
EECS 16B Designing Information Devices and Systems II
Spring 2021 UC Berkeley

Homework 9

This homework is due on Sunday, March 21, 2021, at 11:00PM. Self-grades and HW Resubmission are due on Tuesday, March 30, 2021, at 11:00PM.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: [Note 8](#) and [Note 9](#)

- (a) What is the matrix test for controllability of a general linear discrete time system $\vec{x}[t + 1] = A\vec{x}[t] + \vec{b}u[t]$?

2. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}[t + 1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t].$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[t + 1] = A\vec{x}[t] + Bu[t]$.

(a) **Is this system controllable?**

(b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

(c) Suppose we use state feedback of the form $u[t] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[t]$

Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

(d) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix A and B from (a) for the system

$$\vec{x}[t + 1] = A\vec{x}[t] + Bu[t] + w[t]$$

into the Jupyter notebook “**eigenvalue_placement.ipynb**” and use the random input $w[t]$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}[t]$ evolves over time for the given A. **What do you see happening to the norm of the state?**

(e) Add the feedback computed in part (c) to the system in the notebook and **explain how the norm of the state changes.**

(f) **[OPTIONAL]** Now suppose we’ve got a different system described by the controlled scalar difference equation $z[t + 1] = z[t] + 2z[t - 1] + u[t]$. To convert this second-order difference equation to a two-dimensional discrete time system, we will let $\vec{y}[t] = \begin{bmatrix} z[t - 1] \\ z[t] \end{bmatrix}$ **Write down the system representation for \vec{y} in the following matrix form:**

$$\vec{y}(t + 1) = A_y \vec{y}(t) + B_y u(t). \quad (1)$$

Please specify what the matrix A_y and the vector B_y .

(g) **[OPTIONAL]** We will now show how the initial system for $\vec{x}[t]$ can be converted to the system for $\vec{y}[t]$ using a change of basis. Suppose we change coordinates with the transformation $\vec{y}[t] = P\vec{x}[t]$. **Write down the state-transition matrices of $\vec{y}[t]$ in terms of the state transition matrices of $\vec{x}[t]$, i.e., express A_y and B_y in terms of A, B , and P .** Additionally, confirm that for $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, the resulting state space representation of $\vec{y}[t]$ is the same as in the previous part (i.e. we get the same A_y, B_y).

(h) **[OPTIONAL]** For the \vec{y} system from part (f), **design a feedback gain matrix $\begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix}$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Confirm that $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P$.**

3. Tracking a Desired Trajectory in Continuous Time

The treatment in 16B so far has treated closed-loop control as being about holding a system steady at some desired operating point, by placing the eigenvalues of the state transition matrix. This control used the actual current state to apply a control signal designed to bring the eigenvalues in the region of stability. Meanwhile, the idea of controllability itself was more general and allowed us to make an open-loop trajectory that went pretty much anywhere. This problem is about combining these two ideas together to make feedback control more practical — how we can get a system to more-or-less closely follow a desired trajectory, even though it might not start exactly where we wanted to start and in principle could be affected by small disturbances throughout.

In this question, we will also see that everything that you have learned to do closed-loop control in discrete-time can also be used to do closed-loop control in continuous time.

Consider the specific 2-dimensional system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + \vec{w}(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + \vec{w}(t) \quad (2)$$

where $u(t)$ is a scalar valued continuous control input and $\vec{w}(t)$ is a bounded disturbance (noise).

- (a) **Would the given system be controllable if we viewed the parameters A and \vec{b} as the parameters of a discrete-time system, i.e. $\vec{x}_d[t+1] = A\vec{x}_d[t] + \vec{b}u_d[t]$?**
- (b) In an ideal noiseless scenario, the desired control signal $u^*(t)$ makes the system follow the desired trajectory $\vec{x}^*(t)$ that satisfies the following dynamics:

$$\frac{d}{dt}\vec{x}^*(t) = A\vec{x}^*(t) + \vec{b}u^*(t) \quad (3)$$

The presence of the bounded noise term $\vec{w}(t)$ makes the actual state $\vec{x}(t)$ deviate from the desired $\vec{x}^*(t)$ and follow (2) instead. In the following subparts, we will analyze how we can adjust the desired control signal $u^*(t)$ in (3) to the control input $u(t)$ in (2) so that the deviation in the state caused by $\vec{w}(t)$ remains bounded.

Represent the state as $\vec{x}(t) = \vec{x}^*(t) + \vec{v}(t)$ and $u(t) = u^*(t) + u_v(t)$. Using (2) and (3), we can represent the evolution of the trajectory deviation $\vec{v}(t)$ as a function of the control deviation $u_v(t)$ and the bounded disturbance $\vec{w}(t)$ as:

$$\frac{d}{dt}\vec{v}(t) = A_v\vec{v}(t) + \vec{b}_v u_v(t) + \vec{w}(t) \quad (4)$$

What are A_v and \vec{b}_v in terms of the original system parameters A and \vec{b} ? (HINT: Write out equation (2) in terms of $\vec{x}^*(t)$, $\vec{v}(t)$, $u^*(t)$ and $u_v(t)$.)

- (c) **Are the dynamics that you found for $\vec{v}(t)$ in part (b) stable? Based on this, in the presence of bounded disturbance $\vec{w}(t)$, will $\vec{x}(t)$ in (2) follow the desired trajectory $\vec{x}^*(t)$ closely if we just apply the control $u(t) = u^*(t)$ to the original system in (2), i.e. $u_v(t) = 0$?**
(HINT: Use the numerical values of A and \vec{b} from (2) in the solution from part (b) to determine stability of $\vec{v}(t)$.)

Now, we want to apply state feedback control to the system using $u_v(t)$ to get it to more or less follow the desired trajectory $\vec{x}^*(t)$.

- (d) **[OPTIONAL]** For the $\vec{v}(t), u_v(t)$ system, **apply feedback control by choosing $u_v(t)$ as a function of $\vec{v}(t)$ that would place both the eigenvalues of the closed-loop $\vec{v}(t)$ system at -10 .** (*HINT: $u_v(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \vec{v}(t)$. Find f_0 and f_1 .*)
- (e) **[OPTIONAL]** Based on what you did in the previous parts, and given access to the desired trajectory $\vec{x}^*(t)$, the desired control $u^*(t)$, and the actual measurement of the state $\vec{x}(t)$, **come up with a way to do feedback control that will keep the trajectory staying close to the desired trajectory no matter what the small bounded disturbance $\vec{w}(t)$ does.** (*HINT: Express the control input $u(t)$ in terms of $u^*(t)$, $\vec{x}^*(t)$, and $\vec{x}(t)$.*)

4. [OPTIONAL] Group Re-assignment Survey

How are your study groups working out? We hope they have been helpful so far. If you feel things are not going as well as you hoped and you would prefer to be assigned to a new group, please fill out the following form:

[Group Re-assignment Survey - Google Form](#)

5. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**

List names and student ID's. (In case of homework party, you can also just describe the group.)

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