1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 5 and Note 6.

(a) Consider an RC circuit with a sinusoidal voltage input $V(t) = A \cos(\omega t)$. We are interested in finding the voltage on the capacitor in steady state (after a long time has passed). Can we solve this using our standard differential equation techniques? Can we solve this with phasors? Which one is more concise and why?

**Solution:**

**Standard Differential Equations:** We can solve this using our standard differential equation techniques. Recall that if the D.E. takes the form $\frac{d}{dt} x(t) = \lambda x(t) + u(t)$, we can solve using $x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda (t-\theta)} u(\theta) \, d\theta$. This works, but requires us to perform an integral.

**Phasor Analysis:** Using phasor analysis, we can solve this by using circuit analysis techniques. Phasor analysis will be more concise because fundamentally, phasor analysis is built for sinusoidal inputs (since they are complex exponentials) at steady state. Phasor analysis only requires one to solve a simple RC circuit using standard circuit analysis techniques. **Phasor analysis is much more concise than solving the D.E..**

(b) There are two ways to make a low pass filter (discussed in the notes). What are they?

**Solution:** The first method is to construct a series R-C circuit (measuring the voltage across C). The second method is to construct a series L-R circuit (measuring the voltage across R). These are extremely similar. In fact, the transfer function for both cases has the same form: $H(j\omega) = \frac{1}{1+j\omega \omega_c}$. The only difference is that $\omega_c = \frac{1}{RC}$ for R-C circuits and $\omega_c = \frac{R}{L}$ for L-R circuits.

(c) Draw the voltage sources between terminals $a$ and $b$ in figure 1 as a single equivalent voltage source between terminals $a$ and $b$, and label its voltage value. How does this equivalence relate to filtering and phasor analysis?

**Solution:**

![Diagram of voltage sources](image_url)

Figure 1: Three voltage sources in series.
The equivalence between a single voltage source that is the sum of many different sinusoids of different frequencies and multiple voltage sources in series allows us to perform phasor analysis separately through superposition for each frequency to predict how much of each frequency component will appear in some output voltage or current. Each of the different frequencies will be affected independently, so that certain frequency signals can be affected more than others, achieving filtering of signals as a function of frequency.

(d) **How we can address filter loading?**

**Solution:** One way to address loading is to pass a voltage signal from the output of a filter to the input of the next filter by using a unity gain buffer. The second method of addressing filter loading is by choosing the component values in such a way that the impedance of the following filters at the frequencies of interest are large relative to the component from which the output is taken.
2. **Group Re-assignment Survey**

   **How are your study groups working out?**

   (If you don’t have a study group you can just say so for full credit.)

   We hope they have been helpful so far. If you feel things are not going as well as you hoped and you would prefer to be assigned to a new group, or if you did not request a group before but have decided you would like one going forward, please fill out the following form:

   (a) [Group Re-assignment Survey - Google Form](#)
3. Low-pass Filter

You have a 1 kΩ resistor and a 1 μF capacitor wired up as a low-pass filter.

(a) Draw the filter circuit, labeling the input node, output node, and ground.

Solution:

(b) Write down the transfer function of the filter, $H(j\omega)$ that relates the output voltage phasor to the input voltage phasor. Be sure to use the given values for the components.

Solution: First, we convert everything into the phasor domain. We have,

\[
Z_R = R = 1 \times 10^3 \, \Omega \\
Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega \times 10^{-6}} \, \Omega
\]

In phasor domain, we can treat these impedances essentially like we treat resistors and recognize the voltage divider. Hence,

\[
\tilde{V}_{\text{out}} = \frac{Z_C}{Z_C + Z_R} \tilde{V}_{\text{in}}
\]

\[
\frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}
\]

\[
= \frac{1}{1 + j\omega RC}
\]

\[
= \frac{1}{1 + j\omega \times 10^{-3}}
\]

Hence, the corner frequency $\omega_C = \frac{1}{RC} = \frac{1}{10^{-3} \times 10^{-6}} = 10^3 \, \text{rad/s}$.

(c) Write an exact expression for the magnitude of $H(j\omega = j10^6)$, and give an approximate numerical answer.

Solution: We obtained this expression for the transfer function’s magnitude above:

\[
|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2/\omega_C^2}}
\]
Plugging in for $\omega = 10^6$:

$$|H(j\omega = j10^6)| = \frac{1}{\sqrt{1 + (10^6)^2/(10^3)^2}}$$  \hspace{1cm} (10)

$$|H(j\omega = j10^6)| = \frac{1}{\sqrt{1 + 10^6}}$$  \hspace{1cm} (11)

Approximately:

$$|H(j\omega = j10^6)| \approx \frac{1}{\sqrt{10^6}} = \frac{1}{10^3} = 10^{-3}$$ \hspace{1cm} (12)

$$|H(j\omega = j10^6)| \approx 10^{-3}$$ \hspace{1cm} (13)

(d) **Write an exact expression for the phase of $H(j\omega = j1)$, and give an approximate numerical answer.**

**Solution:** We obtained this expression for the transfer function’s phase above:

$$\angle H(j\omega) = \text{atan2}\left(-\frac{\omega}{\omega_c}, 1\right)$$ \hspace{1cm} (14)

Plugging in for $\omega = 1$:

$$\angle H(j\omega = j1) = \text{atan2}\left(-\frac{10^6}{10^3}, 1\right)$$ \hspace{1cm} (15)

By the small angle approximation, this is:

$$\angle H(j\omega = j1) \approx -10^{-3}\text{rad}$$ \hspace{1cm} (16)

(e) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1 \sin(1000t)\text{ V}$. You can assume that any transients have died out — we are interested in the steady-state waveform.**

**Solution:** We can find the transfer function at this point:

$$|H(j\omega = j10^3)| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$ \hspace{1cm} (17)

$$\angle H(j\omega = j10^3) = \text{atan}2(-1, 1) = -45^o = -\frac{\pi}{4}\text{rad}$$ \hspace{1cm} (18)

Therefore the output will be:

$$V_{out}(t) = \frac{1}{\sqrt{2}} \sin\left(1000t - \frac{\pi}{4}\right).$$ \hspace{1cm} (19)

How do we know that this is the output? Because we know that $\sin(\omega t) = A e^{j\omega t} + \overline{A} e^{-j\omega t}$ and the point of transfer functions is that we can see the answer by looking at what happens to $A$, the phasor representing $\sin(\omega t)$. This $A$ gets multiplied by $H(j\omega)$ and thus $|H(j\omega)|A e^{j\angle H(j\omega)}$ is the output phasor. We can pull the $e^{j\angle H(j\omega)}$ into the $e^{j\omega t}$ to get $e^{j\omega t + \angle H(j\omega)}$ which means that wherever we had an $\omega t$ in the output sinusoid, we now have $\omega t + \angle H(j\omega)$. Notice here that the exact nature of $A$ didn’t matter at all.
(f) Use a computer or calculator to help you sketch the transfer function (both magnitude and phase) of the filter on the graph paper below.

Log-log plot of transfer function magnitude

Semi-log plot of transfer function phase

Solution:

Log-log plot of transfer function magnitude

Semi-log plot of transfer function phase

(g) Now, let’s consider that the 1 kΩ resistor and the 1 µF capacitor are wired up as a high-pass filter. Draw the high-pass filter circuit with labels. Use a computer or calculator to help you sketch the transfer function (both magnitude and phase) of the high-pass filter on the graph paper below.
Solution:

The transfer function is

$$H(j\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega CR}{1 + j\omega CR} = \frac{1}{1 - j\omega CR} \quad (20)$$

Figure 3: A simple CR circuit
4. Alternative “second order” perspective on solving the RLC circuit

Consider the following circuit like you saw in lecture, discussion, and the previous homework:

Suppose now we insisted on expressing everything in terms of one waveform $V_C(t)$ instead of two of them (voltage across the capacitor and current through the inductor). This is called the “second-order” point of view, for reasons that will soon become clear.

For this problem, use $R$ for the resistor, $L$ for the inductor, and $C$ for the capacitor in all the expressions until the last part.

(a) **Write the current $I_L(t)$ through the inductor in terms of the voltage $V_C(t)$ across the capacitor.**

**Solution:** The current $I_L(t)$ through the inductor $L$ must be the same as the current $I_C(t)$ through $C$, which is $C \frac{d}{dt} V_C(t)$. Hence, we can write

$$I_L(t) = C \frac{d}{dt} V_C(t). \tag{21}$$

(b) Now, notice that the voltage drop across the inductor involves $\frac{d}{dt} I_L(t)$. **Write the voltage drop across the inductor, $V_L(t)$, in terms of the second derivative of $V_C(t)$.**

**Solution:** The voltage drop is

$$V_L(t) = L \frac{d}{dt} I_L(t) = LC \frac{d}{dt} \left( \frac{d}{dt} V_C(t) \right) = LC \frac{d^2}{dt^2} V_C(t). \tag{22}$$

(c) For this part, treat $V_s(t)$ as a generic input waveform. Consider the switch to be in the same configuration for all $t$, corresponding to $t < 0$ for the previous parts.

Now write out a differential equation governing $V_C(t)$ in the form of

$$\frac{d^2}{dt^2} V_C(t) + a \frac{d}{dt} V_C(t) + b V_C(t) + c(t) = 0. \tag{23}$$

where $a$, $b$ and $c(t)$ are terms you need to figure out by analyzing the circuit.

*(HINT: The $c(t)$ needs to involve $V_s(t)$ in some way.)*

**Solution:** Note that the current passing through the resistor is

$$I_R(t) = \frac{V_s(t) - V_C(t) - V_L(t)}{R} = C \frac{d}{dt} V_C(t). \tag{24}$$
or equivalently,
\[ V_L(t) + RC \frac{d}{dt} V_C(t) + V_C(t) - V_s(t) = 0. \]  
\[ (25) \]

Plugging in \( V_L(t) \), we have
\[ LC \frac{d^2}{dt^2} V_C(t) + RC \frac{d}{dt} V_C(t) + V_C(t) - V_s(t) = 0. \]  
\[ (26) \]

Finally, dividing by \( LC \),
\[ \frac{d^2}{dt^2} V_C(t) + \frac{R}{L} \frac{d}{dt} V_C(t) + \frac{1}{LC} V_C(t) - \frac{1}{LC} V_s(t) = 0. \]  
\[ (27) \]

(d) If we hadn’t done earlier homework, we wouldn’t know how to solve equations like eq. (23). But to reduce this to something we know how to solve, we define \( X(t) \) as an additional state, with \( \frac{d}{dt} V_C(t) = X(t) \). Note that this definition directly gives us one equation:
\[ \frac{d}{dt} V_C(t) = X(t). \]  
\[ (28) \]

**Solution:** With our expression of \( X \), we can write
\[ \frac{d}{dt} X(t) + \frac{R}{L} X(t) + \frac{1}{LC} V_C(t) - \frac{1}{LC} V_s(t) = 0 \]  
\[ (29) \]

and
\[ \frac{d}{dt} V_C(t) = X(t). \]  
\[ (30) \]

Then, we can write a matrix
\[ \begin{bmatrix} \frac{d}{dt} X(t) \\ \frac{d}{dt} V_C(t) \end{bmatrix} = \begin{bmatrix} \frac{R}{L} & -\frac{1}{LC} \\ \frac{1}{LC} & 0 \end{bmatrix} \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} V_s(t). \]  
\[ (31) \]

(e) **What is the characteristic polynomial of the matrix \( A \) from eq. (28)? Comment on any relationship it might have to the second-order differential equation eq. (23) you found earlier.**

**Solution:** Finding the characteristic polynomial via \( \det(\lambda I - A) \) gives us
\[ \lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC}. \]  
\[ (32) \]

We notice that the coefficient of \( \lambda^2 \) is 1, which matches the 1 that multiplies the \( \frac{d^2}{dt^2} V_C(t) \) coefficient in eq. (27). Similarly, the coefficient \( \frac{R}{L} \) for \( \lambda \) matches the identical coefficient of \( \frac{d}{dt} V_C(t) \) in eq. (27) and the \( \frac{1}{LC} \) constant term matches the identical coefficient of \( V_C(t) \) in eq. (27).

We can read off the characteristic polynomial (that will give us the eigenvalues) of the matrix directly
from the second-order form of the differential equation! This is always going to happen since

\[
\det \left( \lambda I - \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \right) = \det \left( \begin{bmatrix} \lambda - a & -b \\ -1 & \lambda \end{bmatrix} \right) = \lambda^2 - a\lambda - b,
\]

and in the above strategy, the \( a \) will always be the negative of the coefficient of the first derivative term and the \( b \) will always be the negative of the coefficient of the undifferentiated term.

(f) Find the eigenvalues (and OPTIONALLY eigenvectors) of the matrix \( A \) from eq. (28).

(Hint: use the same trick you did in the previous homework, i.e., look for eigenvectors of the form \( \begin{bmatrix} 1 \\ a \end{bmatrix} \).

Please don’t look for eigenvectors the hard way.)

Solution: Finding the characteristic equation via \( \det(A - \lambda I) = 0 \) gives us

\[
\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0.
\]  

(33)

Then,

\[
\lambda = \frac{-R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}.
\]  

(34)

To find the eigenvectors corresponding to \( \lambda \), we assume the eigenvector is of the form \( \begin{bmatrix} 1 \\ a \end{bmatrix} \). Then,

\[1 = \lambda a\]  

implies \( a = \frac{1}{\lambda} \), and hence, we conclude that

\[
\text{Eigenvalue } \frac{-R}{L} + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \quad \text{corresponds to eigenvector } \begin{bmatrix} 1 \\ \frac{2}{L} + \frac{\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{L} \end{bmatrix} \]  

(35)

\[
\text{Eigenvalue } \frac{-R}{L} - \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \quad \text{corresponds to eigenvector } \begin{bmatrix} 1 \\ \frac{2}{L} - \frac{\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{L} \end{bmatrix} \]  

(36)

(g) Revisit Problem 3 (and OPTIONALLY Problem 5) of the previous homework, and use the values of \( R, L, C \), and the same initial conditions to solve for \( V_C \). Did you get the same answer as in Problem 3 (and optionally Problem 5) of the previous homework?

(For fun: Remember Problem 5 part (g) of HW 3. Can you use those ideas here to solve the problem without having to use the eigenvectors explicitly?)

Solution: We can proceed two ways. Either gives full credit, but the point of the problem was to give you a little bit more practice with the second approach.

The old way (that would need the eigenvectors) is to use the vector-differential-equation approach is to solve the vector differential equation:

\[
\begin{bmatrix} \frac{d}{dt} X(t) \\ \frac{d}{dt} V_C(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix}.
\]  

(37)

For Problem 3 of the previous homework, the eigenvalues are \( \lambda_1 = -1.0 \times 10^5 \) and \( \lambda_2 = -4.0 \times 10^7 \),
For problem 5 of the previous homework, the eigenvalues are $V$.

\[
\lambda_1 = 2.0 \times 10^4 + (2 \times 10^6)j \quad \text{and} \quad \lambda_2 = -2.0 \times 10^4 - (2 \times 10^6)j.
\]

Finally,

\[
\tilde{y}(t) = V\tilde{y}(t) = \begin{bmatrix} 10^6j & -10^6j \\ -10^6j & 10^6j \end{bmatrix} e^{(2 \times 10^4) - (2 \times 10^6)j} t.
\]
Writing out $V_C(t)$, we can simplify it approximately to
\[ V_C(t) = e^{-(0.02\times10^6)t}\cos\left((2 \times 10^6)t\right) + 0.01e^{-(0.02\times10^6)t}\sin\left((2 \times 10^6)t\right). \] (48)

Comparing $V_C(t)$, we see that we get the same answer as in problem 5 of the previous homework. The newer approach avoids having to use or compute eigenvectors for these specific kinds of problems, leveraging what we learned in HW3. It would diverge from the above once we computed the eigenvalues. And notice that since we could’ve gotten the characteristic polynomial directly from the second-order differential equation, we not only don’t need to compute eigenvectors to get these eigenvalues, we don’t even need to do a matrix determinant.

In particular, recall that once we get the eigenvalues $\lambda_1, \lambda_2$ of our $A$ matrix and verify that they are distinct, we know that the solutions are of the form
\[ \tilde{y}(t) = \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} c_0e^{\lambda_1t} + c_1e^{\lambda_2t} \\ c_2e^{\lambda_1t} + c_3e^{\lambda_2t} \end{bmatrix}. \] (49)

Let’s do a little extra work now so we won’t repeat anything later. The way we defined $X(t)$ was $X(t) = \frac{d}{dt}V_C(t)$. We now have an expression for $V_C(t)$, so let’s take the derivative to discover the value of $X(t)$. We have

\[ X(t) = \frac{dV_C(t)}{dt} \] (50)

\[ = \frac{d}{dt}\left(c_2e^{\lambda_1t} + c_3e^{\lambda_2t}\right) \] (51)

\[ = \lambda_1c_2e^{\lambda_1t} + \lambda_2c_3e^{\lambda_2t}. \] (52)

But we already know that
\[ X(t) = c_0e^{\lambda_1t} + c_1e^{\lambda_2t}. \] (53)

Hence, by pattern matching the coefficients of $e^{\lambda_1t}$ and $e^{\lambda_2t}$, we obtain
\[ c_0 = \lambda_1c_2 \quad \text{and} \quad c_1 = \lambda_2c_3. \] (54)

With these substitutions, we see that
\[ \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} \lambda_1c_2e^{\lambda_1t} + \lambda_2c_3e^{\lambda_2t} \\ c_2e^{\lambda_1t} + c_3e^{\lambda_2t} \end{bmatrix}. \] (55)

Now to solve for $c_2$ and $c_3$ we have two equations in two variables. Perfect! Now let’s actually get about solving this system.

For problem 3 of the previous homework, the eigenvalues are $\lambda_1 = -1.0 \times 10^5$ and $\lambda_2 = -4.0 \times 10^7$. So
\[ \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} -(1.0 \times 10^5)c_2e^{-(1.0\times10^5)t} - (4.0 \times 10^7)c_3e^{-(4.0\times10^7)t} \\ c_2e^{-(1.0\times10^5)t} + c_3e^{-(4.0\times10^7)t} \end{bmatrix}. \] (56)

We want to using the initial conditions to solve. We again don’t need the eigenbasis since the initial condition on the voltage is physical, and because of the interpretation of the first derivative of the capacitor voltage being proportional to the inductor current, we also have that since the inductor current starts at zero.
\[
\begin{bmatrix}
0 \\
1
\end{bmatrix} = \vec{y}(0) \tag{57}
\]

\[
\begin{bmatrix}
X(0) \\
V_C(0)
\end{bmatrix} = \begin{bmatrix}
-(1.0 \times 10^5)c_2 e^{-(1.0 \times 10^5)t} + (4.0 \times 10^7)c_3 e^{-(4.0 \times 10^7)t} \\
c_2 e^{-(1.0 \times 10^5)t} + c_3 e^{-(4.0 \times 10^7)t}
\end{bmatrix} \tag{58}
\]

\[
\begin{bmatrix}
-(1.0 \times 10^5)c_2 - (4.0 \times 10^7)c_3 \\
c_2 + c_3
\end{bmatrix}
\]

\[
\Rightarrow c_2 = 1.0025 \approx 1, \quad c_3 \approx -0.0025 
\tag{60}
\]

\[
\vec{y}(t) = \begin{bmatrix}
-(1.0 \times 10^5)e^{-(1.0 \times 10^5)t} + (1.0 \times 10^5)e^{-(4.0 \times 10^7)t} \\
e^{-(1.0 \times 10^5)t} - (2.5 \times 10^{-3})e^{-(4.0 \times 10^7)t}
\end{bmatrix} \tag{61}
\]

This is the same as the previous homework.

For problem 5 of the previous homework, the eigenvalues are \( \lambda_1 = -2.0 \times 10^4 + (2 \times 10^6)j \) and \( \lambda_2 = -2.0 \times 10^4 - (2 \times 10^6)j \). So

\[
\begin{bmatrix}
X(t) \\
V_C(t)
\end{bmatrix} = \begin{bmatrix}
-(2.0 \times 10^4) + (2 \times 10^6)j)c_2 e^{-(2.0 \times 10^4) + (2 \times 10^6)jt} \\
+(2.0 \times 10^4) - (2 \times 10^6)j)c_3 e^{-(2.0 \times 10^4) - (2 \times 10^6)jt} \\
c_2 e^{-(2.0 \times 10^4) + (2 \times 10^6)jt} + c_3 e^{-(2.0 \times 10^4) - (2 \times 10^6)jt}
\end{bmatrix} \tag{63}
\]

Using the initial conditions to solve, we have

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix} = \vec{y}(0) \tag{64}
\]

\[
\begin{bmatrix}
X(0) \\
V_C(0)
\end{bmatrix} = \begin{bmatrix}
-(2.0 \times 10^4) + (2 \times 10^6)j)c_2 + (2.0 \times 10^4) - (2 \times 10^6)j)c_3 \\
c_2 + c_3
\end{bmatrix} \tag{65}
\]

\[
\Rightarrow c_2 = 0.5 - 0.005j, \quad c_3 = 0.5 + 0.005j \tag{66}
\]

Putting these together, we see that

\[
\begin{bmatrix}
X(t) \\
V_C(t)
\end{bmatrix} = \begin{bmatrix}
10^6j e^{-(2.0 \times 10^4) + (2 \times 10^6)jt} - 10^6j e^{-(2.0 \times 10^4) - (2 \times 10^6)jt} \\
(0.5 - 0.005j)e^{-(2.0 \times 10^4) + (2 \times 10^6)jt} + (0.5 + 0.005j)e^{-(2.0 \times 10^4) - (2 \times 10^6)jt}
\end{bmatrix} \tag{68}
\]

From here we can do the same sine and cosine simplification as earlier in the solution in order to get the same answer for \( \vec{y}(t) \).

The alternative approach here won’t let us do the critically damped case, since the eigenvalues aren’t distinct. But it turns out that we could extend it. That is just outside the scope of 16B HW, but you can see how in courses like 120 — or derive it for yourself since you will have the tools to do so.
5. Color Organ Filter Design

The “color organ” is a three-class tone classifier that we will hand design in the fourth lab. We will design low-pass, band-pass, and high-pass filters for our color organ. There will be red, green, and blue LEDs. Each color will correspond to a specified frequency range of the input audio signal. The intensity of the light emitted will correspond to the amplitude of the audio signal.

(a) First, you remember that you saw in lecture that you can build simple filters using a resistor and a capacitor. **Design a simple first-order passive low-pass filter with the following specification using a 1 µF capacitor.** (“Passive” means that the filter does not require any power supply to operate on the input signal. Passive components include resistors, capacitors, inductors, diodes, etc., while an example of an active component would be an op-amp).

- Low-pass filter: cut-off frequency $f_c = 2400 \text{ Hz}$, $\omega_c = 2\pi \cdot 2400 \text{ rad/s}$. Hz can be interpreted as “cycles/sec”, and $\text{rad/s}$ can be interpreted as “2π radians/cycle”.

Recall that the cutoff-frequency of such a filter is just where the magnitude of the filter is $\frac{1}{\sqrt{2}}$ of its peak value.

**Show your work to find the resistor value that creates this low-pass filter. Draw the schematic-level representation of your design. Please mark $V_{\text{in}}$, $V_{\text{out}}$, and the ground node(s) in your schematic.** Round your results to two significant figures.

**Solution:**

Low-pass filter

\[
\omega_c = 2\pi f_c = \frac{1}{RC} \tag{69}
\]

\[
f_c = \frac{1}{2\pi RC} = 2400 \text{ Hz} \tag{70}
\]

\[
R = \frac{1}{2\pi \cdot 1 \mu F \cdot 2400 \text{ Hz}} = 66 \Omega \tag{71}
\]

Therefore, we need a 66 Ω resistor.

![Schematic of Low-pass Filter](image)

(b) **Now design a simple first-order passive high-pass filter with the following specification using a 1 µF capacitor.**

- High-pass filter: cut-off frequency $f_c = 100 \text{ Hz}$, $\omega_c = 2\pi \cdot 100 \text{ rad/s}$

**Show your work to find the resistor value that creates this high-pass filter. Draw the schematic-level representation of your design. Please mark $V_{\text{in}}$, $V_{\text{out}}$, and the ground node(s) in your schematic.** Round your results to two significant figures.

**Solution:**
High-pass filter

\[ f_c = \frac{1}{2\pi RC} = 100 \text{ Hz} \quad (72) \]

\[ R = \frac{1}{2\pi \cdot 1\mu F \cdot 100 \text{ Hz}} = 1.6 \text{ k\Omega} \quad (73) \]

Therefore, we need a 1.6 k\Omega resistor. Note that we want a 24 times lower frequency, which means a 24 times higher time constant, which means a 24 times higher resistor.

(c) You can try to build a bandpass filter by cascading the first-order low-pass and high-pass filters you designed in parts (a) and (b). To do this, you might be tempted to connect the \( V_{out} \) node of your low-pass filter directly to the \( V_{in} \) node of your high-pass filter. However, if you did this, just as you saw in 16A for voltage dividers, the purported high-pass filter would “load” the low-pass filter and you might get some potentially complicated mess instead of what you wanted.

**Show how you can use an ideal op-amp configured as a unity gain buffer to eliminate this loading effect to cascade the low-pass and high-pass filters, and write the resulting transfer function of the combined circuit.** Draw the magnitude and phase transfer functions of the combined circuit.

**What kind of filter is this?** You can optionally use the included Jupyter notebook `plot_tf.ipynb`.

*(HINT: Read Section 2.1 in Note 7.)*

*(NOTE: In Python, use 1j when your transfer function has a j.)*

**Solution:**

Consider the circuit given below, which is the low pass and the high pass, connected with a unity gain buffer:

We know that when we cascade circuits, the combined transfer function is the multiplication of the individual elements. For the Low Pass Filter \( H_L(j\omega) \), Unity Gain Buffer \( H_{unity}(j\omega) \), and High Pass Filter \( H_H(j\omega) \).

\[ H(j\omega) = H_L(j\omega) \cdot H_{unity}(j\omega) \cdot H_H(j\omega) \quad (74) \]
And we know that:

\[ H_L(j\omega) = \frac{1}{1 + j\omega R_L C_L}, \quad H_{\text{unity}}(j\omega) = 1, \quad H_H(j\omega) = \frac{j\omega R_H C_H}{1 + j\omega R_H C_H} \quad (75) \]

Combining the transfer functions, we get:

\[ H(j\omega) = \frac{1}{1 + j\omega R_L C_L} \cdot \frac{j\omega R_H C_H}{1 + j\omega R_H C_H} \quad (76) \]

The magnitude and phase transfer functions are shown below. We can see that this is a band pass filter.

![Magnitude and Phase Plot](image)

**Figure 4:** Magnitude and Phase transfer functions

(d) **Write down an expression for the time-domain output waveform** \( V_{\text{out}}(t) \) **of this filter if the input voltage is** \( V_{\text{in}}(t) = 1 \sin(1000t) \text{V} \). **Round your answer to 2 significant digits.**

**Solution:** We can find the transfer function at this point:

\[ |H(j\omega = j10^3)| = 0.85 \quad (77) \]

\[ \angle H(j\omega = j10^3) = 0.49 \text{ rad} = 28.23^\circ \quad (78) \]

Therefore the output will be:

\[ V_{\text{out}}(t) = 0.85 \sin(1000t + 0.49) \text{V}. \quad (79) \]
6. Phasors and Eigenvalues

Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

\[
\frac{d}{dt} \vec{x}(t) = A \vec{x}(t) + \vec{b}u(t)
\]  

(80)

where for this problem, the matrix \( A \) and the vector \( \vec{b} \) are both real.

(a) **Give a necessary condition on the eigenvalues \( \lambda_k \) of \( A \) such that any impact of an initial condition will eventually completely die out.** (i.e. the system will reach steady-state.)

You don’t have to prove this. The idea here is to make sure that you understand what kind of thing is required. *(HINT: Read Section 2 in Note 5.)*

**Solution:**

(Recall how the diagonalization we have done in the past takes us to an coordinate system where the matrix representing the differential equation has only diagonal entries being the eigenvalues, corresponding to differential equations of the form \( \frac{d}{dt}(t) = \lambda z(t) \)).

The condition is that all eigenvalues must have real parts that are less than zero. In equations

\[
\forall k, \quad \text{Re}(\lambda_k) < 0
\]  

(81)

This condition derives from the fact that the solutions to differential equations in the eigenspace contain terms that look like \( e^{\lambda_k t} \). So, if all the eigenvalues are have strictly negative real parts, then all such exponential terms will die out.

If any of the eigenvalues have strictly positive real parts, then the exponential terms corresponding to them will blow up as growing exponentials.

The case of \( \lambda = 0 \) or having a zero real part in general (purely imaginary eigenvalues) is a little more ambiguous in feeling. This suggests that some constant offset (for the case of \( \lambda = 0 \)) or some steady oscillation at a natural frequency of the system can persist throughout all time. But persisting isn’t dying out and so we really want the eigenvalues to have strictly negative real parts for us to be able to ignore the initial conditions.

The argument above implicitly assumes that we can find enough linearly independent eigenvectors to get a basis. But what if we can’t? We will explicitly address that case later in the course, but so far, we have seen in the cases that we have explored that what seems to happen is that even in the new basis, we seem to get a copy of an existing eigenvalue showing up again. This gives us some confidence that the condition that we are expressing is probably the right one, but we aren’t fully sure yet since we have no proof that covers not having enough eigenvectors. We also know that these kinds of “not enough eigenvectors” cases can occur in physical circuits, since we saw the critically damped case in a previous homework.

(b) Now assume that \( u(t) \) has a phasor representation \( \vec{U} \). In other words, \( u(t) = \vec{U} e^{+j\omega t} + \vec{U} e^{-j\omega t} \).

Assume that the vector solution \( \vec{x}(t) \) to the system of differential equations (80) can also be written in phasor form as

\[
\vec{x}(t) = \vec{X} e^{+j\omega t} + \vec{X} e^{-j\omega t}.
\]  

(82)

**Derive an expression for \( \vec{X} \) involving \( A, \vec{b}, j\omega, \vec{U} \), and the identity matrix \( I \).**

*(HINT: Plug (82) into (80) and simplify, using the rules of differentiation and grouping terms by which exponential \( e^{\pm j\omega t} \) they multiply.)*
Solution: As the hint suggests, plugging back (82) into (80) we get the following:

\[
\frac{d}{dt} \left( \vec{X}e^{j\omega t} + \overline{\vec{X}}e^{-j\omega t} \right) = A(\vec{X}e^{j\omega t} + \overline{\vec{X}}e^{-j\omega t}) + \vec{b}(\overline{U}e^{+j\omega t} + \overline{U}e^{-j\omega t}) \tag{83}
\]

\[
(j\omega \vec{X}e^{j\omega t} - j\omega \overline{\vec{X}}e^{-j\omega t}) = (A\vec{X} + b\overline{U})e^{j\omega t} + (A\overline{\vec{X}} + \overline{bU})e^{-j\omega t} \tag{84}
\]

\[
(85)
\]

Note that \( \vec{X} \) and \( \overline{U} \) do not depend on time since they are phasors. Next, we can group the coefficients with the same exponential terms,

\[
j\omega \vec{X} = A\vec{X} + \overline{bU} \tag{86}
\]

\[
- j\omega \overline{\vec{X}} = A\overline{\vec{X}} + \overline{bU} \tag{87}
\]

\[
\Rightarrow (j\omega)\vec{X} = (A\vec{X} + \overline{bU}) \tag{88}
\]

\[
\Rightarrow (j\omega)\overline{\vec{X}} = (A\overline{\vec{X}} + \overline{bU}) \tag{89}
\]

We see that equations (86) and (89) match, which is good. Note that, here we are assuming \( A \) and \( \overline{b} \) are real. Next, we can solve (86) to get \( \vec{X} \):

\[
j\omega \vec{X} = A\vec{X} + \overline{bU} \tag{90}
\]

\[
\Rightarrow (j\omega I - A)\vec{X} = \overline{bU} \tag{91}
\]

\[
\Rightarrow \vec{X} = (j\omega I - A)^{-1}\overline{bU}. \tag{92}
\]

Notice that we didn’t need to explicitly deal with the conjugate terms. We know that their solution is just going to be the conjugate of what we computed here, because of the properties of complex arithmetic.

It turns out that it is possible to invert a general matrix \( M \) by writing it as some matrix \( M_c \) (that depends on \( M \)) divided by the determinant of \( M \). (This is a fact related to something called the adjoints of matrices that are studied when one considers a combinatorial perspective on determinants, and thinks about things that are sometimes called “cofactors”.) This is not something that is covered in 16AB because it cannot be proved at the level of mathematical maturity that is fair to assume for courses at this level.

However, the above linear-algebraic fact has a consequence for transfer functions. It tells you that all the polynomial terms in the denominators of the transfer functions are going to have the eigenvalues of the system as their roots. Why? Because the roots of \( \det(j\omega I - A) \) tell you the eigenvalues of \( A \). In later courses like 120, 105, 140, and beyond, you will see these roots of the denominators referred to as “poles” based on terminology from complex analysis. When you see them, understand that they are just the eigenvalues of the system in disguise. When you see conversations in later courses (or in your job or research) about understanding the placement of poles, understand that what is being talked about is where the relevant eigenvalues of the system are.
7. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{\text{out}}(t)$.

![Circuit Diagram]

The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2} \cos \left(100t - \frac{\pi}{4}\right) \quad (93)$$

$$R = 5 \ \Omega \quad (94)$$

$$L = 50 \text{ mH} \quad (95)$$

$$C = 2 \text{ mF} \quad (96)$$

(a) Give the amplitude $V_0$, input frequency $\omega$, and phase $\phi$ of the input voltage $V_s$.

**Solution:** A sinusoid takes the form $v(t) = V_0 \cos(\omega t + \phi)$. Given $V_s(t)$, we find:

$$V_0 = 10\sqrt{2} \text{ volt} \quad (97)$$

$$\omega = 100 \text{ rad/sec} \quad (98)$$

$$\phi = \frac{-\pi}{4} \text{ rad} \quad (99)$$

(b) Transform the circuit into the phasor domain. What are the impedances of the resistor, capacitor, and inductor? What is the phasor $V_s^*$ of the input voltage $V_s(t)$?

**Solution:**

$$Z_L = j\omega L = j5\Omega \quad (100)$$

$$Z_C = \frac{1}{j\omega C} = -j5\Omega \quad (101)$$

$$Z_R = R = 5\Omega \quad (102)$$

$$V_s^* = \frac{|V_s|}{2} e^{j\phi} V_s = 5\sqrt{2}e^{-j\pi/4} \quad (103)$$
(c) Use the circuit equations to solve for $\vec{V}_{\text{out}}$, the phasor representing the output voltage.

**Solution:**
We have

\[
\vec{I}_R = \frac{\vec{V}_S - \vec{V}_{\text{out}}}{R} \quad (104)
\]

\[
\vec{I}_L = \frac{\vec{V}_{\text{out}}}{j\omega L} \quad (105)
\]

\[
\vec{I}_C = \vec{V}_{\text{out}} \cdot j\omega C \quad (106)
\]

Rewriting the current relation in terms of voltage phasors gives:

\[
\frac{\vec{V}_S - \vec{V}_{\text{out}}}{R} = \frac{\vec{V}_{\text{out}}}{j\omega L} + \vec{V}_{\text{out}} \cdot j\omega C \quad (107)
\]

\[
\frac{\vec{V}_S}{R} = \vec{V}_{\text{out}} \left( \frac{1}{j\omega L + j\omega C + \frac{1}{R}} \right) \quad (108)
\]

\[
\frac{\vec{V}_S}{R} = \vec{V}_{\text{out}} \left( \frac{R + (j\omega)^2 RLC + j\omega L}{j\omega RL} \right) \quad (109)
\]

Solving for $\vec{V}_{\text{out}}$:

\[
\vec{V}_{\text{out}} = \vec{V}_S \left( \frac{j\omega L}{R - \omega^2 RLC + j\omega L} \right) \quad (110)
\]

Plugging in for the values of $R$, $L$, and $C$:

\[
\vec{V}_{\text{out}} = \vec{V}_S \left( \frac{j \cdot 100 \cdot 50 \times 10^{-3}}{5 - 100^2 \cdot 5 \cdot 50 \times 10^{-3} \cdot 2 \times 10^{-3} + j100 \cdot 50 \times 10^{-3}} \right) \quad (111)
\]

\[
= \vec{V}_S \left( \frac{j5}{5 - 5 + j5} \right) \quad (112)
\]

\[
= \vec{V}_S \left( \frac{j5}{j5} \right) \quad (113)
\]

\[
\vec{V}_{\text{out}} = \vec{V}_S \quad (114)
\]

We found that $\vec{V}_{\text{out}} = \vec{V}_S$ because this circuit is in resonance; i.e., the capacitor and inductor have the exact values that cause current and voltage to endlessly oscillate between them at this frequency. If we chose a different value for $\omega$ with these same component values, the circuit would not be in resonance and $\vec{V}_{\text{out}}$ and $\vec{V}_S$ would no longer be equal.

One may think that this answer seems weird. For $\vec{V}_{\text{out}}$ to equal $\vec{V}_S$ means that no current is flowing through the resistor. This means that somehow, the impedance of the parallel $L$ and $C$ combination would have to be infinity. Let’s check what that is:
\[ Z_L \parallel Z_C = \frac{(j5) \cdot (-j5)}{j5 + (-j5)} = +\infty \]  \hspace{1cm} (115)

Wow! Indeed it is infinity. This shows something counterintuitive that can occur with phasors and impedances. For resistors, one may think that parallel connections always lower the resistance. However, since imaginary impedances can be positive imaginary and negative imaginary, a parallel connection can make the impedance bigger or smaller. The same kind of counterintuitive behavior is also possible for series combinations. Resistors in series always increase the resistance. But the same L and C in series can combine to have a zero impedance at the natural frequency.

If one wants to know why something divided by 0 is \( \infty \) in the complex plane, read this Wiki article: Riemann Sphere. Again, this is another facet of the extreme beauty of complex analysis, and why engineers were drawn to it when modeling physical systems for design purposes.

(d) Convert the phasor \( \tilde{V}_{\text{out}} \) back to get the time-domain signal \( V_{\text{out}}(t) \).

\textbf{Solution:} Since \( \tilde{V}_{\text{out}} = \tilde{V}_S \),

\[ v_{\text{out}}(t) = 10\sqrt{2} \cos \left( 100t - \frac{\pi}{4} \right) \]  \hspace{1cm} (116)
8. RLC filter

Consider the following RLC circuit:

(a) Write down the impedance of a series RLC circuit in the form \( Z_{RLC}(j\omega) = A(\omega) + jX(\omega) \), where \( X(\omega) \) is a real valued function of \( \omega \).

Solution: Since the capacitor, resistor and inductor are in series, the equivalent impedance is given by,

\[
Z_{RLC}(j\omega) = R + Z_L(j\omega) + Z_C(j\omega)
\]

\[
\Rightarrow Z_{RLC}(j\omega) = R + j\omega L + \frac{1}{j\omega C}
\]

Since,

\[
\frac{1}{j} = -j
\]

\[
Z_{RLC}(j\omega) = R + j(\omega L - \frac{1}{\omega C})
\]

Hence,

\[
A(\omega) = R
\]

and

\[
X(\omega) = \omega L - \frac{1}{\omega C}
\]

(b) Write the transfer function from \( V_S \) to \( V_R \) — the voltage drop across the resistor.

Solution: For an impedance divider, we know that:

\[
\tilde{V}_{out}(j\omega) = \frac{R}{Z_{RLC}(j\omega)} \tilde{V}_{in}(j\omega) = \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \tilde{V}_{in}(j\omega)
\]

Giving:

\[
H(j\omega) = \frac{\tilde{V}_{out}(j\omega)}{\tilde{V}_{in}(j\omega)}
\]

Then, divide through by \( R \) to get a simpler expression to work with:

\[
H(j\omega) = \frac{1}{1 + j(\omega R - \frac{1}{\omega RC})}
\]
For the different specific values for $R, L, C$ given by different cases (underdamped, overdamped, and critically damped) in the previous HW, use a computer to sketch plots of the magnitude and phase of the transfer function above. You can optionally use the included Jupyter notebook `plot_tf.ipynb`. \textit{(NOTE: In Python, use 1j when your transfer function has a j.)}

**Solution:**

From the last part, we have the magnitude and phase of the transfer function as:

\[
|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{L}{R} - \frac{1}{\omega RC}\right)^2}}
\]

\[
\angle(H(j\omega)) = -\tan^{-1}\left(\frac{\omega L}{R} - \frac{1}{\omega RC}, 1\right)
\]

Now we will plot all the values.

Overdamped \quad R = 1 \text{k}\Omega, \ C = 10 \text{nF}, \ L = 25 \mu\text{H} \quad (128)

Underdamped \quad R = 1 \text{\Omega}, \ C = 10 \text{nF}, \ L = 25 \mu\text{H} \quad (129)
critically damped \[ R = 2\sqrt{\frac{L}{C}} = 100, C = 10 \text{ nF}, L = 25 \mu\text{H} \] \hspace{1cm} (130)

Figure 7: Magnitude and Phase transfer functions for critically damped case

(d) To see how the values of \( R, L, C \) impact the impedance at different frequencies, run the included Jupyter notebook `hw5rlc_transfer.ipynb`. The script will generate two plots, the transfer function of the circuit as a function of frequency and the location of the eigenvalues in the imaginary, real plane. Explain what happens at the following sets of values as the resistance, inductance, and capacitance vary:

<table>
<thead>
<tr>
<th></th>
<th>( R )</th>
<th>( L )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>2.5E-5</td>
<td>1E-8</td>
</tr>
<tr>
<td>II</td>
<td>10</td>
<td>2.5E-5</td>
<td>1E-8</td>
</tr>
<tr>
<td>III</td>
<td>10</td>
<td>2.5E-5</td>
<td>2E-9</td>
</tr>
<tr>
<td>IV</td>
<td>500</td>
<td>0.0001</td>
<td>2E-8</td>
</tr>
</tbody>
</table>

Table 1: Values for RLC Bandwidth problem, part d

Solution: In this part, the values for I are default, then parts II, III, IV show how the impedance peak can change location and magnitude. Please see the figure below, then explanatory text in the caption.
Figure 8: Here you can see how changing $L, C$ changes the resonant frequency which determines where the peak is, and increasing $R$ increases the width of the resonant peak. The eigenvalues location on the imaginary axis tells you where the peak is going to be, and the distance to the imaginary axis (real part) tells you how wide the peak is going to be.
9. Uniqueness justification for phasor-style solutions

In general, we have seen that we need to justify our methods of solving differential equations with a uniqueness proof. This important so that it tells us we can trust our solution as being the only solution to the problem at hand.

In Note 5, you saw that phasor-style techniques gave rise to convenient solutions to the matrix-vector differential equations of the form:

\[
\frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + \vec{\tilde{U}} e^{st}
\]  

(131)

with some initial condition \( \vec{x}(0) = \vec{x}_0 \).

However, all the uniqueness proofs that you have done for yourself have been concerned with scalar differential equations, and scalar differential equations driven by inputs. So, why can we trust the solutions that we are getting for such matrix-vector differential equations?

This question takes us part of the way to the answer.

(a) Suppose that the \( n \times n \) matrix \( A \) has \( n \) distinct eigenvalues and corresponding eigenvectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \), so that the matrix \( V = [\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n] \) has linearly independent columns.

Show that if you are given any valid solution for the original system (131), you can change coordinates to the eigenbasis and also get a valid solution for the diagonalized system. Additionally, list the new initial conditions that are satisfied in the diagonalized system.

Solution: Because the eigenvector matrix \( V \) has linearly independent columns, we know it is invertible. We can therefore change the coordinates into the eigenbasis coordinates \( \vec{y} \) with the following conversions:

\[
\vec{y} = V^{-1} \vec{x} \quad \vec{x} = V \vec{y}
\]  

(132)

We transform the original differential equation by substituting \( \vec{x} = V \vec{y} \):

\[
\frac{d}{dt} V \vec{y}(t) = AV \vec{y}(t) + \vec{\tilde{U}} e^{st}
\]  

(133)

\[
\frac{d}{dt} \vec{y}(t) = V^{-1} AV \vec{y}(t) + V^{-1} \vec{\tilde{U}} e^{st}
\]  

(134)

\[
\frac{d}{dt} \vec{y}(t) = \Lambda \vec{y}(t) + \vec{\tilde{W}} e^{st}
\]  

(135)

where \( \vec{\tilde{W}} = V^{-1} \vec{\tilde{U}} \) and \( \Lambda = V^{-1} AV \).

We also transform the initial conditions into the eigenbasis:

\[
\vec{y}(0) = \vec{y}_0 = V^{-1} \vec{x}_0
\]  

(136)

Because the matrix \( V \) is composed of the eigenvectors of \( A \), and is invertible, we know that \( \Lambda = V^{-1} AV \) is the diagonal eigenvalue matrix, with each diagonal entry \( \Lambda_{i,i} \) equal to the eigenvalue associated with the \( i \)th eigenvector of \( V \).

Because the \( \Lambda \) matrix is diagonal, the above matrix equation yields a collection of uncoupled scalar differential equations with initial conditions:

\[
\frac{d}{dt} \vec{y}_k = \lambda_k \vec{y}_k + \vec{\tilde{W}}_k e^{st}, \quad \vec{y}_k(0) = \vec{y}_0[k]
\]  

(137)
for $k \in [1, 2, \ldots, n]$ where the subscript indicates indexing into the vector.

Finally, we must show that if we are given a valid solution for (131), then it remains a valid solution for the collection of differential equations in the transformed coordinates (137). Let $\vec{x}_{\text{sol}}(t)$ satisfy (131) and the initial condition $\vec{x}_{\text{sol}}(0) = \vec{x}_0$.

We can transform this solution into the eigenbasis: $\vec{y}_{\text{sol}}(t) = V^{-1} \vec{x}_{\text{sol}}(t)$.

At $t = 0$, $\vec{y}_{\text{sol}}(0) = V^{-1} \vec{x}_{\text{sol}}(0) = V^{-1} \vec{x}_0$, so we see that the initial condition is satisfied in the eigenbasis.

We also check if the transformed solution $\vec{y}_{\text{sol}}(t)$ satisfies the transformed differential equation:

\[
\frac{d}{dt} \vec{y}_{\text{sol}}(t) = V^{-1} A \vec{x}_{\text{sol}}(t) + \vec{\tilde{U}} e^{st} = V^{-1} AV \vec{y}_{\text{sol}}(t) + \vec{\tilde{U}} e^{st} = V^{-1} \Lambda \vec{y}_{\text{sol}}(t) + \vec{\tilde{W}} e^{st}
\]

We see that indeed $\vec{y}_{\text{sol}}(t)$ satisfied the transformed differential equations. Thus a valid solution in the original basis produces a valid solution in the changed basis.

(b) You have already proved the uniqueness of solutions for any scalar differential equation of the form $\frac{d}{dt} x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$. How can you use this fact and the result of the previous part to argue that the solution must be unique for the matrix/vector differential equation?

Give a proof by contradiction argument.

(Hint: Start by assuming that you have two distinct solutions. Use the linear independence of $V$ and the fact that a change of basis is invertible.)

Solution: Let us assume that the solution is not unique, so we have two distinct solutions in the original basis $\vec{x}_1(t)$ and $\vec{x}_2(t)$ which satisfy (131) and the initial condition. Because $V$ has linearly independent columns, the matrix $V$ is invertible. Thus, we can transform the two distinct original solutions into two distinct solutions in the eigenbasis:

\[
\vec{y}_1(t) = V^{-1} \vec{x}_1(t) \quad \vec{y}_2(t) = V^{-1} \vec{x}_2(t) \quad \vec{y}_1(t) \neq \vec{y}_2(t)
\]

We have already proved that the solutions for scalar equations of the form $\frac{d}{dt} x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$ are unique. In the previous part of the problem, we saw that we could transform the original matrix-vector differential equations into a collection of scalar equations of this form. Therefore, the solutions to these scalar equations must be unique.

This now leads to the contradiction: We know there can only be a single unique solution in the eigenbasis, but our initial assumption of two unique solutions in the original basis leads to two distinct solutions in the eigenbasis. Thus our assumption is not valid, and so there must be a unique solution.

We will see later in the course how the assumption we made on the eigenvectors of $A$ is not actually needed for this proof to hold. But for now, it is important to understand this case first.
10. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous “Bloom’s Taxonomy” that lists the levels of learning (from the bottom up) is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don’t want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don’t have to achieve this every week. But unless you try every week, it probably won’t ever happen.

You need to write your own question and provide a thorough solution to it. The scope of your question should roughly overlap with the scope of this entire problem set. This is because we want you to exercise your understanding of this material, and not earlier material in the course. However, feel free to combine material here with earlier material, and clearly, you don’t have to engage with everything all at once. A problem that just hits one aspect is also fine.

Note: One of the easiest ways to make your own problem is to modify an existing one. Ordinarily, we do not ask you to cite official course materials themselves as you solve problems. This is an exception. Because the problem making process involves creative inputs, you should be citing those here. It is a part of professionalism to give appropriate attribution.

Just FYI: Another easy way to make your own question is to create a Jupyter part for a problem that had no Jupyter part given, or to add additional Jupyter parts to an existing problem with Jupyter parts. This often helps you learn, especially in case you have a programming bent.

11. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) What sources (if any) did you use as you worked through the homework?
(b) If you worked with someone on this homework, who did you work with?
   List names and student ID’s. (In case of homework party, you can also just describe the group.)
(c) Roughly how many total hours did you work on this homework? Write it down here where you’ll need to remember it for the self-grade form.

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