $x(0) = 0$. Then in the solution for $x(t)$ we are just left with the integral term. We want to find a bounded input that makes the state blow up. The first thing to try is the simplest example of a bounded input — a constant input. So let’s consider the case when $u(t) = \epsilon \forall t$, giving us

$$x(t) = \int_0^t e^{a(t-\tau)}b\epsilon d\tau.$$  \hfill (15)

Rearranging the integral, we can focus on the exponential term:

$$x(t) = b\epsilon e^{at} \int_0^t e^{-a\tau} d\tau.$$  \hfill (16)

We can evaluate this integral

$$\int_0^t e^{-a\tau} d\tau = \frac{-1}{a}e^{-a\tau}\bigg|_0^t$$

$$= \frac{1 - e^{-at}}{a}$$

$$\implies x(t) = (b\epsilon e^{at}) \frac{1 - e^{-at}}{a}$$

$$= \frac{b\epsilon}{a} \left(e^{at} - 1\right)$$  \hfill (19)

Now let $a = c + dj$ so we have $e^{at} = e^{ct}e^{jdt}$ where $\Re\{a\} = c > 0$. Then as $t \to \infty$, $x(t)$ will be unbounded since the $e^{ct}$ term will grow exponentially. Formally, one could write $|e^{at} - 1| = |e^{ct}e^{jdt} - 1| = |e^{ct} - 1|$ which is clearly going to infinity as $t \to \infty$ since $c > 0$.

(c) Repeat the previous part to show the system is BIBO unstable for the specific case of purely imaginary $a = j2\pi$. Again assume $x(0) = 0$. 

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All the other purely imaginary unstable cases are the same way for the same essential reason.

**Solution:** This part is a little different than the last part, since the last part diverged due to a positive exponential term. Here if you try the same input equal to all \( \epsilon \), it won’t work to show that the state grows without bound. Recall the solution of \( x(t) \) with the initial condition at zero

\[
x(t) = \int_0^t e^{a(t-\tau)}bu(\tau) \, d\tau.
\]

(21)

Remember, the style of argumentation here is the “counterexample” style. The question asks you to show that some bounded input exists that will make the state grow without bound. Because we know we can get an integral to diverge if we are just integrating a nonzero constant, we decide to try the bounded input \( u(t) = \epsilon e^{j2\pi t} \), whose magnitude is equal to \( \epsilon \) for all \( t \).

Plugging this input and a value in, we see

\[
x(t) = \int_0^t e^{j2\pi(t-\tau)}be^{j2\pi\tau} \, d\tau = \int_0^t e^{j2\pi t}be \, d\tau.
\]

(22)

Factoring out the terms that do not depend on \( \tau \), we are left with

\[
x(t) = be^{j2\pi t} \int_0^t \, d\tau.
\]

(23)

Solving this integral, we get

\[
x(t) = bete^{j2\pi t}.
\]

(24)

Now taking the magnitude of \( x(t) \) using the fact that \( |e^{j\omega t}| = 1 \) for all \( \omega \), we get \( |x(t)| = \epsilon |b| t \) which clearly diverges as \( t \rightarrow \infty \).

(d) We now consider the discrete-time system

\[
x[i+1] = -3x[i] + u[i]
\]

(25)

with \( x[0] = 0 \).

**Is this system stable or unstable?** If stable, prove it. If unstable, find a bounded input sequence \( u[i] \) that causes the system to grow unbounded.

**Solution:** The system is unstable since the eigenvalue \(-3\) has magnitude \( \geq 1 \). This can be formally seen by considering the input

\[
u[i] = 1, 0, 0, 0, 0, \ldots
\]

(26)

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x[i] )</td>
<td>0</td>
<td>1</td>
<td>-3</td>
<td>9</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( u[i] )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>(-3x[i] + u[i] )</td>
<td>1</td>
<td>-3</td>
<td>9</td>
<td>-27</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

This results in the state at times \( i \geq 1 \) to be \( x[i] = (-3)^i \) which has magnitude \( 3^i \). This is clearly exploding exponentially with \( i \), not staying bounded.

(e) For the example in the previous part, **give an explicit sequence of inputs that are not zero but for which the state \( x[i] \) will always stay bounded.** (**HINT:** See if you can find any input pattern that results in an oscillatory behavior.)
Solution:
Consider the following input
\[ u[i] = 1, 3, 1, 3, 1, 3, \ldots \] (27)

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>x[i]</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>\ldots</td>
</tr>
<tr>
<td>u[i]</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>\ldots</td>
</tr>
<tr>
<td>-3x[i] + u[i]</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

In this case, we get \( x[i] = 0 \) when \( t \) is even, and \( x[i] = 1 \) when \( i \) is odd. In fact, there are an infinite number of input sequences that would result in bounded outputs. But because we can find a single example of a bounded input sequence that leads to an unbounded output, the system is deemed unstable. We can’t trust that we will only get nice inputs in engineering contexts.

(f) Consider the discrete-time real system with known solution:
\[ x[i + 1] = ax[i] + bu[i] \]
\[ x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^{i-1-k}bu[k] \] (28)

Show that if \(|a| > 1\), then a bounded input can result in an unbounded output, i.e. the system is BIBO unstable. Assume that \( x[0] = 0 \).

Solution: For simplicity, let’s say that \( u[i] = 1 \) \( \forall i \). This gives us a new expression of
\[ x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^{i-1-k}bu[k] \] (29)
\[ = \sum_{k=0}^{i-1} a^{i-1-k}b \] (30)

Note that the summation now is a sum of a geometric series. Thus, our solution simplifies to
\[ x[i] = \sum_{k=0}^{i-1} a^{i-1-k}b = b \frac{a^i - 1}{a - 1} \] (31)

This is diverging since \( a^i \) has magnitude that grows without bound if \(|a| > 1\), and thus the whole term does as well.

(g) [Optional] since this part was derived in lecture. Now consider the discrete-time stable case where \( a \) is complex and has \(|a| < 1\). Show that as long as \(|u[i]| < \epsilon \) for some \( \epsilon \), that the solution \( x[i] \) will be bounded for all time \( i \).

(HINT: There are a few helpful facts about absolute values and inequalities that are helpful in such proofs. First: \( \sum_j a_j \leq \sum_j |a_j| \). Second: \(|ab| = |a| \cdot |b| \). Third: \(|e^{j\theta}| = 1 \) no matter what real number \( \theta \) is. And fourth, if \( a_i > 0 \) and \( b_i > 0 \), and \( b_i \leq B \), then \( \sum_i a_i b_i \leq \sum_i a_i B = B \sum a_i \).)

Solution: We want to show that all bounded inputs will result in a bounded output. First, we need to think about whether we want to consider the initial condition. Any reasonable definition of stability must let the bound on the output depend on the initial condition, otherwise there would always be a large enough initial condition that would violate the bound even if the system were obviously stable.
Let’s say that \( 0 \leq |x[0]| < \infty \) (a finite initial condition with a value that may be nonzero) and that \( u[i] \leq k \) for all \( t \) (the input is bounded at all timesteps). If we can find a bound for \( |x[i]| \) for all timesteps (i.e. \( |x[i]| \leq \alpha \) where \( \alpha \) is some positive value) then we have shown that this system is stable. The \( \alpha \) is allowed to depend on both \( k \) and \( x[0] \).

Once again, it is good to first understand why this is true before setting out to prove it.

\[
x[1] = ax[0] + bu[0] \quad (32)
\]

\[
\]

\[
\]

Following this pattern we find:

\[
x[i] = a^ix[0] + bu[i - 1] + abu[i - 2] + \cdots + a^{i-1}bu[0] = a^ix[0] + \sum_{k=0}^{i-1} bu[k]a^{i-1-k} \quad (35)
\]

We now want to find an upper bound for the magnitude of \( |x[i]| \). To simplify our upper bound, we will use many of the hints from the question:

\[
|x[i]| = \left| a^ix[0] + \sum_{k=0}^{i-1} bu[k]a^{i-1-k} \right| \quad (36)
\]

\[
\leq \left| a^ix[0] \right| + \sum_{k=0}^{i-1} \left| bu[k]a^{i-1-k} \right| \quad (37)
\]

\[
= \left| a^i \right| \left| x[0] \right| + \sum_{k=0}^{i-1} |b||u[k]| |a|^{i-1-k} \quad (38)
\]

\[
\leq |a|^i \left| x[0] \right| + \sum_{k=0}^{i-1} |b| \epsilon |a|^{i-1-k} \quad (39)
\]

\[
= \left| a^i \right| \left| x[0] \right| + |b| \epsilon \sum_{k=0}^{i-1} |a|^{i-1-k} \quad (40)
\]

\[
\leq |a|^i \left| x[0] \right| + |b| \epsilon \sum_{k=0}^{i-1} |a|^m \quad (41)
\]

\[
= \left| a^i \right| \left| x[0] \right| + |b| \epsilon \frac{1 - |a|^i}{1 - |a|} \quad (42)
\]

\[
< \left| x[0] \right| + |b| \epsilon \frac{1}{1 - |a|} \quad (43)
\]

We first bound an absolute value of a sum by the sum of absolute values in the first inequality (step (36) to (37)). Then we use the fact that the absolute value of a product is the product of absolute values (step (37) to (38)). Then we use our bound on \( |u[i]| \) to upper bound the overall summation (step (38) to (39)). Then we pull out the constants from the summation (step (39) to (40)).

We then convert the index of the summation from \( k \) to \( m \) using the change of variable that \( m = i - 1 - k \) (step (40) to (41)). To find the new bounds of the summation, we will plug in \( k = 0 \) and \( k = i - 1 \) into the expression for \( m \). Thus \( m \) will iterate from \( i - 1 - (0) = i - 1 \) to \( i - 1 - (i - 1) = 0 \) so actually
the bounds of the summation don’t change. We then use the formula for a finite geometric series to remove the summation (step (41) to (42)).

Finally, we can use the key assumption that $|a| < 1$ (note that we haven’t used that fact yet in our analysis). Since $|a| < 1$, we know that $|a|^i < 1$ and $0 \leq 1 - |a|^i < 1$. Plugging this in gets us to our final bound which is clearly finite (step (42) to (43)).

As we have found a bound for $|x[i]|$ given any initial condition and any bounded input, we have shown that this system must be BIBO stable.

These kind of manipulations of sums and inequalities are very common in more advanced mathematical courses. You have seen some of this in your basic calculus courses, and we need to keep up the practice so that you get to the right level if you want to take later courses that touch probability (70 and then 126), optimization (127 and then 189), control (128, 221A), signal processing (120 and then 123), etc. The ideas of bounding are also critical for doing more advanced circuit analysis and design.
4. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

\[
\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i].
\]  \hspace{1cm} (44)

In standard language, we have \( A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) in the form: \( \vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \)

(a) **Is this system controllable?**

**Solution:** We calculate the controllability matrix

\[
C = [\vec{b} \quad A\vec{b}] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}
\]  \hspace{1cm} (45)

Observe that the \( C \) matrix has linearly independent columns and hence our system is controllable.

(b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

**Solution:** We have to calculate the eigenvalues of matrix \( A \). Thus,

\[
0 = \det(\lambda I - A)
\]  \hspace{1cm} (46)

\[
= \det\begin{bmatrix} \lambda + 2 & -2 \\ 2 & \lambda - 3 \end{bmatrix}
\]  \hspace{1cm} (47)

\[
= \lambda^2 - \lambda - 2
\]  \hspace{1cm} (48)

\[
\implies \lambda_1 = 2, \quad \lambda_2 = -1
\]  \hspace{1cm} (49)

Since at least one eigenvalue has a magnitude that is greater than or equal to 1, the discrete-time system is unstable. In this case, both of the eigenvalues are unstable.

(c) Suppose we use state feedback of the form \( u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] = F\vec{x}[i]. \)

**Find the appropriate state feedback constants, \( f_1, f_2 \) so that the state space representation of the resulting closed-loop system has eigenvalues at \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}. \)**

**Solution:** The closed loop system using state feedback has the form

\[
\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]
\]  \hspace{1cm} (50)

\[
= \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]
\]  \hspace{1cm} (51)

\[
= \left( \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i]
\]  \hspace{1cm} (52)
Thus, the closed loop system has the form

\[
\bar{x}[i + 1] = \begin{bmatrix}
-2 + f_1 & 2 + f_2 \\
-2 + f_1 & 3 + f_2
\end{bmatrix} \bar{x}[i]
\]

(53)

Finding the characteristic polynomial of the above system, we have

\[
\det \left( \lambda I - \begin{bmatrix}
-2 + f_1 & 2 + f_2 \\
-2 + f_1 & 3 + f_2
\end{bmatrix} \right) = (\lambda + 2 - f_1)(\lambda - 3 - f_2) - (-2 - f_2)(2 - f_1)
\]

(54)

\[
= \lambda^2 - f_1 \lambda - f_2 \lambda - \lambda + f_1 f_2 - 6 - 2 f_2 + 3 f_1
\]

(55)

\[
- (-4 + f_1 f_2 + 2 f_1 - 2 f_2)
\]

(56)

\[
= \lambda^2 - (1 + f_1 + f_2) \lambda + f_1 - 2
\]

(57)

However, we want to place the eigenvalues at \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \). That means we want

\[
\lambda^2 - (1 + f_1 + f_2) \lambda + f_1 - 2 = \left( \lambda + \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right)
\]

(58)

or equivalently:

\[
\lambda^2 - (1 + f_1 + f_2) \lambda + f_1 - 2 = \lambda^2 - \frac{1}{4}
\]

(59)

Equating the coefficients of the different powers of \( \lambda \) on both sides of the equation, we get,

\[
1 + f_1 + f_2 = 0
\]

(60)

\[
f_1 - 2 = -\frac{1}{4}
\]

(61)

Solving the above system of equations gives us \( f_1 = \frac{7}{4}, f_2 = -\frac{11}{4} \).

(d) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrices \( A \) and \( B \) from (a) for the system \( \bar{x}[i + 1] = A\bar{x}[i] + B\bar{u}[i] + \bar{w}[i] \) into the Jupyter notebook “eigenvalue_placement.ipynb” and use the randomly generated \( \bar{w}[i] \) as the disturbance introduced into the state equation. Observe how the norm of \( \bar{x}[i] \) evolves over time for the given \( A \). What do you see happening to the norm of the state?

**Solution:** See Jupyter notebook “eigenvalue_placement_sol.ipynb” for solution. The norm of \( \bar{x}(t) \) increases with time for the given \( A \). This is because the matrix \( A \) has eigenvalues with magnitude greater than one as we discussed in (b) and thus the state keeps growing at each time step.

(e) Add the feedback computed in part (c) to the system in the notebook and explain how the norm of the state changes.

**Solution:** The eigenvalues of the closed loop system are at \( \frac{1}{2} \) and \( -\frac{1}{2} \). Thus, the norm of the state variable is now bounded with time. Check the solution in the Jupyter notebook.

(f) Now suppose we’ve got a different system described by the controlled scalar difference equation \( z[i + 1] = z[i] + 2z[i - 1] + u[i] \). To convert this second-order difference equation to a two-dimensional discrete time system, we will let \( \bar{y}[i] = \begin{bmatrix} z[i - 1] \\ z[i] \end{bmatrix} \).
Write down the system representation for $\vec{y}$ in the following matrix form:

$$\vec{y}[i + 1] = A_y \vec{y}[i] + B_y u[i].$$

(62)

Specify the values of the matrix $A_y$ and the vector $B_y$.

Solution: From the problem, we have $z[i + 1] = 2z[i - 1] + z[i] + u[i]$, which will become the second row of our system. We can then write the equation in matrix form as

$$\vec{y}[i + 1] = \begin{bmatrix} z[i] \\ z[i + 1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} z[i - 1] \\ z[i] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$$

(63)

where $A_y = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(g) It turns out that the original $\vec{x}[i]$ system can be converted to the $\vec{y}[i]$ system using a change of basis $P$. Let this coordinate change be written as $\vec{y}[i] = P \vec{x}[i]$. First express $A_y$ and $B_y$ symbolically in terms of $A$, $B$, and $P$. Then, confirm numerically that $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ is the correct change of basis matrix between the two systems.

Solution: As we know from before, $\vec{x}[i + 1] = A \vec{x}[i] + B u[i]$. Then,

$$\vec{y}[i + 1] = P \vec{x}[i + 1]$$

(64)

$$= P(A \vec{x}[i] + B u[i])$$

(65)

$$= P A \vec{x}[i] + P B u[i]$$

(66)

$$= P A P^{-1} \vec{y}[i] + P B u[i]$$

(67)

Thus,

$$A_y = P A P^{-1}$$

(68)

$$B_y = P B.$$

(69)

We confirm that,

$$P A P^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

(70)

$$= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

(71)

$$= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = A_y$$

(72)

(Note: the above is not a typo. The inverse of this particular $P$ matrix is really itself.)

We also confirm that

$$P B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B_y$$

(73)
(h) For the \( \vec{y} \) system from part (f), design a feedback gain matrix \( \begin{bmatrix} f_1 & f_2 \end{bmatrix} \) to place the closed-loop eigenvalues at \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \). Additionally, confirm that this matrix is just a change of basis of the gain matrix from part (c), i.e. \( \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P \). 

Note that this means you can solve for the closed-loop gains of your system in any basis, and then transform it to the basis you care about.

**Solution:** Solving for the new feedback matrix: The closed loop system using state feedback has the form

\[
\vec{y}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{y}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] \tag{74}
\]

\[
= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{y}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left( \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} \vec{y}[i] \right) \tag{75}
\]

\[
= \left( \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} \right) \vec{y}[i] \tag{76}
\]

\[
= \begin{bmatrix} 0 & 1 \\ 2 + \bar{f}_1 & 1 + \bar{f}_2 \end{bmatrix} \lambda_i \vec{y}[i] \tag{77}
\]

Thus, finding the eigenvalues of the above system we have

\[
det(\lambda I \quad \begin{bmatrix} 0 & 1 \\ 2 + \bar{f}_1 & 1 + \bar{f}_2 \end{bmatrix}) = 0 \Rightarrow \lambda^2 - (1 + \bar{f}_2)\lambda - (2 + \bar{f}_1) = 0 \tag{78}
\]

However, we want to place the eigenvalue at \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \). Thus, this means that

\[
\lambda^2 - (1 + \bar{f}_2)\lambda - \bar{f}_1 - 2 = \left( \lambda + \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right) \tag{79}
\]

\[
= \lambda^2 - \frac{1}{4}. \tag{80}
\]

Equating the co-efficients of \( \lambda \) on both sides, we get

\[
1 + \bar{f}_2 = 0 \tag{81}
\]

\[
-\bar{f}_1 - 2 = -\frac{1}{4} \tag{82}
\]

The above system of equations gives us \( \bar{f}_1 = -\frac{7}{4}, \bar{f}_2 = -1 \).

Matrix multiplication by the basis \( P \) confirms that

\[
\begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P = \begin{bmatrix} -\frac{7}{4} & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & -\frac{11}{4} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \tag{83}
\]
5. Tracking a Desired Trajectory in Continuous Time

The treatment in 16B so far has treated closed-loop control as being about holding a system steady at some desired operating point, by placing the eigenvalues of the state transition matrix. This control used something proportional to the actual present state to apply a control signal designed to bring the eigenvalues in the region of stability. Meanwhile, the idea of controllability itself was more general and allowed us to make an open-loop trajectory that went pretty much anywhere. This problem is about combining these two ideas together to make feedback control more practical — how we can get a system to more-or-less closely follow a desired trajectory, even though it might not start exactly where we wanted to start and in principle could be affected by small disturbances throughout.

In this question, we will also see that everything that you have learned to do closed-loop control in discrete-time can also be used to do closed-loop control in continuous time.

Consider the specific 2-dimensional system

\[
\frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + \vec{w}(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + \vec{w}(t) \quad (84)
\]

where \( u(t) \) is a scalar valued continuous control input and \( \vec{w}(t) \) is a bounded disturbance (noise).

(a) In an ideal noiseless scenario, the desired control signal \( u^*(t) \) makes the system follow the desired trajectory \( \vec{x}^*(t) \) that satisfies the following dynamics:

\[
\frac{d}{dt} \vec{x}^*(t) = A\vec{x}^*(t) + \vec{b}u^*(t). \quad (85)
\]

The presence of the bounded noise term \( \vec{w}(t) \) makes the actual state \( \vec{x}(t) \) deviate from the desired \( \vec{x}^*(t) \) and follow (84) instead. In the following subparts, we will analyze how we can adjust the desired control signal \( u^*(t) \) in (85) to the control input \( u(t) \) in (84) so that the deviation in the state caused by \( \vec{w}(t) \) remains bounded.

Represent the state as \( \vec{x}(t) = \vec{x}^*(t) + \vec{v}(t) \) and \( u(t) = u^*(t) + u_v(t) \). Using (84) and (85), we can represent the evolution of the trajectory deviation \( \vec{v}(t) \) as a function of the control deviation \( u_v(t) \) and the bounded disturbance \( \vec{w}(t) \) as:

\[
\frac{d}{dt} \vec{v}(t) = A_v\vec{v}(t) + \vec{b}_vu_v(t) + \vec{w}(t) \quad (86)
\]

What are \( A_v \) and \( \vec{b}_v \) in terms of the original system parameters \( A \) and \( \vec{b} \)? (HINT: Write out equation (84) in terms of \( \vec{x}^*(t) \), \( \vec{v}(t) \), \( u^*(t) \) and \( u_v(t) \).)

Solution: Using the change of variables \( \vec{x}(t) = \vec{x}^*(t) + \vec{v}(t) \) and \( u(t) = u^*(t) + u_v(t) \) in (84), we get

\[
\frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + \vec{w}(t) \quad (87)
\]

\[
\Rightarrow \frac{d}{dt} \vec{x}^*(t) + \frac{d}{dt} \vec{v}(t) = A\vec{x}^*(t) + A\vec{v}(t) + \vec{b}u^*(t) + \vec{b}_vu_v(t) + \vec{w}(t) \quad (88)
\]

\[
\Rightarrow \frac{d}{dt} \vec{v}(t) = A\vec{v}(t) + \vec{b}_vu_v(t) + \vec{w}(t) + \left( A\vec{x}^*(t) + \vec{b}u^*(t) - \frac{d}{dt} \vec{x}^*(t) \right) \quad (89)
\]
Using (85) we know that the last term in parenthesis is zero, so

\[ \frac{d}{dt} \vec{v}(t) = A \vec{v}(t) + b \vec{u}_v(t) + \vec{w}(t) \]  

(90)

By pattern matching with (86), we can see that \( A_v = A, \ b_v = \vec{b}. \)
Note that this implies the disturbance \( \vec{w}(t) \) is entirely something that must be dealt with in the \( \vec{v} \) dynamics. It doesn’t affect the desired trajectory at all.

(b) **Are the dynamics that you found for \( \vec{v}(t) \) in part (a) stable?** Based on this, in the presence of bounded disturbance \( \vec{w}(t) \), will \( \vec{x}(t) \) in (84) follow the desired trajectory \( \vec{x}^*(t) \) closely if we just apply the control \( u(t) = u^*(t) \) to the original system in (84), i.e. \( u_v(t) = 0? \)

(*HINT: Use the numerical values of \( A \) and \( \vec{b} \) from (84) in the solution from part (b) to determine stability of \( \vec{v}(t) \).*)

**Solution:** If we just set \( u_v(t) = 0 \), then our \( v(t) \) dynamics becomes open-loop so

\[ \frac{d}{dt} \vec{v}(t) = A \vec{v}(t) + \vec{w}(t). \]  

(91)

Recall that the condition for stability in the continuous-time case is that the real part of the eigenvalues of the state transition matrix \( A \) must be less then zero. Note that since \( A \) is an upper-triangular matrix, its eigenvalues lie on the diagonal, so they are 2 and 2. In this case, they have real parts greater than zero so the open-loop \( \vec{v}(t) \) system is unstable.

\( \vec{v}(t) \) will then follow a growing exponential trajectory in the form of \( e^{2t} \), and will thus amplify any disturbance \( \vec{w}(t) \) to the state. Therefore, \( \vec{v}(t) \) will not go to \( \vec{0} \) and we will not end up following the intended trajectory \( \vec{x}^*(t) \).

Now, we want to apply state feedback control to the system using \( u_v(t) \) to get our system to follow the desired trajectory \( \vec{x}^*(t) \).

(c) For the \( \vec{v}(t), u_v(t) \) system, **apply feedback control by letting** \( u_v(t) = F \vec{v}(t) = \left[ f_0 \quad f_1 \right] \vec{v}(t) \) **that would place both the eigenvalues of the closed-loop \( \vec{v}(t) \) system at \(-10.\)** Find \( f_0 \) and \( f_1 \).

**Solution:**

With the new input, the system equation for \( \vec{v}(t) \) is given by:

\[ \frac{d}{dt} \vec{v}(t) = A_v \vec{v}(t) + \vec{b} \left[ f_0 \quad f_1 \right] \vec{v}(t) + \vec{w}(t) \]  

(92)

\[ \implies \frac{d}{dt} \vec{v}(t) = \begin{bmatrix} 2 + f_0 & 1 + f_1 \\ f_0 & 2 + f_1 \end{bmatrix} \vec{v}(t) + \vec{w}(t) \]  

(93)

where we denote \( A_{cl} = \begin{bmatrix} 2 + f_0 & 1 + f_1 \\ f_0 & 2 + f_1 \end{bmatrix} \) as the state matrix for the closed loop system. The characteristic polynomial for finding the eigenvalues of \( A_{cl} \) is given by:

\[ \det(\lambda I - A_{cl}) = \begin{bmatrix} \lambda - 2 - f_0 & -1 - f_1 \\ -f_0 & \lambda - 2 - f_1 \end{bmatrix} \]  

(94)

\[ = \lambda^2 - (4 + f_0 + f_1) \lambda + f_0 + 2f_1 + 4 \]  

(95)
To set the eigenvalues to be where we want, we set this equal to \((\lambda + 10)(\lambda + 10) = \lambda^2 + 20\lambda + 100\). By comparing the coefficients, we have:

\[-(4 + f_0 + f_1) = 20 \tag{96}\]
\[f_0 + 2f_1 + 4 = 100 \tag{97}\]

Solving the above system of equations, we can find \(f_0 = -144, f_1 = 120\). Therefore, we can design the state-feedback \(u_v(t) = \begin{bmatrix} -144 & 120 \end{bmatrix} \vec{v}(t)\) which will place both the eigenvalues of the closed loop system at \(-10\).

Why did we pick \(-10\)? So that our closed-loop system would converge faster and aggressively reject disturbances.

(d) Based on what you did in the previous parts, and given access to the desired trajectory \(\vec{x}^*(t)\), the desired control \(u^*(t)\), and the actual measurement of the state \(\vec{x}(t)\), come up with a way to do feedback control that will keep the trajectory staying close to the desired trajectory no matter what the small bounded disturbance \(\vec{w}(t)\) does. (HINT: Express the control input \(u(t)\) in terms of \(u^*(t), x^*(t), \) and \(\vec{x}(t)\).

Solution:

From the previous parts, we have successfully found a feedback control law \(u_v(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \vec{v}(t)\) such that the closed-loop system for \(\vec{v}(t)\) is stable and converging to \(\vec{0}\) as long as the disturbances are bounded. As a result, by changing variables \(\vec{x}(t) = x^*(t) + \vec{v}(t)\) and \(u(t) = u^*(t) + u_v(t)\) that we performed in (b), we can infer that the state \(\vec{x}(t)\) will stay close to the desired trajectory \(x^*(t)\) no matter what the bounded disturbance \(\vec{w}(t)\) does.

From our initial change of variables, we want to set

\[u(t) = u^*(t) + u_v(t) = u^*(t) + \begin{bmatrix} -144 & 120 \end{bmatrix} \vec{v}(t) \tag{98}\]

\[= u^*(t) + \begin{bmatrix} -144 & 120 \end{bmatrix} (\vec{x}(t) - x^*(t)) \tag{99}\]

as our overall system input to achieve this.

This lets us have our cake and eat it too! We can use the desired system dynamics from (85) to plan, and by using closed-loop feedback we can make sure that we mostly follow our plan even in the face of disturbances.
6. Miscellaneous Practice Problems for Midterm

(a) You are given the graph in Figure 2. **Express the coordinates of vectors \( \vec{v} \) and \( \vec{w} \) in both Cartesian \((x, y)\) and Polar \((re^{i\theta})\) forms.**

You may use the atan2() or \(\tan^{-1}\) function for angle \((\theta)\) as necessary.

![Figure 2: Vectors in the x − y plane](image)

i. Label \( \vec{v} \) with its corresponding Cartesian \((x, y)\) and Polar \((re^{i\theta})\) coordinates, in the given form.

**Solution:** Vector \( \vec{v} \) Cartesian = \((-4, 2)\)

Vector \( \vec{v} \) Polar = \(\sqrt{20}e^{j\text{atan2}(2, -4)} \equiv 2\sqrt{5}e^{-j\tan^{-1}(1/2)}\)

ii. Label \( \vec{w} \) with its corresponding Cartesian \((x, y)\) and Polar \((re^{i\theta})\) coordinates, in the given form.

**Solution:** Vector \( \vec{w} \) Cartesian = \((0, -1)\)

Vector \( \vec{w} \) Polar = \(1e^{-j\pi/2}\)

(b) You are given an input voltage signal below:

\[
v_{in}(t) = -1.5 \sin(\omega t - \frac{\pi}{3}). \quad (100)
\]

**Convert the signal of eq. (100) to its phasor representation. That is, find \( \vec{V}_{in} \). Justify your answer.**

**Solution:**

\[
\vec{V}_{in} = -0.75e^{-j\frac{5\pi}{6}}. \quad (101)
\]

We can use Euler’s formulae here, which states that:

\[
\cos(x) = \frac{1}{2} \left( e^{ix} + e^{-ix} \right), \quad (102)
\]

\[
\sin(x) = \frac{1}{2j} \left( e^{ix} - e^{-ix} \right). \quad (103)
\]

There are many ways to proceed. All of them should count for credit and will give you the same answer.
One way is to remember that a sine is just a phase-shifted cosine. This would let us applying the second of these formulae as follows:

\[
v_{in}(t) = -1.5 \sin \left( \omega t - \frac{\pi}{3} \right) \tag{104}
\]

\[
= -1.5 \cos \left( \omega t - \frac{\pi}{3} - \frac{\pi}{2} \right) \tag{105}
\]

\[
= -1.5 \cos \left( \omega t - \frac{5\pi}{6} \right) \tag{106}
\]

\[
= -1.5 \cdot \frac{1}{2} \left( e^{j(\omega t - \frac{5\pi}{6})} + e^{-j(\omega t - \frac{5\pi}{6})} \right) \tag{107}
\]

\[
= -0.75 \left( e^{j\omega t} e^{-j\frac{5\pi}{6}} + e^{-j\omega t} e^{j\frac{5\pi}{6}} \right) \tag{108}
\]

\[
= \left( -0.75 e^{-j\frac{5\pi}{6}} \right) e^{j\omega t} + \left( -0.75 e^{j\frac{5\pi}{6}} \right) e^{-j\omega t} \tag{109}
\]

When we have a term of the form \( u(t) = \tilde{U} e^{j\omega t} + \tilde{U} e^{-j\omega t} \), we denote \( \tilde{U} \) as the phasor for the time-domain signal. So, by pattern matching:

\[
\tilde{V}_{in} = -0.75 e^{-j\frac{5\pi}{6}} \tag{110}
\]

We also could have proceeded using the second formula eq. (103) as follows:

\[
v_{in}(t) = -1.5 \sin \left( \omega t - \frac{\pi}{3} \right) \tag{111}
\]

\[
= -1.5 \cdot \frac{1}{2j} \left( e^{j(\omega t - \frac{\pi}{3})} - e^{-j(\omega t - \frac{\pi}{3})} \right) \tag{112}
\]

\[
= 0.75j \left( e^{j\omega t} e^{-j\frac{\pi}{6}} - e^{-j\omega t} e^{j\frac{\pi}{6}} \right) \tag{113}
\]

\[
= 0.75e^{\frac{\pi}{6}} \left( e^{j\omega t} e^{-j\frac{\pi}{6}} - e^{-j\omega t} e^{j\frac{\pi}{6}} \right) \tag{114}
\]

\[
= 0.75 \left( e^{j\omega t} e^{j(\frac{\pi}{6} - \frac{\pi}{6})} - e^{-j\omega t} e^{j(\frac{\pi}{6} + \frac{\pi}{6})} \right) \tag{115}
\]

\[
= 0.75 \left( e^{j\omega t} e^{j(\frac{\pi}{6} - \frac{\pi}{6})} + e^{-j\pi} e^{-j\omega t} e^{j(\frac{\pi}{6} + \frac{\pi}{6})} \right) \tag{116}
\]

\[
= 0.75 \left( e^{j\omega t} e^{j\frac{\pi}{6}} + e^{-j\omega t} e^{-j\frac{\pi}{6}} \right) \tag{117}
\]

which gives us

\[
\tilde{V}_{in} = 0.75 e^{j\frac{\pi}{6}}. \tag{118}
\]

This is the same answer and is in a sense, more standard in its form because all of the phase is showing up where you expect to see it, instead of \( \pi \) hiding in the minus sign up front.
(c) You decided to analyze the transfer function of a band-pass filter, and have generated the following Bode plots for \( H(j\omega) \). If your input voltage signal is

\[
v_{\text{in}}(t) = 4 \cos \left( \omega_s t + \frac{2\pi}{3} \right),
\]

where \( \omega_s = 1 \times 10^4 \text{ rad/s} \), what is the approximate value of \( v_{\text{out}}(t) \) based on the Bode plots? Since the original transfer function is not provided, you cannot numerically compute the exact values of magnitude and phase. Just read the approximate values from the Bode plot.

**Solution:**

\[
v_{\text{out}}(t) = 0.4 \cos \left( 10^4 t - \frac{\pi}{3} \right)
\]

Given the Bode plots, we need to examine how the transfer function affects two quantities: the magnitude of the input voltage, and the phase of the input voltage. The Magnitude Bode Plot reveals that at \( \omega = 10^4 \), the value is \( 10^{-1} \). The Phase Bode Plot reveals that at \( \omega = 10^4 \), the value is \(-\pi\) radians. The general form of the output voltage is:

\[
v_{\text{out}}(t) = |H(j\omega)||v_{\text{in}}(t)| \cos \left( \omega t + \phi + \angle H(j\omega) \right)
\]

where \( \phi \) is the phase of the input voltage (here, \( \frac{2\pi}{3} \)). Combining these results, we find:

\[
v_{\text{out}}(t) = 0.1 \cdot 4 \cos \left( 10^4 t + \frac{2\pi}{3} - \pi \right)
\]

\[
= 0.4 \cos \left( 10^4 t - \frac{\pi}{3} \right)
\]

(d) Assume that the overall transfer function of a new filter, \( H(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} \), is given by

\[
H(j\omega) = \left( \frac{1}{1 + j\frac{\omega}{\omega_c/1}} + \frac{j\frac{\omega}{\omega_c/2}}{1 + j\frac{\omega}{\omega_c/2}} \right),
\]
where $\omega_c = 100\omega_{c1}$. **Qualitatively describe the magnitude of the transfer function $|H(j\omega)|$ in three regions:** frequencies below $\omega_{c1}$, frequencies between $\omega_{c1}$ and $\omega_{c2}$, and frequencies above $\omega_{c2}$. **Identify the filter type by explaining what it is doing qualitatively** (for example, a low-pass filter passes low frequencies but does not pass high frequencies).

**Solution:** We can qualitatively analyze the behavior of the transfer function by evaluating the transfer function at $\omega \to 0$, $\omega = 10\omega_{c1}$, and $\omega \to \infty$.

Note that the first term of $H(j\omega)$ is a low pass filter, and we can denote it as $H_{LPF}(j\omega)$. Similarly, the second term of $H(j\omega)$ is a high pass filter and we can denote it as $H_{HPF}(j\omega)$, so

$$H(j\omega) = H_{LPF}(j\omega) + H_{HPF}(j\omega) \quad (125)$$

When $\omega \to 0$, we know that the LPF will be approximately 1 and the HPF will be approximately 0. Thus the overall sum $H(j\omega)$ will be approximately 1 so the magnitude will be about 1.

When $\omega \to \infty$, we know the LPF will be approximately 0 and the HPF will be approximately 1, so the overall sum and magnitude is still approximately 1.

Finally when $\omega = 10\omega_{c1} = \frac{1}{10} \omega_{c2}$, we are above the cutoff of the low pass and below the cutoff of the high pass, meaning we are in the attenuation region of both filters and so both filters will be much less than 1. Numerically,

$$H_{LPF}(j10\omega_{c1}) = \frac{1}{1 + 10j} \implies |H_{LPF}(j10\omega_{c1})| \ll 1 \quad (126)$$

$$H_{HPF}(j\frac{1}{10} \omega_{c2}) = \frac{1.1j}{1 + 1.1j} \implies |H_{HPF}(j\frac{1}{10} \omega_{c2})| \ll 1 \quad (127)$$

and so the total magnitude $|H(j\omega)| \ll 1$ as well.

Therefore, this filter attenuates the frequencies between $\omega_{c1}$ and $\omega_{c2}$ and passes frequencies outside of this range. This can be thought of as the opposite of a band-pass filter, and is commonly called a **band-stop** filter. (You don’t need to know the name for credit, just how the filter acts on the various frequency ranges).
7. [Optional] Op-Amp Practice Problem

We are going to analyze the following op-amp circuit in the phasor domain. All the voltages and currents in the problem are phasors and all the $Z_i$ are impedances.

(a) Treat all the Op-Amps as being in negative feedback and therefore following the Golden Rules. **What are the voltages at $V_1$, $V_2$, and $V_3$ in terms of $V_s$?**

**Solution:** Op-amp will keep the voltage difference between the positive and negative polarity input at 0 V if the op-amp has negative feedback. Therefore, we get:

$$V_1 = V_2 = V_3 = V_s \quad (128)$$

(b) **Express $I_s$ in terms of $V_s$, $V_a$, $Z_1$.**

**Solution:**

$$I_s = \frac{V_s - V_a}{Z_1} \quad (129)$$
(c) The impedance $Z_s$ is defined as $\left( Z_s = \frac{V_s}{I_s} \right)$. **Find $Z_s$ in terms of $Z_1, Z_2, Z_3, Z_4, Z_5$.**

**Solution:** (Long version:)

We can first find $V_b$ in terms of $V_s, Z_4$ and $Z_5$ as follows:

$$\frac{V_b - V_3}{Z_4} = \frac{V_3}{Z_5} \quad (130)$$

$$\frac{V_b}{Z_4} = V_3 \left( \frac{1}{Z_5} + \frac{1}{Z_4} \right) = V_s \left( \frac{1}{Z_5} + \frac{1}{Z_4} \right) \quad (131)$$

$$V_b = V_s \left( \frac{Z_4}{Z_5} + 1 \right) \quad (132)$$

Once we express $V_b$ in terms of $V_s, Z_4$ and $Z_5$, we can express $V_a$ in terms of $Z_2, Z_3, Z_4, Z_5$ as follows:

$$\frac{V_a - V_2}{Z_2} = \frac{V_2 - V_b}{Z_3} \quad (133)$$

LHS: $\frac{V_a - V_2}{Z_2} = \frac{V_a - V_s}{Z_2}$

RHS: $\frac{V_2 - V_b}{Z_3} = \frac{V_s - V_b}{Z_3} = \frac{V_s}{Z_3} - \frac{V_4}{Z_3}$

$$V_a = V_s \left( 1 - \frac{Z_2 \cdot Z_4}{Z_3 \cdot Z_5} \right) \quad (136)$$

Using the answer from part (b), we get:

$$I_s = \frac{V_s - V_a}{Z_1} = V_s \left( \frac{Z_2 \cdot Z_4}{Z_1 \cdot Z_3 \cdot Z_5} \right) \quad (137)$$

$$\Rightarrow Z_s = \frac{V_s}{I_s} = \frac{Z_1 \cdot Z_3 \cdot Z_5}{Z_2 \cdot Z_4} \quad (138)$$

(Short version:)

Since $V_1 = V_2 = V_3 = V_s$, we do know that

$$\frac{V_b - V_3}{Z_4} = \frac{1}{Z_5} V_s \quad \Rightarrow V_b - V_s = \frac{Z_4 \cdot V_s}{Z_5} \quad (139)$$

$$\frac{V_a - V_2}{Z_2} = \frac{V_3 - V_s}{Z_3} = -\frac{Z_4 \cdot V_s}{Z_3 \cdot Z_5} \Rightarrow V_a - V_s = -\frac{Z_2 \cdot Z_4}{Z_3 \cdot Z_5} V_s \quad (140)$$

$$I_s = \frac{V_s - V_a}{Z_1} = \frac{Z_2 \cdot Z_4}{Z_1 \cdot Z_3 \cdot Z_5} \quad V_s \quad (141)$$

$$Z_s = \frac{V_s}{I_s} = \frac{Z_1 \cdot Z_3 \cdot Z_5}{Z_2 \cdot Z_4} \quad (142)$$

\[ \therefore Z_s = \frac{Z_1 \cdot Z_3 \cdot Z_5}{Z_2 \cdot Z_4} \quad (143) \]

(d) Assume the following:

$$Z_1 = R_1 \quad (144)$$
\( Z_2 = \frac{1}{j\omega C_2} \)  
\( Z_3 = R_3 \)  
\( Z_4 = R_4 \)  
\( Z_5 = R_5 \)  

Evaluate \( Z_s \) for the above case.

**Solution:**

\[
Z_s = \frac{Z_1 \cdot Z_3 \cdot Z_5}{Z_2 \cdot Z_4} = \frac{R_1 \cdot R_3 \cdot R_5}{j\omega C_2 \cdot R_4} = j\omega \left( \frac{R_1 \cdot R_3 \cdot R_5 \cdot C_2}{R_4} \right)
\]  

(c) Is \( Z_s \) inductive or capacitive? If it is inductive, find its inductance. If it is capacitive, find its capacitance.

**Solution:** It is inductive since it is a positive imaginary number like of the form of \( j\omega L \). We then get \( L = \frac{R_1 \cdot R_3 \cdot R_5 \cdot C_2}{R_4} \).
8. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous “Bloom’s Taxonomy” that lists the levels of learning (from the bottom up) is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don’t want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don’t have to achieve this every week. But unless you try every week, it probably won’t ever happen.

You need to write your own question and provide a thorough solution to it. The scope of your question should roughly overlap with the scope of this entire problem set. This is because we want you to exercise your understanding of this material, and not earlier material in the course. However, feel free to combine material here with earlier material, and clearly, you don’t have to engage with everything all at once. A problem that just hits one aspect is also fine.

Note: One of the easiest ways to make your own problem is to modify an existing one. Ordinarily, we do not ask you to cite official course materials themselves as you solve problems. This is an exception. Because the problem making process involves creative inputs, you should be citing those here. It is a part of professionalism to give appropriate attribution.

Just FYI: Another easy way to make your own question is to create a Jupyter part for a problem that had no Jupyter part given, or to add additional Jupyter parts to an existing problem with Jupyter parts. This often helps you learn, especially in case you have a programming bent.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) What sources (if any) did you use as you worked through the homework?

(b) If you worked with someone on this homework, who did you work with?

   List names and student ID’s. (In case of homework party, you can also just describe the group.)

(c) Roughly how many total hours did you work on this homework? Write it down here where you’ll need to remember it for the self-grade form.

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