EECS 16B Spring 2021 — Designing Information Devices and Systems II, UC Berkeley

Homework 8

This homework is due on Friday, March 12, 2021, at 11:00PM. Self-grades and HW Resubmission are due on Tuesday, March 16, 2021, at 11:00PM. Note that to save time for midterm prep, the solutions will be released early on Monday, but we still will require a HW submission on Gradescope.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 7B, Note 8

   (a) For the overdetermined system $A\vec{x} = \vec{b}$, what condition is needed to use least squares to estimate $\hat{x}$?

   (b) What are the eigenvalue tests for stability for both discrete-time systems and continuous-time systems?

   (c) How do you use feedback control to change the eigenvalues of a closed-loop continuous-time system?

2. System Identification

You are given a discrete-time system as a black-box. You don't know the specifics of the system but you know that it takes one scalar input and has two states that you can observe. You assume that the system is linear and of the form

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] + \vec{w}[t], \tag{1}$$

where $\vec{w}[t]$ is an external unseen disturbance that you hope is small, $u[t]$ is a scalar input, and

$$A = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \quad x[t] = \begin{bmatrix} x_0[t] \\ x_1[t] \end{bmatrix}. \tag{2}$$

You want to identify the system parameters from measured data. You need to find the unknowns: a_0, a_1, a_2, a_3, b_0 and b_1. However, you can only interact with the system via a blackbox model, i.e. you can see the states $\vec{x}[t]$ and set the inputs $u[t]$ that allow the system to move to the next state.

(a) You observe that the system has state $\vec{x}[t] = [x_0[t], x_1[t]]^T$ at time t. You pass input $u[t]$ into the blackbox and observe the next state of the system: $\vec{x}[t+1] = [x_0[t+1], x_1[t+1]]^T$.

Write scalar equations for the new states, $x_0[t+1]$ **and** $x_1[t+1]$. Write these equations in terms of the a_i, b_i, the states $x_0[t]$, $x_1[t]$ and the input $u[t]$. Here, assume that $\vec{w}[t] = \vec{0}$ (i.e. the model is perfect).

(b) Now we want to identify the system parameters. We observe the system at the start state $\vec{x}[0] = \begin{bmatrix} x_0[0] \\ x_1[0] \end{bmatrix}$.

We can then input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_0[1] \\ x_1[1] \end{bmatrix}$. We can continue this for an m long sequence of inputs.

Let us define an m long trajectory to be $[x_0[0], x_1[0], u[0], x_0[1], x_1[1], u[1], x_0[2], x_1[2], u[2], \ldots, x_0[m-1], x_1[m-1], u[m-1], x_0[m], x_1[m]]$. **Assuming there is no noise ($\vec{w}[t] = \vec{0}$), what is the minimum value of m you need to identify the system parameters?**

(c) Now assume that there is a nonzero noise/disturbance $\vec{w}[t]$. **Would using more than equations than in part (b) help you in this case? If so, explain why.**

(d) Say we feed in a total of 4 inputs $[u[0], u[1], u[2], u[3]]$ into our blackbox. This allows us to observe the following states $[x_0[0], x_0[1], x_0[2], x_0[3], x_0[4]]$ and $[x_1[0], x_1[1], x_1[2], x_1[3], x_1[4]]$, which we can use to identify the system.

To identify the system we need to set up an approximate (because of potential disturbances) matrix equation

$$D\vec{p} \approx \vec{y}$$

using the observed values above and the unknown parameters we want to find. We know our parameter vector should be $\vec{p} = \begin{bmatrix} a_0 & a_1 & b_0 & a_2 & a_3 & b_1 \end{bmatrix}^T$. **Find the corresponding D and $\vec{y}$ to do system indentification. Write out both explicitly.**

(e) Now that we have set up $D\vec{p} \approx \vec{y}$, **explain how you would use this approximate equation to estimate the unknown values $a_0, a_1, a_2, a_3, b_0, b_1$.** In particular, give an expression for your estimate $\widehat{\vec{p}}$ in terms of the D and $\vec{y}$. Assume that the columns of D are linearly independent.

(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.)

3. Identifying systems from their responses to known inputs

In many problems, we have an unknown system, and would like to characterize it. One of the ways of doing so is to observe the system response with different initial conditions (or inputs). This problem is also called system identification. It is a prototypical example of a problem that today is called machine learning — inferring an underlying pattern from data, and doing so well enough to be able to exploit that pattern in some practical setting. Go through the attached Jupyter notebook "DemoSystemID_16b.ipynb" and answer the following questions.

(a) In Example 2, we assume that instead of measuring the state $\vec{x}$, we are instead measuring a transformation of the state $\vec{y} = T\vec{x}$ where T is a full rank matrix. Assume that we perform system ID on our observations $\vec{y}[t]$ to recover A_y, B_y such that $\vec{y}[t+1] = A_y\vec{y}[t] + B_y u[t]$. **How do the identified A_y and B_y matrices relate to the original A and B matrices in the dynamics of $\vec{x}$?** Remember that our original state dynamics are $\vec{x}[t+1] = A\vec{x} + B\vec{u}$.

HINT: The answer is given in the Jupyter notebook but remember to show your work.

(b) **Please share your observations on Example 2.**

(c) **Prove that for any full rank transformation matrix T, the eigenvalues of A_y and A from part (a) are the same.**

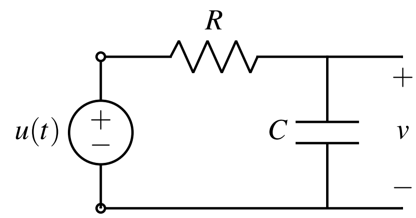
(d) **Please share your observations on Example 3.**

(e) **Please share your observations on Example 4.**

(f) **Please share your observations on Example 5.**

4. BIBO Stability

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5$F. Furthermore assume that $v(0) = 0$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{d}{dt}v(t) = -2v(t) + 2u(t) \tag{3}$$

Show that $v(t)$ remains bounded for all time if the input $u(t)$ is bounded, i.e. $|u(t)| < k, \forall t \geq 0$. *HINT: You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then use the triangle inequality to prove that $x(t)$ is bounded. Also use the fact that the norm of an integral is $\geq$ the integral of a norm. In other words $|\int_a^b f(\tau)d\tau| \leq \int_a^b |f(\tau)|d\tau$. This inequality is just the regular triangle inequality you have seen before, generalized to integrals. The proof strategy used in lecture for the discrete time case will extend here as well with the modified triangle inequality usage.*

Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

(b) Consider a continuous-time scalar real differential equation with known solution

$$\frac{d}{dt}x(t) = ax(t) + bu(t) \qquad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau.$$

Show that if the system has $\text{Re}\{a\} > 0$, then there exists a bounded input ($|u(t)| \leq \varepsilon$) that can result in an unbounded output, even with a zero initial condition. In other words, show the system is BIBO unstable.

(c) **Repeat the previous part for the specific case of complex $a = r + j2\pi$ where $r > 0$ and zero initial condition $x(0) = 0$.**

All the other truly complex unstable cases are the same way for the same essential reason.

(d) **[Optional] Repeat the previous part for the specific case of purely imaginary $a = j2\pi$ with zero initial condition $x(0) = 0$.**

All the other purely imaginary unstable cases are the same way for the same essential reason.

(e) Consider the discrete-time system

$$x[t+1] = -2x[t] + 2u[t] \tag{4}$$

with $x[0] = 0$.

Is this system stable or unstable? If stable, prove it. If unstable, find a bounded input sequence $u[t]$ that causes the system to 'blow up'.

(f) For the example in the previous part, **give an explicit sequence of inputs that are not zero but for which the state $x[t]$ will always stay bounded.** *(HINT: building off of the previous part see if you can find any input pattern that result in an oscillatory behavior.)*

(g) Consider the discrete-time real system with known solution:

$$x[t+1] = ax[t] + bu[t] \qquad x[t] = a^t x[0] + \sum_{\ell=0}^{t-1} a^{t-1-\ell} bu[\ell]$$

Show that if the system is unstable (has $|a| > 1$), then a bounded input can result in an unbounded output. Assume a zero initial condition here.

(h) **[Optional] Repeat the previous part for the specific case of $a = -1$.**

(i) [Optional: this part was derived in lecture] Now consider the discrete-time stable case where a is complex and has $|a| < 1$. **Show that as long as $|u[t]| < k$ for some k, that the solution $x[t]$ will be bounded for all time t.**

(HINT: There are a few helpful facts about absolute values and inequalities that are helpful in such proofs. First: $|\sum_j a_j| \leq \sum_j |a_j|$. Second $|ab| = |a| \cdot |b|$. Third: $|e^{j\theta}| = 1$ no matter what real number θ is. And fourth, if $a_i > 0$ and $b_i > 0$, and $b_i \leq B$, then $\sum_i a_i b_i \leq \sum_i a_i B = B \sum a_i$.)

5. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

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