Homework 11

This homework is due on Saturday, April 13, 2023 at 11:59PM.

1. SVD of a Matrix with Orthogonal Columns

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ where $\vec{a}_i^\top \vec{a}_j = 0$ for all $1 \le i, j \le n$ such that $i \ne j$, and $\vec{a}_i^\top \vec{a}_i \ne 0$ for all $i = 1, \ldots, n$. **What is the set of singular values for *any* such matrix A?**

(Please fill in one of the circles for the options below.)

(a) $\{0\}$ (all zero)

(b) $\{\sqrt{\|\vec{a}_1\|}, \ldots, \sqrt{\|\vec{a}_n\|}\}$

(c) $\{\|\vec{a}_1\|, \ldots, \|\vec{a}_n\|\}$

(d) $\{\|\vec{a}_1\|^2, \ldots, \|\vec{a}_n\|^2\}$

(e) $\{1\}$ (all one)

Option	a	b	c	d	e
Answer	◯	◯	◯	◯	◯

2. SVD Computation

(a) Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 5 \\ 3 & 1 & -1 \\ 2 & -1 & 4 \end{bmatrix}.$$

Observe that the columns of matrix A are mutually orthogonal with norms $\sqrt{14}, \sqrt{3}, \sqrt{42}$.

Verify numerically that columns $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **and** $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ **are orthogonal to each other.**

(b) **Write $A = BD$, where B is an orthonormal matrix and D is a diagonal matrix.** What is B? What is D?

(c) **Write out a valid singular value decomposition of $A = U\Sigma V^\top$ using the result from the previous part.** Note that the singular values in Σ should be ordered from largest to smallest.
(HINT: There is no need to compute any eigenvalues.)

(d) Given a new matrix

$$A = \frac{1}{\sqrt{50}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{3}{\sqrt{50}} \begin{bmatrix} -4 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{1}$$

write out a singular value decomposition of matrix A in the form $U\Sigma V^\top$. Remember that the singular values in Σ should be ordered from the largest to smallest.
(HINT: You don't have to compute any eigenvalues for this. Some useful observations are that

$$\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0, \quad \left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\| = 5, \quad \left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \sqrt{2}.$$

)

(e) Let us define a new matrix

$$A = \begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix}.$$

Find the SVD of A by following the standard algorithm introduced in the notes (i.e. by computing the eigendecomposition of $A^\top A$). Also find the eigenvectors and eigenvalues of A. Is there a relationship between the eigenvalues or eigenvectors of A with the SVD of A?

3. SVD and the Fundamental Subspaces

Consider a matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$. The compact SVD of A is given by $A = U_r \Sigma_r V_r^\top$ where

$$U_r = \begin{bmatrix} \vec{u}_1 \cdots \vec{u}_r \end{bmatrix} \in \mathbb{R}^{m \times r}, \quad \Sigma_r = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \in \mathbb{R}^{r \times r}, \quad V_r = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_r \end{bmatrix} \in \mathbb{R}^{n \times r}$$

with $\sigma_1 \geq \cdots \geq \sigma_r > 0$ being the singular values of A.

(a) **Which one of the following sets is always guaranteed to form an *orthonormal* basis for** $\text{Col}(A)$**?**
(Please fill in one of the circles for the options below.)

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$
ii. $\{\sigma_1 \vec{u}_1, \dots, \sigma_r \vec{u}_r\}$
iii. $\{\vec{v}_1, \dots, \vec{v}_r\}$
iv. $\{\sigma_1 \vec{v}_1, \dots, \sigma_r \vec{v}_r\}$

Option	i	ii	iii	iv
Answer	◯	◯	◯	◯

(b) **Which one of the following sets is always guaranteed to form an *orthonormal* basis for** $\text{Col}(A^\top)$**?**
(Please fill in one of the circles for the options below.)

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$
ii. $\{\sigma_1 \vec{u}_1, \dots, \sigma_r \vec{u}_r\}$
iii. $\{\vec{v}_1, \dots, \vec{v}_r\}$
iv. $\{\sigma_1 \vec{v}_1, \dots, \sigma_r \vec{v}_r\}$

Option	i	ii	iii	iv
Answer	◯	◯	◯	◯

Now suppose that the considered A matrix has the following compact SVD components:

$$U_r = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma_r = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(c) **Using the given compact SVD, state α, where α is the tightest upper bound $\|A\vec{x}\| \leq \alpha$ for any $\vec{x}$ such that $\|\vec{x}\| \leq 1$.**

(d) **Given the compact SVD, which of the following provides a valid full SVD for $A = U \Sigma V^\top$?**
(Please fill in one of the circles for the options below.)

i. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

iv. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Option	i	ii	iii	iv
Answer	◯	◯	◯	◯

4. Frobenius Norm

In this problem we will investigate the basic properties of the Frobenius norm.

Similar to how the norm of vector $\vec{x} \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, the Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2}. \tag{2}$$

A_{ij} is the entry in the i^{th} row and the j^{th} column. This is basically the norm that comes from treating a matrix like a big vector filled with numbers.

(a) With the above definitions, **show that for a 2×2 matrix A:**

$$\|A\|_F = \sqrt{\operatorname{tr}(A^\top A)}. \tag{3}$$

Note: The trace of a matrix is the sum of its diagonal entries. For example, for $A \in \mathbb{R}^{m\times n}$,

$$\operatorname{tr}(A) = \sum_{i=1}^{\min(n,m)} A_{ii} \tag{4}$$

Think about how/whether this expression eq. (3) generalizes to general $m \times n$ matrices.

(b) **Show that for any matrix $A \in \mathbb{R}^{m\times n}$:**

$$\|A\|_F = \left\|A^\top\right\|_F \tag{5}$$

(HINT: The definition from eq. (2) can help interpret this mathematically.)

(c) **Show that if U and V are square orthonormal matrices, then**

$$\|UA\|_F = \|AV\|_F = \|A\|_F\,. \tag{6}$$

(HINT: Use the trace interpretation from part (a) and the equation from part (b).)

(d) **Use the SVD decomposition to show that $\|A\|_F = \sqrt{\sum_{i=1}^n \sigma_i^2}$, where $\sigma_1, \ldots, \sigma_n$ are the singular values of A.**

(HINT: The previous part might be quite useful.)

(e) **(OPTIONAL) Show that for any matrix $A \in \mathbb{R}^{m\times n}$ and any vector $\vec{x} \in \mathbb{R}^n$:**

$$\|A\vec{x}\|^2 \leq \|A\|_F^2 \|\vec{x}\|^2 \tag{7}$$

(HINT: Use the summation form of matrix multiplication to find an expression for each element of $A\vec{x}$ and use this to find the expression for $\|A\vec{x}\|^2$. Then, use the fact that $|\sum ab|^2 \leq \left(\sum |a|^2\right)\left(|\sum |b|^2\right)$ (called the Cauchy-Schwarz inequality).)

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