EECS 16B Designing Information Devices and Systems II
Spring 2021 UC Berkeley

Homework 5

This homework is due on Friday, February 19, 2021, at 11:00PM. Self-grades and HW Resubmission are due on Tuesday, February 23, 2021, at 11:00PM.

1. Reading Lecture Notes

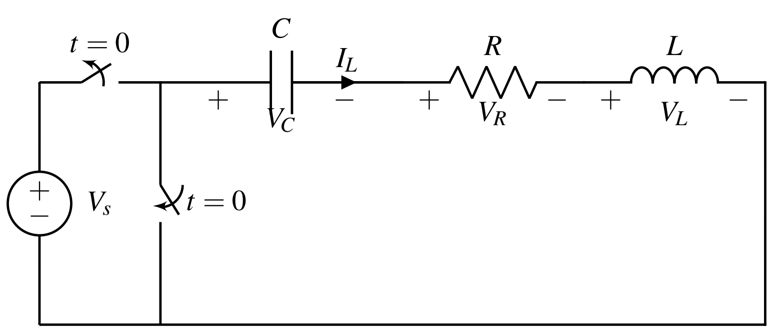
Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 3B.

(a) What is the I-V relationship of an inductor? What is the behavior of the inductor under DC current (i.e. constant current)?

Solution: An inductor's I-V relationship is $V_L = L\frac{dI_L}{dt}$ where L is the inductance of the inductor in Henries. In DC, the current through the inductor isn't changing and so its derivative is 0. Thus, the voltage across the inductor is 0 and it acts as a wire.

2. RLC Responses: Initial Part

Consider the following circuit like you saw in lecture:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 6 combined will try to show you the various RLC system responses and how they relate to how the eigenvalues of the A matrix changes. Note that the work you will do will also hold for any second-order system, like a mass-spring-damper, and is very common to study in controls as we'll see later on in Module 2 of the course.

(a) We first need to construct our state space system. Our natural state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the values that are changing in our circuit. Now, find the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of V_s, L, R, and C.

Solution: For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{d}{dt}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{d}{dt}V_C(t) = I_L(t) \tag{1}$$

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t). \tag{2}$$

If we use the state variable names, we can write this as

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t), \tag{3}$$

so now we have one differential equation.

For the other differential equation, we can apply KVL around the single loop in this circuit. (Alternatively, we could just solve it directly and substitute in for the desired voltage on the capacitor, which is a state variable.) Going clockwise, we have

$$V_C(t) + V_R(t) + V_L(t) = 0. \tag{4}$$

Using Ohm's Law and the inductor equation $V_L = L\frac{d}{dt}I_L(t)$, we can write this as

$$V_C(t) + RI_L(t) + L\frac{d}{dt}I_L(t) = 0, \tag{5}$$

which we can rewrite as

$$\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t) - \frac{1}{L}V_C(t). \tag{6}$$

If we use the state variable names, this becomes

$$\frac{d}{dt}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t), \tag{7}$$

and we have a second differential equation.

To summarize the final system is

$$\frac{d}{dt}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{8}$$

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t). \tag{9}$$

(b) Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$. This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

Solution: By inspection from the previous part, we have

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{10}$$

which is in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{11}$$

(c) Find the eigenvalues of the A matrix symbolically.

(Hint: the quadratic formula will be involved.)

Solution: To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{12}$$

$$= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \tag{13}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{14}$$

The Quadratic Formula gives

$$\lambda = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{15}$$

(d) Under what condition on the circuit parameters R, L, C are there going to be a pair of distinct purely real eigenvalues of A?

Solution: For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \tag{16}$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \tag{17}$$

(e) Under what condition on the circuit parameters R, L, C are there going to be a pair of purely imaginary eigenvalues of A? What will the eigenvalues be in this case?

Solution: The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm j\sqrt{\frac{1}{LC}}. \tag{18}$$

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex when conditions of part (d) fails) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, find eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ corresponding to them.

(HINT: Rather than trying to find the relevant nullspaces, etc., try to find eigenvectors of the form $\begin{bmatrix} 1 \\ y \end{bmatrix}$ *where we just want to find the missing entry y. This works because we know the first entry of the eigenvector can not be 0 and we want to normalize it to so that first entry is 1.)*

Solution:

The easy way is just to remember what an eigenvector is. We want $A\vec{v}_{\lambda_i} = \lambda_1\vec{v}_{\lambda_i}$. So, we can try to follow the hint:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ (\lambda_i)(y) \end{bmatrix} \tag{19}$$

We also know that:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix} \tag{20}$$

Equating the two equations from above gives:

$$\begin{bmatrix} \lambda_i \\ (\lambda_i)(y) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix} \tag{21}$$

From the second row we see that $y = \frac{1}{\lambda_i C}$. Now we find the eigenvectors as:

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix}$$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}$$

Alternatively, you can try to use the standard approach of finding the nullspace of $A - \lambda_i I$ to arrive at the same answer as above.

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = [\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}]$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. What is the $\widetilde{A}$ so that $\frac{d}{dt}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$? It is fine to have your answer expressed symbolically using λ_1, λ_2.

Solution: V is given by:

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

We know that V transforms from the $\tilde{x}$ coordinate frame to the x coordinate frame, V^{-1} transforms back, and A takes gives the relationship from x to $\frac{d}{dt}x$.
Therefore to go from $\widetilde{x}$ to $\frac{d}{dt}\widetilde{x}$:

$$\widetilde{A} = V^{-1}AV = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

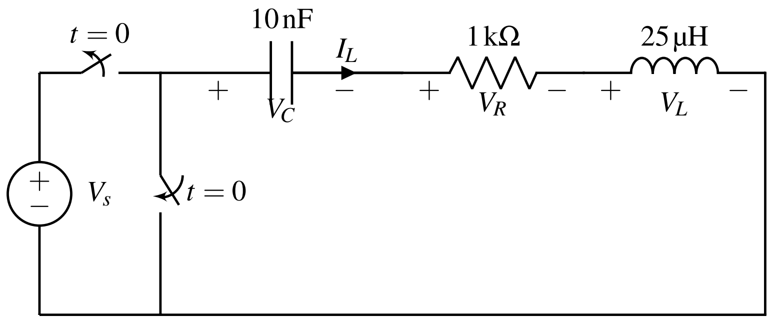
$$\widetilde{A} = V^{-1}AV = \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2 C} & -1 \\ -\frac{1}{\lambda_1 C} & 1 \end{bmatrix} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

$$\widetilde{A} = V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

You didn't have to multiply things out explicitly. You could have just noticed that the eigenvector matrix is special as it will diagonalize the A matrix such that $AV = V\Lambda$ or $V^{-1}AV = \Lambda$.

3. RLC Responses: Overdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached 'RLC_Calc.ipynb' Jupyter Notebook for numerical calculations.

(a) Suppose $R = 1\,\text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. Find the initial conditions for $\vec{\tilde{x}}(0)$. Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

Solution: First of all, we must state the initial conditions for $\vec{x}(0)$. If the circuit is in steady state before $t = 0$, then no current is flowing and the entire voltage drop is across the capacitor. Therefore:

$$x_1(0) = I_L(0) = 0$$
$$x_2(0) = V_C(0) = V_s = 1$$

Under these conditions, we can solve for

$$\lambda_1 = -1.0 \times 10^5,\ \lambda_2 = -4.0 \times 10^7$$

$$V^{-1} = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix}$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.001 \\ 0.001 \end{bmatrix}$$

(b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

Solution: Plugging in for the component values gives:

$$\widetilde{A} = \begin{bmatrix} -1.0 \times 10^5 & 0 \\ 0 & -4.0 \times 10^7 \end{bmatrix}$$

These eigenvalues are the negative reciprocals of the relevant time constants for these modes.

$$\begin{bmatrix} \frac{d}{dt}\widetilde{x}_1(t) \\ \frac{d}{dt}\widetilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1.0\times 10^5 & 0 \\ 0 & -4.0\times 10^7 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix}, \tag{22}$$

Therefore:

$$\widetilde{x}_1(t) = K_1 e^{-1.0\times 10^5 t}$$
$$\widetilde{x}_2(t) = K_2 e^{-4.0\times 10^7 t}$$

Solving for K with the initial condition gives:

$$\widetilde{x}_1(t) = -0.001 e^{-1.0\times 10^5 t}$$
$$\widetilde{x}_2(t) = 0.001 e^{-4.0\times 10^7 t}$$

Converting back to the $(\vec{x})$ coordinates:

$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -1000 & -2.5 \end{bmatrix} \vec{\widetilde{x}}(t)$$

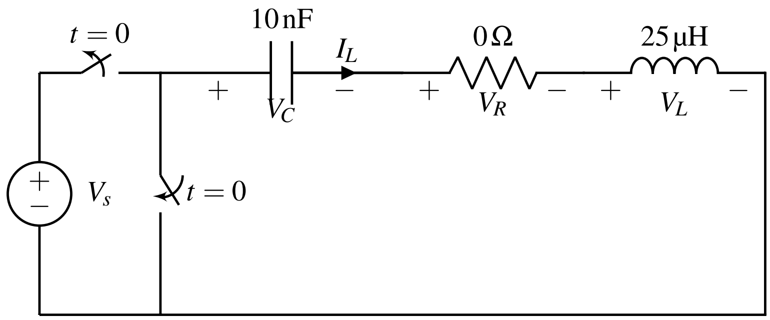
$$x_1(t) = -0.001 e^{-1.0\times 10^5 t} + 0.001 e^{-4.0\times 10^7 t}$$
$$x_2(t) = e^{-1.0\times 10^5 t} - 0.0025 e^{-4.0\times 10^7 t}$$

(c) In the ‘RLCSliders.ipynb‘ Jupyter notebook, move the sliders to approximately $R = 1k\Omega$ and $C = 10nF$. Comment on the graph of $V_c(t)$ and the location of the eigenvalues on the complex plane.

Solution: $V_c(t)$ should look like a decaying exponential. The eigenvalues lie on the real axis at coordinates $(-1\times 10^5, 0)$ and $(-4\times 10^7, 0)$.

4. RLC Responses: Undamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume that the capacitor is charged to V_s and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached 'RLC_Calc.ipynb' Jupyter Notebook for numerical calculations.

(a) Suppose $R = 0\,\text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. Find the initial conditions for $\vec{\tilde{x}}(0)$. Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

Solution: Under these conditions, we can solve for $\lambda = \pm j\sqrt{\frac{1}{LC}} = \pm j\sqrt{\frac{1}{250\times 10^{-15}}} = \pm j(2\times 10^6)$

$$\lambda_1 = j(2\times 10^6),\ \lambda_2 = -j(2\times 10^6)$$

Using the rule we derived earlier for finding V, we have

$$V = \begin{bmatrix} 1 & 1 \\ -j50 & j50 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 0.5 & j(0.01) \\ 0.5 & -j(0.01) \end{bmatrix}$$

which lets us say

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 0.5 & j(0.01) \\ 0.5 & -j(0.01) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j(0.01) \\ -j(0.01) \end{bmatrix}$$

(b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$. Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

Solution:

Plugging in for the component values gives:

$$\tilde{A} = \begin{bmatrix} j(2\times 10^6) & 0 \\ 0 & -j(2\times 10^6) \end{bmatrix}$$

$$\begin{bmatrix} \frac{d}{dt}\widetilde{x}_1(t) \\ \frac{d}{dt}\widetilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} j(2\times 10^6) & 0 \\ 0 & -j(2\times 10^6) \end{bmatrix} \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix}, \tag{23}$$

Therefore:

$$\widetilde{x}_1(t) = K_1 e^{+j(2\times 10^6)t}$$
$$\widetilde{x}_2(t) = K_2 e^{-j(2\times 10^6)t}$$

Solving for K with the initial condition gives:

$$\widetilde{x}_1(t) = j(0.01)e^{+j(2\times 10^6)t}$$
$$\widetilde{x}_2(t) = -j(0.01)e^{-j(2\times 10^6)t}$$

Converting back to the $(\vec{x})$ coordinates:

$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -j50 & j50 \end{bmatrix} \begin{bmatrix} j(0.01)e^{+j(2\times 10^6)t} \\ -j(0.01)e^{-j(2\times 10^6)t} \end{bmatrix}$$

$$x_1(t) = j(0.01)e^{+j(2\times 10^6)t} - j(0.01)e^{-j(2\times 10^6)t} = -0.02\sin\left(2\times 10^6 t\right)$$
$$x_2(t) = 0.5e^{+j(2\times 10^6)t} + 0.5e^{-j(2\times 10^6)t} = \cos\left(2\times 10^6 t\right)$$

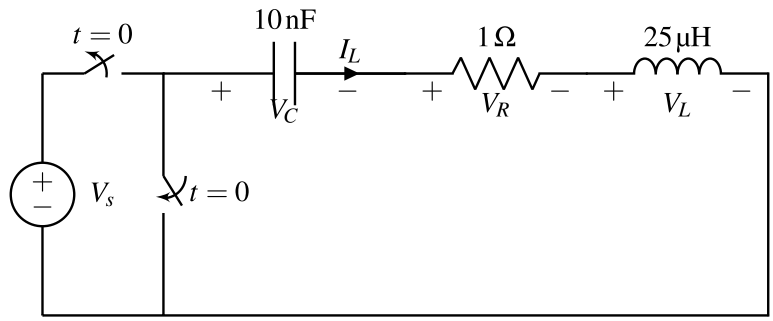
(c) In the ‘RLCSliders.ipynb‘ Jupyter notebook, move the sliders to approximately $R = 0\Omega$ and $C = 10nF$. Comment on the graph of $V_c(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?

Note: Because there is no resistance, this is called the “undamped” case.

Solution: No, these waveforms are sinusoids and do not die out over time. The eigenvalues are located on the imaginary axis at coordinates $(0, -2\times 10^6)$ and $(0, 2\times 10^6)$.

5. RLC Responses: Underdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple. You may use a calculator or the attached 'RLC_Calc.ipynb' Jupyter Notebook for numerical calculations.

(a) Now suppose that $R = 1\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ V. Find the initial conditions for $\vec{\tilde{x}}(0)$. Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

Solution: Under these conditions, we can solve for

$$\lambda_1 = -0.02\times10^6 + j(2\times10^6),\ \lambda_2 = -0.02\times10^6 - j(2\times10^6)$$

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{-0.0002+j(0.02)} & \frac{1}{-0.0002-j(0.02)} \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} .5+j(0.005) & j(0.01) \\ .5-j(0.005) & -j(0.01) \end{bmatrix}$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = V^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j(0.01) \\ -j(0.01) \end{bmatrix}$$

(b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$. Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).
(HINT: Remember that $e^{a+jb} = e^a e^{jb}$.)

Solution:

$$\widetilde{A} = \begin{bmatrix} -0.02\times10^6 + j(2\times10^6) & 0 \\ 0 & -0.02\times10^6 - j(2\times10^6) \end{bmatrix}$$

$$\begin{bmatrix} \frac{d}{dt}\widetilde{x}_1(t) \\ \frac{d}{dt}\widetilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.02\times10^6 + j(2\times10^6) & 0 \\ 0 & -0.02\times10^6 - j(2\times10^6) \end{bmatrix} \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix}, \tag{24}$$

Therefore:

$$\widetilde{x}_1(t) = K_1 e^{(-0.02\times 10^6 + j(2\times 10^6))t}$$
$$\widetilde{x}_2(t) = K_2 e^{(-0.02\times 10^6 - j(2\times 10^6))t}$$

Solving for K with the initial condition gives:

$$\widetilde{x}_1(t) = j(0.01) e^{(-0.02\times 10^6 + j2\times 10^6)t}$$
$$\widetilde{x}_2(t) = -j(0.01) e^{(-0.02\times 10^6 - j(2\times 10^6))t}$$

Converting back to the $(\vec{x})$ coordinates:

$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -0.5 - j50 & -0.5 + j50 \end{bmatrix} \begin{bmatrix} j(0.01)e^{(-0.02\times 10^6 + j(2\times 10^6))t} \\ -j(0.01)e^{(-0.02\times 10^6 - j(2\times 10^6))t} \end{bmatrix}$$

$$\begin{aligned} x_1(t) &= j(0.01)e^{(-0.02\times 10^6 + j(2\times 10^6))t} - j(0.01)e^{(-0.02\times 10^6 - j(2\times 10^6))t} \\ &= j(0.01)e^{-0.02\times 10^6 t} e^{j(2\times 10^6)t} - j(0.01)e^{-0.02\times 10^6 t} e^{-j(2\times 10^6)t} \\ &= e^{-0.02\times 10^6 t}\left(j(0.01)e^{j(2\times 10^6)t} - j(0.01)e^{-j(2\times 10^6)t}\right) \\ &= -0.020002 e^{-0.02\times 10^6 t} \sin\left(2\times 10^6 t\right) \end{aligned}$$

$$\begin{aligned} x_2(t) &= (0.5 - j(0.005))e^{(-0.02\times 10^6 + j(2\times 10^6))t} + (0.5 + j(0.005))e^{(-0.02\times 10^6 - j(2\times 10^6))t} \\ &= (0.5 - j(0.005))e^{-0.02\times 10^6 t} e^{j(2\times 10^6)t} + (0.5 + j(0.005))e^{-0.02\times 10^6 t} e^{-j(2\times 10^6)t} \\ &= e^{-0.02\times 10^6 t}\left((0.5 - j(0.005))e^{j(2\times 10^6)t} + (0.5 + j(0.005))e^{-j(2\times 10^6)t}\right) \\ &= e^{-0.02\times 10^6 t}\cos\left(2\times 10^6 t\right) + 0.01\cdot e^{-0.02\times 10^6 t}\sin\left(2\times 10^6 t\right). \end{aligned}$$

(c) In the 'RLCSliders.ipynb' Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. Comment on the graph of $V_c(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?

Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

Solution: Yes, the waveforms decay to 0. They appear to be sinusoids that are decaying exponentially. The eigenvalues should be located at coordinates $(-0.02\times 10^6, 2\times 10^6)$ and $(-0.02\times 10^6, -2\times 10^6)$.

(d) Notice that you got answers in terms of complex exponentials. Why did the final voltage and current waveforms end up being purely real?

Solution: In this case, it's because of the complex conjugacy of the quantities in the problem. The eivenvalues and their associated eigenvectors were complex conjugates, as were the transformed solutions $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$. When we applied the inverse transformation to $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$, we added together many complex conjugate terms, and the imaginary parts cancelled out.

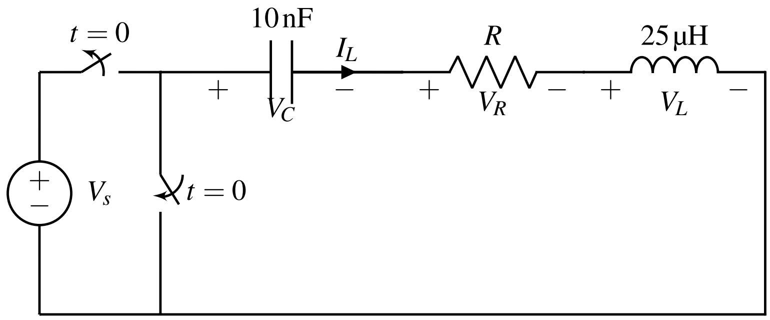
Now, was this just a fluke that just happened to line up perfectly? Is there some A matrix out there with real-valued entries that will result in a complex solution? Or is something more profound going on?

It turns to be no fluke. If the entries in the A matrix are real, and the initial condition $\vec{x}_0$ is real, then the solution to the differential equation $\frac{d}{dt}\vec{x} = A\vec{x}$ with $\vec{x}(0) = \vec{x}_0$ will also be real, *regardless of whether the eigenvalues of A are real, imaginary, or complex.* If a matrix $A \in \mathbf{R}^{n\times n}$ has some complex eigenvalues, then those eigenvalues will always arise in complex conjugate pairs. Furthermore, the eigenvectors associated to those eigenvalues arise on complex conjugate pairs. This will lead to the kind of cancellation that you saw in here, every single time.

After all, the quantities that we observe in the world are always purely real, so we would expect that the solutions to our models would also be real-valued.

6. RLC Responses: Critically Damped Case

Building on the previous problem, consider the following circuit with specified component values: (Notice R is not specified yet. You'll have to figure out what that is.)



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached 'RLC_Calc.ipynb' Jupyter Notebook for numerical calculations.

(a) For what value of R is there going to be a single eigenvalue of A?

Solution: If the terms under the square root, i.e., the discriminant of the quadratic formula, is 0, then we have a single value. More concretely,

$$\frac{R^2}{L^2} - \frac{4}{LC} = 0 \tag{25}$$

$$\frac{R^2}{L^2} = \frac{4}{LC} \tag{26}$$

$$R = 2\sqrt{\frac{L}{C}} \tag{27}$$

(b) Find the eigenvalues and eigenspaces of A. What are the dimensions of the corresponding eigenspaces? (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?)
For this part, assume the given values for the capacitor and the inductor, as well as the critical value for the resistance R that you found in the previous part. It is easier to do the algebra with a non-symbolic matrix to work with.

Solution: Our system's matrix becomes,

$$A = \begin{bmatrix} -4\times 10^6 & -4\times 10^4 \\ 10^8 & 0 \end{bmatrix} \tag{28}$$

Our single eigenvector is,

$$\lambda = -\frac{R}{2L} = -2\times 10^6 \tag{29}$$

Hence, the eigenvector is a basis of the nullspace of $A - \lambda I$,

$$\begin{bmatrix} -2\times 10^6 & -4\times 10^4 \\ 10^8 & 2\times 10^6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{30}$$

Hence, the eigenvector, $\vec{v} = \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -50 \end{bmatrix}$. We have only one eigenvector, since we have a single dimensional nullspace.

(c) We now create a new coordinate system V, with the first vector being $\vec{v}_\lambda$ — the eigenvector you found for the single eigenvalue λ above. For the second vector, just pick $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We will see later why we chose such a vector. We then apply a change of basis to define $\vec{\widetilde{x}}$ in the transformed coordinates such that $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. What is the resulting $\widetilde{A}$ matrix defining the system of differential equations in the transformed coordinates?

Solution: We want to find $\tilde{A}$ such that,

$$\frac{d}{dt}\vec{x} = A\vec{x} \tag{31}$$

$$\frac{d}{dt}V\tilde{x} = AV\tilde{x} \tag{32}$$

$$V\frac{d}{dt}\tilde{x} = AV\tilde{x} \tag{33}$$

$$\frac{d}{dt}\tilde{x} = V^{-1}AV\tilde{x} = \widetilde{A}\tilde{x} \tag{34}$$

Note that we went from step 2 to 3 using the fact that V is constant and doesn't get affected by a time derivative. Finally, we have

$$\widetilde{A} = V^{-1}AV = \begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4\times10^6 & -4\times10^4 \\ 1\times10^8 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} -2\times10^6 & -4\times10^4 \\ 0 & -2\times10^6 \end{bmatrix} \tag{36}$$

(d) Notice that the second differential equation for $\frac{d}{dt}\widetilde{x_2}(t)$ in the above coordinate system only depends on $\widetilde{x_2}(t)$ itself. There is no cross-term dependence. This happened because we earlier chose $\vec{v}_2$ such that the transformation $\widetilde{A}$ becomes upper traiangular and results in the removal of the cross-dependency. We will later see that in fact, we had many other choices for $\vec{v}_2$. Now, compute the initial condition for $\widetilde{x_2}(0)$ and write out the solution to this scalar differential equation for $\widetilde{x_2}(t)$ for $t \geq 0$.

Solution: First let's compute $x_2(0)$,

$$\vec{\tilde{x}} = V^{-1}\vec{x} \tag{37}$$

$$\begin{bmatrix} \widetilde{x_1}(0) \\ \widetilde{x_2}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{38}$$

$$\widetilde{x_2}(0) = 1 \tag{39}$$

Solving the following differential equation,

$$\frac{d}{dt}\widetilde{x_2}(t) = -2\times10^6\widetilde{x_2}$$

$$\widetilde{x_2}(t) = k_1 e^{-2\times10^6 t}$$

Substituting for the initial condition, we get, $\widetilde{x_2}(t) = e^{-2\times10^6 t}$

(e) With an explicit solution to $\widetilde{x}_2(t)$ in hand, substitute this in and write out the resulting scalar differential equation for $\widetilde{x}_1(t)$. This should effectively have a time-dependent input in it.

Note: this is just the differential-equations counterpart to the back-substitution step from Gaussian Elimination in 16A, once you had done one full downward pass of Gaussian Elimination. You went upwards and just substituted in the solution that you found to remove this dependence from the equations above. This is the exact same design pattern, except for a system of linear differential equations.

Solution: The differential equation for $\frac{d}{dt}\widetilde{x}_1(t)$ is

$$\frac{d}{dt}\widetilde{x}_1(t) = \left(-2\times 10^6\right)\widetilde{x}_1(t) - \left(4\times 10^4\right)\widetilde{x}_2(t), \tag{40}$$

which is just the top row of the matrix equation $\frac{d}{dt}\widetilde{x}(t) = A\widetilde{x}(t)$. Substituting in the solution we found for $\widetilde{x}_2(t)$ gives

$$\frac{d}{dt}\widetilde{x}_1(t) = \left(-2\times 10^6\right)\widetilde{x}_1(t) - \left(4\times 10^4\right)e^{\left(-2\times 10^6\right)t}. \tag{41}$$

Just like we expected, this is a scalar differential equation with an input.

(f) Solve the above scalar differential equation with input and write out what $\widetilde{x}_1(t)$ is for $t \geq 0$.

(HINT: You might want to look at a problem on an earlier homework for help with this.)

Solution:

Recall from an earlier homework, we proved that the differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t), \tag{42}$$

with initial value $x(0) = x_0$, has the unique solution

$$x(t) = e^{\lambda t}x_0 + \int_0^t e^{\lambda(t-\tau)}u(\tau)d\tau. \tag{43}$$

Now, the differential equation we found for $\widetilde{x}_1(t)$ in the previous part has this form, with $u(t) = ae^{\lambda t}$, $\lambda = -2\times 10^6$, $a = -4\times 10^4$, and $x_0 = 0$. Then solving this for our specific $u(t)$,

$$\widetilde{x}_1(t) = e^{\lambda t}x_0 + \int_0^t e^{\lambda t - \lambda\tau}(ae^{\lambda\tau})d\tau \tag{44}$$

$$= e^{\lambda t}x_0 + e^{\lambda t}\int_0^t a\,d\tau \tag{45}$$

$$= e^{\lambda t}x_0 + ate^{\lambda t} \tag{46}$$

$$= -\left(4\times 10^4\right)te^{\left(-2\times 10^6\right)t}. \tag{47}$$

(g) Find $x_1(t)$ and $x_2(t)$ for $t \geq 0$ based on the answers to the previous three parts.

This particular case is called the "critically damped case" for an RLC circuit. It is called this because the R value you found demarcates the boundary between solutions of the underdamped and overdamped variety.

Solution: Now that we have $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$, all we need to do to find $x_1(t)$ and $x_2(t)$ is to reverse the coordinate change we made. In other words, we can find $x(t)$ as

$$x(t) = V\widetilde{x}(t). \tag{48}$$

This gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix} \begin{bmatrix} -\left(4\times 10^4\right) t e^{\left(-2\times 10^6\right)t} \\ e^{\left(-2\times 10^6\right)t} \end{bmatrix} \tag{49}$$

$$= \begin{bmatrix} -\left(4\times 10^4\right) t e^{\left(-2\times 10^6\right)t} \\ \left(1+2\times 10^6 t\right) e^{\left(-2\times 10^6\right)t} \end{bmatrix} \tag{50}$$

(h) In the 'RLCSliders.ipynb' Jupyter notebook, move the sliders to the resistance value you found in the first part and $C = 10nF$. Comment on the graph of $V_c(t)$ and the location of the eigenvalues on the complex plane. What happens to the voltage and eigenvalues as you slightly increase or decrease R?

Solution: At the $R = 100$, $V_c(t)$ appears to decay exponentially. A slight increase in R causes the voltage to decay more slowly. A slight decrease in R causes a voltage undershoot and eventually oscillations. The eigenvalues have converged into the same point at $(-2\times 10^6, 0)$. Increasing R makes them split into two points, and both points remain on the real axis. One point goes towards the origin, while the other goes towards negative infinity. Decreasing R splits the eigenvalues back into their complex conjugates.

(i) In part (c) we saw that A only had one eigenvalue, λ, and one eigenvector, $\vec{v}_\lambda$. This meant that we had a choice for $\vec{v}_2$ in the expression $V = \begin{bmatrix} \vec{v}_\lambda & \vec{v}_2 \end{bmatrix}$. We seemingly arbitrarily chose $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We claim that there are many correct choices of $\vec{v}_2$ that will result in $\tilde{A} = V^{-1}AV = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a, c \neq 0$ (i.e. it is upper triangular). Remember, we want $\tilde{A}$ to be upper triangular so that we have an uncoupled differential equation for $\vec{\tilde{x}}_2(t)$: $\frac{d}{dt}\vec{\tilde{x}}(t) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} \implies \frac{d}{dt}\tilde{x}_2(t) = c\tilde{x}_2(t)$

In fact, it turns out that we can pick any $\vec{v}_2$ as long as $\vec{v}_2 \neq k\vec{v}_\lambda$ for some $k \in \mathbb{R}$. We will try and prove this very claim. More concisely, prove the statement below:

if $V = \begin{bmatrix} \vec{v}_\lambda & \vec{v}_2 \end{bmatrix}$ **and** $\vec{v}_2 \neq k\vec{v}_\lambda$ **for some** $k \in \mathbb{R}$**, then** $\tilde{A} = V^{-1}AV = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ **where** $a, c \neq 0$

Solution:

We claim that if $V = \begin{bmatrix} \vec{v}_\lambda & \vec{v}_2 \end{bmatrix}$ and $\vec{v}_2 \neq k\vec{v}_\lambda$ for some $k \in \mathbb{R}$, then $\tilde{A} = V^{-1}AV = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a, c \neq 0$

We immediately notice that V^{-1} cannot exist if $\vec{v}_2 = k\vec{v}_\lambda$ since this implies that the columns of V would be linearly dependent. Now we must show that every other choice of $\vec{v}_2$ is fine:

$$\tilde{A} = V^{-1}AV \tag{51}$$

$$= V^{-1}A \begin{bmatrix} \vec{v}_\lambda & \vec{v}_2 \end{bmatrix} \tag{52}$$

$$= V^{-1} \begin{bmatrix} A\vec{v}_\lambda & A\vec{v}_2 \end{bmatrix} \tag{53}$$

$$= V^{-1} \begin{bmatrix} \lambda\vec{v}_\lambda & A\vec{v}_2 \end{bmatrix} \tag{54}$$

Here we notice that $V^{-1}V = \begin{bmatrix} V^{-1}\vec{v}_\lambda & V^{-1}\vec{v}_2 \end{bmatrix} = I_{2\times 2} \implies V^{-1}\vec{v}_\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{e}_1$. So,

$$\tilde{A} = \begin{bmatrix} \lambda V^{-1}\vec{v}_\lambda & V^{-1}A\vec{v}_2 \end{bmatrix} \tag{55}$$

$$= \begin{bmatrix} \lambda \vec{e}_1 & V^{-1}A\vec{v}_2 \end{bmatrix} \tag{56}$$

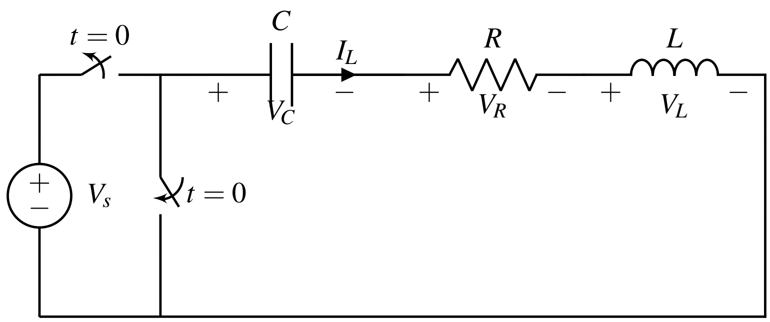
Let us denote $V^{-1}A\vec{v}_2 = \begin{bmatrix} b \\ c \end{bmatrix}$. So, we can write

$$\tilde{A} = \begin{bmatrix} \lambda & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \tag{57}$$

And we are almost done! All that is left is to show that $a, c \neq 0$. We know that $a \neq 0$ since $a = \lambda$. We also know that $c \neq 0$. To see why, assume that c was zero. If $c = 0$, then $\tilde{A} = V^{-1}AV = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ would have a row of zeros, meaning it has a nullspace. However, since A and V are invertible, we know this cannot be the case! So we finally succeed in showing that $\vec{v}_2$ can really be any vector that is not a scalar multiple of $\vec{v}_\lambda$

7. Second order ODE perspective on solving the RLC circuit

Consider the following circuit like you saw in lecture, discussion, and the previous few problems:



Suppose now we insisted on expressing everything in terms of one waveform $V_C(t)$ instead of two of them (voltage across the capacitor and current through the inductor). This is called the "second-order" point of view, for reasons that will soon become clear.

For this problem, use R for the resistor, L for the inductor, and C for the capacitor in all the expressions until the last part.

(a) **Write the current I_L through the inductor in terms of the voltage through the capacitor.**

Solution: The current I_L through the inductor must be the same as the current through C, which is $C\frac{d}{dt}V_C$. Hence, we can write

$$I_L = C\frac{d}{dt}V_C.$$

(b) Now, notice that the voltage drop across the inductor involves $\frac{d}{dt}I_L$. **Write the voltage drop across the inductor in terms of the second derivative of V_C.**

Solution: The voltage drop is

$$V_L = L \cdot \frac{d}{dt}I_L = LC\frac{d}{dt}\left(\frac{d}{dt}V_C\right) := LC\frac{d^2}{dt^2}V_C.$$

(c) For this part, treat $V_s(t)$ as a generic input waveform — don't necessarily view the switch as being thrown, etc.

Now write out a differential equation governing $V_C(t)$ in the form of

$$\frac{d^2}{dt^2}V_C(t) + a \cdot \frac{d}{dt}V_C(t) + b \cdot V_C(t) + c(t) = 0. \tag{58}$$

where a, b and $c(t)$ are terms you need to figure out by analyzing the circuit.

This is called a second-order system here as there is a 2nd derivative in the differential equation.

(HINT: The $c(t)$ needs to involve $V_s(t)$ in some way.)

Solution: Note that the current passing through the resistor is

$$I_R = \frac{V_s - V_C - V_L}{R} = C\frac{d}{dt}V_C.$$

or equivalently,

$$V_L + RC\frac{d}{dt}V_C + V_C - V_s = 0.$$

Plugging in V_L, we have

$$LC\frac{d^2}{dt^2}V_C + RC\frac{d}{dt}V_C + V_C - V_s = 0.$$

Finally, dividing by LC,

$$\frac{d^2}{dt^2}V_C + \frac{R}{L}\cdot\frac{d}{dt}V_C + \frac{1}{LC}\cdot V_C - \frac{1}{LC}\cdot V_s = 0.$$

(d) We don't know how to solve equations like Eq. (58). To reduce this to something we know how to solve, we define the first derivative $\frac{d}{dt}V_C(t)$ as an additional state and label it as $X(t)$. Note that this directly gives us one equation in our matrix: $\frac{d}{dt}V_C(t) = X(t)$. This leaves us needing an equation for $\frac{d}{dt}X(t)$. **Express $\frac{d}{dt}X(t)$ in terms of $X(t)$, $V_C(t)$, and $V_s(t)$. Write a matrix differential equation in terms of $V_C(t)$ and $X(t)$.** Your answer should be in the form:

$$\frac{d}{dt}\begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{A}\begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix}\cdot V_s(t). \tag{59}$$

Solution: With our expression of X, we can write

$$\frac{d}{dt}X + \frac{R}{L}\cdot X + \frac{1}{LC}\cdot V_C - \frac{1}{LC}\cdot V_s = 0$$

and

$$\frac{d}{dt}V_C = X.$$

Then, we can write a matrix

$$\begin{bmatrix} \frac{d}{dt}X \\ \frac{d}{dt}V_C \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} X \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{LC} \\ 0 \end{bmatrix}\cdot V_s.$$

(e) **Find the eigenvalues and eigenvectors of the matrix A from Eq. (59). Compare what you got to your answers for Problem 2 and explain why this is the case.**
(Hint: use the same trick you did in problem 2. Don't do this the hard way.)

Solution: $\det(A - \lambda I) = 0$ gives us

$$\lambda^2 + \frac{R}{L}\cdot\lambda + \frac{1}{LC} = 0.$$

Then,

$$\lambda = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}.$$

To find the eigenvectors corresponding to λ_i, we assume the eigenvector is of the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$. Then,

$$A\begin{bmatrix} 1 \\ a \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = \lambda_i\begin{bmatrix} 1 \\ a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}^T\begin{bmatrix} 1 \\ a \end{bmatrix} = 1 = \lambda_i a \implies a = \frac{1}{\lambda_i}$$

Thus,

$$\text{Eigenvalue } \frac{-\frac{R}{L}+\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}}{2} \text{ corresponds to eigenvector } \begin{bmatrix} 1 \\ \frac{2}{-\frac{R}{L}+\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}} \end{bmatrix}$$

$$\text{Eigenvalue } \frac{-\frac{R}{L}-\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}}{2} \text{ corresponds to eigenvector } \begin{bmatrix} 1 \\ \frac{2}{-\frac{R}{L}-\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}} \end{bmatrix}.$$

Note that we got the exact same eigenvalues as in Problem 2, which is because both A matrices have the same characteristic equation. However, we got different eigenvectors due to the difference in A matrices.

(f) **Now use the same values of R, L, C and initial conditions from Problem 3 and 4 to solve for V_C in this system.** Did you get the same answer as in problems 3 and 4? Remember that you can use the attatched 'RLC_Calc.ipynb' to help with computation.

Solution: Let's first solve the differential equation

$$\begin{bmatrix} \frac{d}{dt}X \\ \frac{d}{dt}V_C \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ V_C \end{bmatrix},$$

With problem 3 parameters, the eigenvalues are $\lambda_1 = -1.0 \times 10^5$ and $\lambda_2 = -4.0 \times 10^7$, with

$$V^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}^{-1} = \frac{1}{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ -\frac{1}{\lambda_1} & 1 \end{bmatrix} = \begin{bmatrix} -0.0025 & 1.0 \times 10^5 \\ 1.0025 & -1.0 \times 10^5 \end{bmatrix}$$

For ease of notations, let $\vec{y} := \begin{bmatrix} X \\ V_C \end{bmatrix}$. The initial conditions are

$$\vec{\tilde{y}}(0) = V^{-1}\vec{y}(0) = \begin{bmatrix} -0.0025 & 1.0 \times 10^5 \\ 1.0025 & -1.0 \times 10^5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.0 \times 10^5 \\ -1.0 \times 10^5 \end{bmatrix}$$

Using the initial conditions with the eigenvalues gives us

$$\vec{\tilde{y}}(t) = \begin{bmatrix} 1.0 \times 10^5 \cdot e^{-1.0 \times 10^5 t} \\ -1.0 \times 10^5 \cdot e^{-4.0 \times 10^7 t} \end{bmatrix}$$

Converting back to the original coordinates,

$$\vec{y}(t) = V\vec{\tilde{y}}(t) = \begin{bmatrix} 1 & 1 \\ -1.0 \times 10^{-5} & -2.5 \times 10^{-8} \end{bmatrix} \begin{bmatrix} 1.0 \times 10^5 \cdot e^{-1.0 \times 10^5 t} \\ -1.0 \times 10^5 \cdot e^{-4.0 \times 10^7 t} \end{bmatrix}$$

and hence

$$\vec{y}(t) = \begin{bmatrix} 1.0 \times 10^5 \cdot e^{-1.0 \times 10^5 t} - 1.0 \times 10^5 \cdot e^{-4.0 \times 10^7 t} \\ -1.0 \times e^{-1.0 \times 10^5 t} + 2.5 \times 10^{-3} \cdot e^{-4.0 \times 10^7 t} \end{bmatrix}$$

Comparing $V_C(t)$, we see that we get the same answer as in problem 3.

With problem 4 parameters, the eigenvalues are $\lambda_1 = j(2.0 \times 10^6)$ and $\lambda_2 = -j(2.0 \times 10^6)$, with

$$V^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}^{-1} = \frac{1}{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ -\frac{1}{\lambda_1} & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & j(10^6) \\ 0.5 + j(0.5) & -j(10^6) \end{bmatrix}$$

For ease of notations, let $\vec{y} := \begin{bmatrix} X \\ V_C \end{bmatrix}$. The initial conditions are

$$\vec{\tilde{y}}(0) = V^{-1}\vec{y}(0) = \begin{bmatrix} 0.5 & j(10^6) \\ 0.5 + j(0.5) & -j(10^6) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j(10^6) \\ -j(10^6) \end{bmatrix}$$

Using the initial conditions with the eigenvalues gives us

$$\vec{\tilde{y}}(t) = \begin{bmatrix} j(10^6) \cdot e^{j(2.0\times10^6)t} \\ -j(10^6) \cdot e^{-j(2.0\times10^6)t} \end{bmatrix}.$$

Finally,

$$\begin{aligned} \vec{y}(t) = V\vec{\tilde{y}}(t) &= \begin{bmatrix} 1 & 1 \\ \frac{1}{j(2.0\times10^6)} & \frac{1}{-j(2.0\times10^6)} \end{bmatrix} \begin{bmatrix} j(10^6) \cdot e^{j(2.0\times10^6)t} \\ -j(10^6) \cdot e^{-j(2.0\times10^6)t} \end{bmatrix} \\ &= \begin{bmatrix} j(10^6)(e^{j(2.0\times10^6)t} - e^{-j(2.0\times10^6)t}) \\ \frac{1}{2}e^{j(2.0\times10^6)t} + \frac{1}{2}e^{-j(2.0\times10^6)t} \end{bmatrix} \end{aligned}$$

Using Euler's Formula, we can rewrite the second row $V_C(t) = \cos\left(2.0 \times 10^6 t\right)$, which is exactly what we got in problem 4.

8. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

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