EECS 16B Designing Information Devices and Systems II
Spring 2021 UC Berkeley

Homework 4

This homework is due on Friday, February 12, 2021, at 11:00PM. Self-grades and HW Resubmission are due on Tuesday, February 16, 2021, at 11:00PM.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 3A

(a) Explain the process to solve a general vector differential equation $\frac{d}{dt}\vec{x} = A\vec{x}$ where $x \in \mathbb{R}^n$ and $\vec{x}(t_0) = \vec{x}_0$, including any necessary conditions.

Solution:

i. Find the eigenvalues and eigenvectors of A. If the number of linearly independent eigenvectors is not equal to n, then we cannot proceed further.

ii. Create the eigenvalue matrix $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$ and eigenvector matrix $V = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$.

iii. Construct a new vector ODE system $\vec{z} = \Lambda\vec{z}$ where $\vec{z} = V^{-1}x$. Solving it gives for all i, $z_i(t) = c_i e^{\lambda_i t}$ where c_i is some constant.

iv. Find the initial condition in the new system with $\vec{z}(t_0) = V^{-1}\vec{x}(t_0) = V^{-1}\vec{x}_0$. Use the initial condition to solve for all constants $c_i = x_0[i]e^{-\lambda_i t_0}$ where $x_0[i]$ is the ith element of x_0.

v. Change coordinates back to the original system with $\vec{x}(t) = V\vec{z}(t)$.

2. Tracking Terry

Terry is a mischievous child, and his mother is interested in tracking him.

For this problem, the $\mathbb{R}^2$ standard basis vectors will be denoted by

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(a) Terry texts his current location $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with these coordinates in the basis $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. Write Terry's location in the standard basis in terms of $\vec{v}_1$ and $\vec{v}_2$.

Solution: The location in the standard basis is just the coordinates multiplied by the basis vectors:

$$\vec{r}_m = 2\vec{v}_1 + 3\vec{v}_2$$

(b) Terry's friend tells you that Terry's location in the standard basis is $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$. Determine the basis vectors he is using, or if it is impossible, explain why.

Solution: Solving for the basis vectors Terry is using (or in other words how he sees things in his coordinate space) is the same as determining the values of $\vec{v}_1, \vec{v}_2$ in the matrix vector equation:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

There are 4 unknowns and only two equations, so this task is impossible.

(c) Terry's basis vectors get leaked to his mom on accident, so she knows they are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In order to do this, he needs a way to convert coordinates from the V basis to the P basis. Thus, find the matrix T such that if $\vec{x}_v$ is a location in V coordinates and $\vec{x}_p$ is the same location in P coordinates, then $\vec{x}_p = T\vec{x}_v$.

Solution: The problem can be formulated as a change of basis problem. Since both $\vec{x}_v$ and $\vec{x}_p$ correspond to the same point, then

$$V\vec{x}_v = P\vec{x}_p$$

Since we want to find $\vec{x}_p$,

$$\begin{aligned} \vec{x}_p &= P^{-1}V\vec{x}_v \\ T &= P^{-1}V \\ &= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \end{aligned}$$

3. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \left[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right]$.

(i) Show that $AV = V\Lambda$, where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

Solution:

$$\begin{aligned} AV &= A[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n] \\ &= [A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n] \\ &= [\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \ldots, \lambda_n\vec{v}_n] \\ &= [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \\ &= V\Lambda \end{aligned}$$

(ii) Argue that V is invertible, and therefore, $A = V\Lambda V^{-1}$.

Solution: Columns of V are eigenvectors of A which are known to be linearly independent. Since V has linearly independent columns, it has full column rank, and therefore, is invertible.

$$\begin{aligned} AV &= V\Lambda \\ AVV^{-1} &= V\Lambda V^{-1} \\ A &= V\Lambda V^{-1} \end{aligned}$$

(b) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A * A * \cdots * A * A}_{k \text{ times}} \tag{1}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Conclude that A^k is diagonalizable.

Solution: Consider the i^{th} eigenvector of A, $\vec{v}_i$ and the corresponding eigenvalue λ_i.

$$\begin{aligned}
A^k\vec{v}_i &= A^{k-1} * A\vec{v}_i \\
&= A^{k-1}\lambda_i\vec{v}_i \\
&= \lambda_i A^{k-2} * A\vec{v}_i \\
&= \lambda_i^2 A^{k-3} * A\vec{v}_i \\
&\vdots \\
&= \lambda_i^k\vec{v}_i
\end{aligned}$$

Thus by definition, v_i is an eigenvector of A^k with corresponding eigenvalue λ_i^k.

Alternate solution: Since A is diagonalizable, we can express A as

$$A = V\Lambda V^{-1} \tag{2}$$

Substituting A as shown in Equation 2 in 1, we get

$$\begin{aligned}
A^k &= \underbrace{A * A * \cdots * A * A}_{k \text{ times}} \\
&= \underbrace{V\Lambda V^{-1} * V\Lambda V^{-1} * \cdots * V\Lambda V^{-1} * V\Lambda V^{-1}}_{k \text{ times}} \\
&= \underbrace{V\Lambda\left(V^{-1} * V\right)\Lambda V^{-1} * \cdots * V\Lambda\left(V^{-1} * V\right)\Lambda V^{-1}}_{k \text{ times}} \\
&= V\underbrace{\Lambda * \Lambda * \cdots * \Lambda * \Lambda}_{k \text{ times}}V^{-1} \\
&= V\Lambda^k V^{-1}
\end{aligned}$$

Since Λ is a diagonal matrix,

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

Thus, A^k is clearly diagonalizable, where the eigenvectors of A^k are just the eigenvectors of A, and the eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k$.

4. Vector Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x} \tag{3}$$

where x_1, x_2 are variables depending on time t, and A is a 2×2 matrix with constant coefficients. We call (3) a vector differential equation.

(a) Suppose we have a system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 7x_1 - 8x_2 \tag{4}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_1 - 5x_2 \tag{5}$$

Write this in the form of (3) and compute the eigenvalues of the A matrix.

Solution:

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned} \det\left(\begin{bmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{bmatrix}\right) &= (7-\lambda)(-5-\lambda)+32 \\ &= \lambda^2 - 7\lambda + 5\lambda - 35 + 32 \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda+1)(\lambda-3). \end{aligned}$$

Thus the eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 3$.

(b) Compute the eigenvectors of the matrix A. For consistency, assume that the smaller eigenvalue is λ_1 and the larger is λ_2.

Solution: We will use the standard null space approach.

$$(A - \lambda_1 I)v_1 = (A+I)v_1 = \begin{bmatrix} 8 & -8 \\ 4 & -4 \end{bmatrix} v_1 = 0$$

$$(A - \lambda_2 I)v_2 = (A-3I)v_2 = \begin{bmatrix} 4 & -8 \\ 4 & -8 \end{bmatrix} v_2 = 0$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(c) We now want to transform our current system to a new coordinate system in order to simplify our differential equation. What basis B should we use so that in the new coordinates $\vec{z} = B^{-1}\vec{x}$, the new A matrix is diagonal? Write out what this new system becomes in the z coordinates.

Solution: We will let our basis B be the eigenvector matrix $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$. Then, since A can be diagonalized, we have $A = B\Lambda B^{-1}$.

$$\frac{d}{dt}x = Ax = B\Lambda B^{-1}x = B\Lambda z$$
$$B^{-1}\frac{d}{dt}x = B^{-1}B\Lambda z$$
$$\frac{d}{dt}B^{-1}x = \frac{d}{dt}z = \Lambda z = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} z$$

(d) Solve the new system in the z coordinates, using the initial conditions that $x_1(0) = 1, x_2(0) = -1$.

Solution: Now z_1 and z_2 are their own separated differential equations so $\frac{d}{dt}z_1 = -z_1$ and $\frac{d}{dt}z_2 = 3z_2$. We know the general form solution of these differential equations:

$$z_1(t) = k_1 e^{-t}$$
$$z_2(t) = k_2 e^{3t}$$

We now find the initial conditions in the z coordinates with

$$z(0) = B^{-1}x(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

This means that $k_1 = z_1(0) = -3$ and $k_2 = z_2(0) = 2$. Thus, our final solutions for $z(t)$ are

$$z_1(t) = -3e^{-t}$$
$$z_2(t) = 2e^{3t}$$

(e) Now convert your solution from the z coordinates back to the original x coordinates.

Solution: We can just use the coordinate transformation we defined in the beginning, that $x = Bz$. Then,

$$\vec{x}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3e^{-t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} -3e^{-t} + 4e^{3t} \\ -3e^{-t} + 2e^{3t} \end{bmatrix}$$

(f) It turns out that all 2nd order linear differential equations with distinct eigenvalues will have this common form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

We will use this method to solve a different system. Consider another second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 6y(t) = 0, \tag{6}$$

First to make the problem familiar, write the system in the form of (3), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

Solution: If we set $x_1(t) = y(t)$, $x_2(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}$, then we have

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2(t) \tag{7}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} = 5\frac{\mathrm{d}y(t)}{\mathrm{d}t} - 6y(t) = 5x_2(t) - 6x_1(t) \tag{8}$$

We can write this in the form of (3) as follows

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9}$$

(g) Now solve the system in (6) with the initial conditions $y(0) = 1$, $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$, using the method from part (f).

Solution: We first compute the eigenvalues of the matrix from the previous part. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix}\right) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)\,.$$

Thus the eigenvalues are $\lambda_1 = 2, \lambda_2 = 3$.
From part (g), the solution for $x_1(t), x_2(t)$ is of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0e^{2t} + c_1e^{3t} \\ c_2e^{2t} + c_3e^{3t} \end{bmatrix}.$$

At $t = 0$, we have $y(0) = 1$ and $\frac{dy}{dt}(0) = 1$. Using our differential equation (6), we can get $\frac{d^2y}{dt^2}(0) = 5\frac{dy}{dt}(0) - 6y(0) = -1$. Plugging these in,

$$x_1(0) = y(0) = 1 = c_0 + c_1 \tag{10}$$

$$x_2(0) = \frac{dy}{dt}(0) = 1 = c_2 + c_3 \tag{11}$$

$$\frac{dx_1}{dt}(0) = \frac{dy}{dt}(0) = 1 = 2c_0 + 3c_1 \tag{12}$$

$$\frac{dx_2}{dt}(0) = \frac{d^2y}{dt^2}(0) = -1 = 2c_2 + 3c_3 \tag{13}$$

This gives $c_0 = 2, c_1 = -1, c_2 = 4, c_3 = -3$. Alternatively, you could've seen that $c_2 = 2c_0$ and $c_3 = 3c_1$ since $x_2(t)$ is the derivative of $x_1(t)$ which makes it solvable with just the first 2 equations. Thus we have

$$x_1(t) = y(t) = 2e^{2t} - e^{3t} \tag{14}$$

$$x_2(t) = \frac{dy(t)}{dt} = 4e^{2t} - 3e^{3t} \tag{15}$$

5. Op-Amp Integrators: A continuation from the previous HW

In this question we will continue on from our analysis in Homework 3 and look at the eigenvalues of the integrator circuit (refer to Figure 3) in both non-ideal and ideal situations.

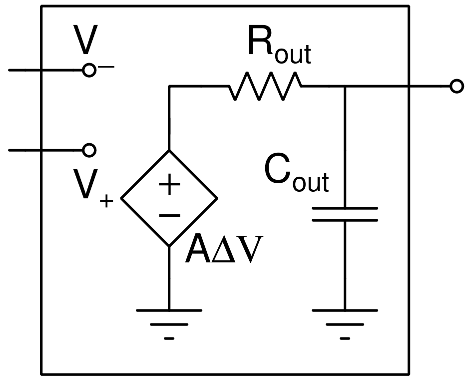
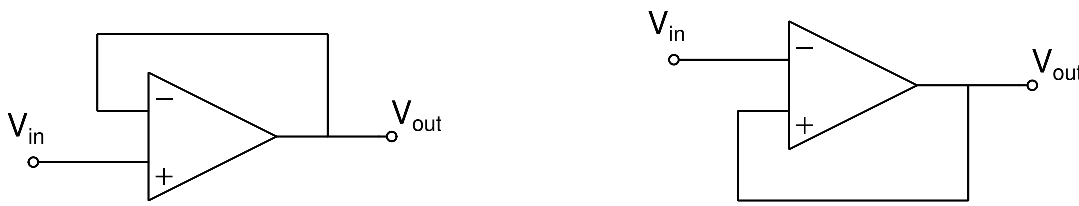


Figure 1: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback

(b) Op-amp in positive feedback

Figure 2: Op-amp in positive and negative feedback

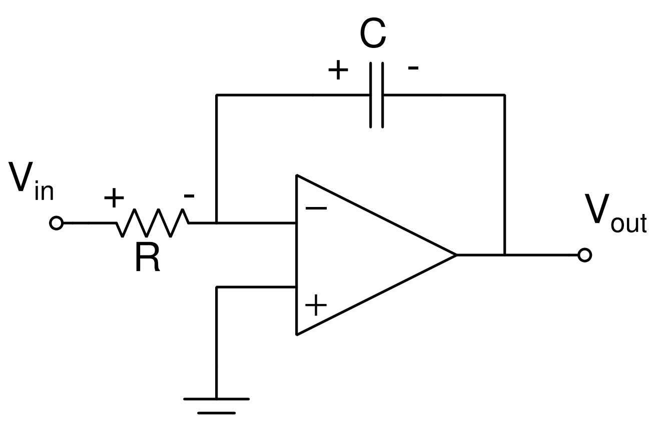


Figure 3: Integrator circuit

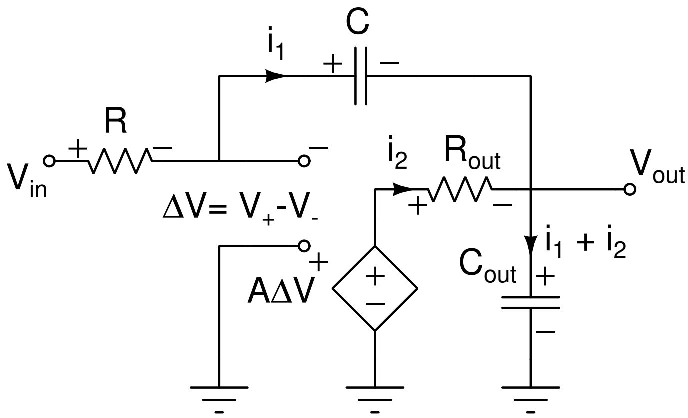


Figure 4: Integrator circuit with Op-amp model

(a) Recall from Homework 3 we had the following analysis to the integrator circuit shown in Figure 4.

$$\frac{d}{dt}\begin{bmatrix} V_{out} \\ V_C \end{bmatrix} = \begin{bmatrix} -\left(\frac{A+1}{R_{out}C_{out}} + \frac{1}{RC_{out}}\right) & -\left(\frac{1}{RC_{out}} + \frac{A}{R_{out}C_{out}}\right) \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} V_{out} \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{out}} \\ \frac{1}{RC} \end{bmatrix} V_{in} \tag{16}$$

Solve for the eigenvalues for the matrix/vector differential equation in Eq. (16).
For simplicity, assume $C_{out} = C = 0.01F$ and $R = 1\Omega$ and looking at the datasheet for the TI LMC6482 (the op-amps used in lab), we have $A = 10^6$ and $R_{out} = 100\Omega$.
Feel free to assume $A+1 \approx 10^6$ when you finally need to plug in values, but do not make any other approximations. (Of course, such an approximation is not valid if you have a $A+1-A$ term showing up somewhere.) Feel free to use a scientific calculator or Jupyter to find the eigenvalues.

Solution: We can find the characteristic equation and substituting the given values, we get

$$\lambda^2 + \left(\frac{A+1}{R_{out}C_{out}} + \frac{1}{RC_{out}} + \frac{1}{RC}\right)\lambda + \frac{1}{R_{out}C_{out}RC} = 0$$
$$\lambda^2 + (10^6 + 2\times 10^2)\lambda + 10^2 = 0$$

Hence, the eigenvalues are

$$\lambda_\pm = \frac{-(10^6 + 2\times 10^2) \pm \sqrt{(10^6 + 2\times 10^2)^2 - 400}}{2}$$
$$\therefore \lambda_+ = -1\times 10^{-4}$$
$$\lambda_- = -1\times 10^{6}$$

You should see that one eigenvalue corresponds to a slowly dying exponential and is close to 0. The other corresponds to a much faster dying exponential. The very slowly dying exponential is what corresponds to the desired integrator-like behavior. This is what lets it “remember.” (If you don’t understand why, think back to the HW problem you saw in a previous HW where you proved the uniqueness of the integral-based solution to a scalar differential equation with an input waveform.)

(b) Again, assume we have an ideal op-amp, *i.e.*, $A \to \infty$. **Find the eigenvalues under this limit.** Feel free to make any reasonable approximations.

Solution: With the given assumptions, we can rewrite our characteristic equation as

$$\lambda^2 + \frac{A}{R_{out}C_{out}}\lambda + \frac{1}{R_{out}C_{out}RC} = 0$$

Hence, we can find the eigenvalues as

$$\begin{aligned}\lambda_{\pm} &= \frac{-A}{2R_{out}C_{out}} \pm \sqrt{\frac{A^2}{4R_{out}^2C_{out}^2} - \frac{1}{R_{out}C_{out}RC}} \\ \therefore \lambda_{+} &\to 0 \\ \lambda_{-} &\to -\infty\end{aligned}$$

Here, you should see that the eigenvalue that used to be a slowly dying exponential stops dying out at all — corresponding to the ideal integrator's behavior of remembering forever.

6. Multi-Capacitor Circuit

Consider the circuit below

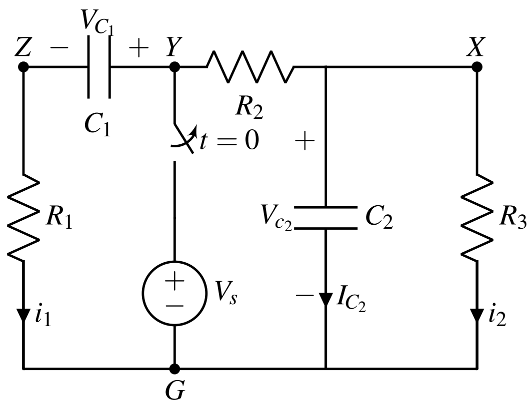


Figure 5: Circuit with multiple capacitors.

The resistors shown in the circuit have the same value $R_1 = R_2 = R_3 = R$. Capacitors C_1 and C_2 have the same capacitance $C_1 = C_2 = C$. Further, $RC = 1$s.

(a) Assume that the switch shown in Figure 5 was held in the closed position for a long time before $t = 0$. At $t = 0$, immediately after the switch is opened, what are the capacitor voltages $V_{C_1}(0)$ and $V_{C_2}(0)$?

Solution: In steady state, capacitors act as open circuits.

Immediately before $t = 0$,

$$
\begin{aligned}
V_Y &= V_s\,, \\
i_2 &= \frac{V_s}{R_2 + R_3} = \frac{V_s}{2R}\,, \\
V_X &= i_2 R = \frac{V_s}{2}\,, \\
i_1 &= 0\,, \\
V_Z &= i_1 R_1 = 0.
\end{aligned}
$$

Since the voltage across the capacitors cannot change immediately in this circuit, the capacitor voltages immediately after the switch is opened at $t = 0$ will be

$$
\begin{aligned}
V_{C_1}(0) &= V_s\,, \\
V_{C_2}(0) &= \frac{V_s}{2}.
\end{aligned}
$$

(b) How are i_2 and V_{C_2} related? Using this, how are i_1 and V_{C_2} related?

Solution: Since the resistor R_3 and capacitor C_2 are connected in parallel, they will have the same voltage drop across them. We can write the voltage drop across the resistor as $V_R = i_2 R$. This gives us $i_2 R = V_{C_2}$. Then, KCL at node G gives us

$$I_{C_2} + i_2 + i_1 = 0$$

Plugging in our expression for i_2 and using the IV relationship of capacitor C_2, we get

$$C_2 \frac{d}{dt} V_{C_2} + \frac{V_{C_2}}{R} + i_1 = 0 \tag{17}$$

(c) Using KVL on the loop comprising of both capacitors C_1 and C_2, find a relationship between V_{C_1}, V_{C_2} and i_1.

Solution: KVL on the mentioned loop gives us

$$V_{C_2} - V_{R_2} - V_{C_1} - V_{R_1} = 0$$

$$V_{C_2} - i_1 R_2 - V_{C_1} - i_1 R_1 = 0$$

$$V_{C_2} - V_{C_1} - 2 i_1 R = 0 \tag{18}$$

(d) Rewrite the equations derived above, eliminating the current i_1 to obtain a system of differential equations involving V_{C_1} and V_{C_2}. Write this system of equations in a matrix form

$$\frac{d}{dt} \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} = A \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix}$$

What is the matrix A and what are its eigenvalues?
Hint: You can use the relation $i_1 = C_1 \frac{d}{dt} V_{C_1}$ in addition to the relations we have derived so far.

Solution: We can use KCL at node Z to write $i_1 = C_1 \frac{d}{dt} V_{C_1}$ and use it to eliminate i_1. Substituting this in Equation 18, we get

$$V_{C_2} - V_{C_1} - 2 C_1 R \frac{d}{dt} V_{C_1} = 0$$

$$\frac{d}{dt} V_{C_1} = \frac{-V_{C_1} + V_{C_2}}{2RC},$$

and substituting $i_1 = \frac{V_{C_2} - V_{C_1}}{2R}$ from Equation 18 in 17, we get

$$C_2 \frac{d}{dt} V_{C_2} + \frac{V_{C_2}}{R} + \frac{V_{C_2} - V_{C_1}}{2R} = 0$$

$$\frac{d}{dt} V_{C_2} + \frac{3 V_{C_2} - V_{C_1}}{2RC} = 0$$

$$\frac{d}{dt} V_{C_2} = \frac{V_{C_1} - 3 V_{C_2}}{2RC}$$

Combining the relations above, we can write

$$\frac{d}{dt} \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2RC} & \frac{1}{2RC} \\ \frac{1}{2RC} & -\frac{3}{2RC} \end{bmatrix} \cdot \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} \tag{19}$$

Plugging in $RC = 1$, this simplifies to

$$\frac{d}{dt} \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -1.5 \end{bmatrix} \cdot \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} \tag{20}$$

The characteristic polynomial for A is

$$\begin{aligned}\det\left(\begin{bmatrix}\lambda+0.5 & -0.5\\ -0.5 & \lambda+1.5\end{bmatrix}\right) &= 0\\ (\lambda+0.5)\cdot(\lambda+1.5)-0.25 &= 0\\ \lambda^2+2\lambda+0.75-0.25 &= 0\\ \lambda^2+2\lambda+0.5 &= 0\\ \lambda &= \frac{-2\pm\sqrt{2}}{2}\end{aligned}$$

Eigenvalues for the matrix A are $\lambda_1=\frac{-2+\sqrt{2}}{2}$, and $\lambda_2=\frac{-2-\sqrt{2}}{2}$.

(e) Immediately after the switch is opened, what are the voltage derivatives for the two capacitors, $\frac{\mathrm{d}V_{C_1}}{\mathrm{d}t}(0)$ and $\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t}(0)$?

Hint: We have already solved for $V_{C_1}(0)$ and $V_{C_2}(0)$ in part (a) and the system of differential equations describing V_{C_1} and V_{C_2} in part (d) .

Solution: From part (a), at time $t=0$, we have

$$\begin{aligned}V_{C_1}(0) &= V_s\,,\\ V_{C_2}(0) &= \frac{V_s}{2}.\end{aligned}$$

Since the system evolves according to the following system of equations as we solved in part (d),

$$\frac{d}{dt}\begin{bmatrix}V_{C_1}\\ V_{C_2}\end{bmatrix}=\begin{bmatrix}-\frac{1}{2RC} & \frac{1}{2RC}\\ \frac{1}{2RC} & -\frac{3}{2RC}\end{bmatrix}\cdot\begin{bmatrix}V_{C_1}\\ V_{C_2}\end{bmatrix} \tag{21}$$

we can plug in the results $V_{C_1}(0)=V_s$ and $V_{C_2}(0)=\frac{V_s}{2}$ into equations 21 and get

$$\begin{aligned}\begin{bmatrix}\frac{d}{dt}V_{C_1}(0)\\ \frac{d}{dt}V_{C_2}(0)\end{bmatrix} &= \begin{bmatrix}-\frac{1}{2RC} & \frac{1}{2RC}\\ \frac{1}{2RC} & -\frac{3}{2RC}\end{bmatrix}\cdot\begin{bmatrix}V_{C_1}(0)\\ V_{C_2}(0)\end{bmatrix}\\ \frac{d}{dt}V_{C_1}(0) &= -\frac{1}{2RC}V_{C_1}(0)+\frac{1}{2RC}V_{C_2}(0)\\ &= -\frac{1}{2RC}V_s+\frac{1}{2RC}\frac{V_s}{2}\\ &= -\frac{V_s}{4RC}\\ \frac{d}{dt}V_{C_1}(0) &= \frac{1}{2RC}V_{C_1}(0)-\frac{3}{2RC}V_{C_2}(0)\\ &= \frac{1}{2RC}V_s-\frac{3}{2RC}\frac{V_s}{2}\\ &= -\frac{V_s}{4RC}\end{aligned}$$

7. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

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