1. SVD of a Matrix with Orthogonal Columns

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ where $\vec{a}_i^\top \vec{a}_j = 0$ for all $1 \leq i, j \leq n$ such that $i \neq j$, and $\vec{a}_i^\top \vec{a}_i \neq 0$ for all $i = 1, \ldots, n$. What is the set of singular values for any such matrix $A$?

(Please fill in one of the circles for the options below.)

(a) $\{0\}$ (all zero)
(b) $\{\sqrt{\|\vec{a}_1\|}, \ldots, \sqrt{\|\vec{a}_n\|}\}$
(c) $\{|\vec{a}_1|, \ldots, |\vec{a}_n|\}$
(d) $\{|\vec{a}_1|^2, \ldots, |\vec{a}_n|^2\}$
(e) $\{1\}$ (all one)

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2. SVD Computation

(a) Consider the matrix
\[
A = \begin{bmatrix}
-1 & 1 & 5 \\
3 & 1 & -1 \\
2 & -1 & 4
\end{bmatrix}.
\]
Observe that the columns of matrix \(A\) are mutually orthogonal with norms \(\sqrt{14}, \sqrt{3}, \sqrt{42}\).

Verify numerically that columns
\[
\begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
5 \\
-1 \\
4
\end{bmatrix}
\]
are orthogonal to each other.

(b) Write \(A = BD\), where \(B\) is an orthonormal matrix and \(D\) is a diagonal matrix. What is \(B\)? What is \(D\)?

(c) Write out a valid singular value decomposition of \(A = U\Sigma V^\top\) using the result from the previous part. Note that the singular values in \(\Sigma\) should be ordered from largest to smallest.

\(HINT: \) There is no need to compute any eigenvalues.

(d) Given a new matrix
\[
A = \frac{1}{\sqrt{50}} \begin{bmatrix}
3 \\
4
\end{bmatrix} \begin{bmatrix}
1 & -1
\end{bmatrix} + \frac{3}{\sqrt{50}} \begin{bmatrix}
-4 \\
3
\end{bmatrix} \begin{bmatrix}
1 & 1
\end{bmatrix},
\]
write out a singular value decomposition of matrix \(A\) in the form \(U\Sigma V^\top\). Remember that the singular values in \(\Sigma\) should be ordered from the largest to smallest.

\(HINT: \) You don’t have to compute any eigenvalues for this. Some useful observations are that
\[
\begin{bmatrix}
3 & 4 \\
3 & -4
\end{bmatrix} = 0, \quad \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} = 0,
\|
\begin{bmatrix}
3 \\
4
\end{bmatrix}
\|
= \|
\begin{bmatrix}
-4 \\
3
\end{bmatrix}
\|
= 5,
\|
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\|
= \|
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\|
= \sqrt{2}.
\]

(e) Let us define a new matrix
\[
A = \begin{bmatrix}
-1 & 4 \\
1 & 4
\end{bmatrix}.
\]

Find the SVD of \(A\) by following the standard algorithm introduced in the notes (i.e. by computing the eigendecomposition of \(A^\top A\)). Also find the eigenvectors and eigenvalues of \(A\). Is there a relationship between the eigenvalues or eigenvectors of \(A\) with the SVD of \(A\)?
3. SVD and the Fundamental Subspaces

Consider a matrix \( A \in \mathbb{R}^{m \times n} \) with \( \text{rank}(A) = r \). The compact SVD of \( A \) is given by \( A = U_r \Sigma_r V_r^\top \)

where

\[
U_r = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_r \end{bmatrix} \in \mathbb{R}^{m \times r}, \quad \Sigma_r = \begin{bmatrix} \sigma_1 & \cdots & \sigma_r \end{bmatrix} \in \mathbb{R}^{r \times r}, \quad V_r = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_r \end{bmatrix} \in \mathbb{R}^{n \times r}
\]

with \( \sigma_1 \geq \cdots \geq \sigma_r > 0 \) being the singular values of \( A \).

(a) Which one of the following sets is always guaranteed to form an orthonormal basis for \( \text{Col}(A) \)?

(Please fill in one of the circles for the options below.)

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(b) Which one of the following sets is always guaranteed to form an orthonormal basis for \( \text{Col}(A^\top) \)?

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Now suppose that the considered \( A \) matrix has the following compact SVD components:

\[
U_r = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma_r = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

(c) Using the given compact SVD, state \( \alpha \), where \( \alpha \) is the tightest upper bound \( \|A\vec{x}\| \leq \alpha \) for any \( \vec{x} \) such that \( \|\vec{x}\| \leq 1 \).

(d) Given the compact SVD, which of the following provides a valid full SVD for \( A = U \Sigma V^\top \)?

(Please fill in one of the circles for the options below.)

i. \[
U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

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ii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

iv. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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4. Frobenius Norm

In this problem we will investigate the basic properties of the Frobenius norm.

Similar to how the norm of vector $\mathbf{x} \in \mathbb{R}^n$ is defined as $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$, the Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2}. \quad (2)$$

$A_{ij}$ is the entry in the $i$th row and the $j$th column. This is basically the norm that comes from treating a matrix like a big vector filled with numbers.

(a) With the above definitions, **show that for a $2 \times 2$ matrix $A$:***

$$\|A\|_F = \sqrt{\text{tr}(A^\top A)}. \quad (3)$$

**Note:** The trace of a matrix is the sum of its diagonal entries. For example, for $A \in \mathbb{R}^{m \times n}$,

$$\text{tr}(A) = \sum_{i=1}^{\min(m,n)} A_{ii}. \quad (4)$$

Think about how/whether this expression eq. (3) generalizes to general $m \times n$ matrices.

(b) **Show that for any matrix $A \in \mathbb{R}^{m \times n}$:**

$$\|A\|_F = \left\| A^\top \right\|_F \quad (5)$$

**HINT:** The definition from eq. (2) can help interpret this mathematically.

(c) **Show that if $U$ and $V$ are square orthonormal matrices, then**

$$\|UA\|_F = \|AV\|_F = \|A\|_F. \quad (6)$$

**HINT:** Use the trace interpretation from part (a) and the equation from part (b).

(d) **Use the SVD decomposition to show that $\|A\|_F = \sqrt{\sum_{i=1}^n \sigma_i^2}$, where $\sigma_1, \ldots, \sigma_n$ are the singular values of $A$.**

**HINT:** The previous part might be quite useful.

(e) **(OPTIONAL) Show that for any matrix $A \in \mathbb{R}^{m \times n}$ and any vector $\mathbf{x} \in \mathbb{R}^n$:***

$$\|A\mathbf{x}\|^2 \leq \|A\|_F^2 \|\mathbf{x}\|^2 \quad (7)$$

**HINT:** Use the summation form of matrix multiplication to find an expression for each element of $A\mathbf{x}$ and use this to find the expression for $\|A\mathbf{x}\|^2$. Then, use the fact that $|\sum a_i b_i|^2 \leq \left(\sum |a_i|^2\right) \left(\sum |b_i|^2\right)$ (called the Cauchy-Schwarz inequality).
• Tanmay Gautam.
• Anirudh Rengarajan.
• Aditya Arun.
• Kourosh Hakhamaneshi.
• Antroy Roy Chowdhury.
• Nikhil Jain.
• Chancharik Mitra.