Homework 7

This homework is due on Friday, October 14, 2022, at 11:59PM. Self-grades and HW re-submissions are due on the following Friday, October 21, 2022, at 11:59PM.

1. **Existence and uniqueness of solutions to differential equations**

When doing circuits or systems analysis, we sometimes model our system via a differential equation, and would often like to solve it to get the system trajectory. To this end, we would like to verify that a solution to our differential equation exists and is unique, so that our model is physically meaningful. There is a general approach to doing this, which is demonstrated in this problem.

We would like to show that there is a unique function $x\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{1}$$

$$x(0) = x_0. \tag{2}$$

In order to do this, we will first verify that a solution x_d exists. To show that x_d is the unique solution, we will take an arbitrary solution y and show that $x_d(t) = y(t)$ for every t.

(a) First, let us show that a solution to our differential equation exists. **Verify that $x_d(t) := x_0 e^{\alpha t}$ satisfies eq. (1) and eq. (2).**

Solution: We first verify eq. (1).

$$\frac{\mathrm{d}}{\mathrm{d}t}x_d(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_0 e^{\alpha t}\right) \tag{3}$$

$$= x_0 \frac{\mathrm{d}}{\mathrm{d}t} e^{\alpha t} \tag{4}$$

$$= x_0 \cdot \alpha e^{\alpha t} \tag{5}$$

$$= \alpha \cdot x_0 e^{\alpha t} \tag{6}$$

$$= \alpha x_d(t). \tag{7}$$

Now we verify eq. (2).

$$x_d(0) = x_0 e^{\alpha \cdot 0} \tag{8}$$

$$= x_0 e^0 \tag{9}$$

$$= x_0. \tag{10}$$

(b) Now, let us show that our solution is unique. As mentioned before, suppose $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eq. (1) and eq. (2).

We want to show that $y(t) = x_d(t)$ for all t. Our strategy is to show that $\frac{y(t)}{x_d(t)} = 1$ for all t.

However, this particular differential equation poses a problem: if $x_0 = 0$, then $x_d(t) = 0$ for all t, so that the quotient is not well-defined. To patch this method, we would like to avoid using any function with x_0 in the denominator. One way we can do this is consider a modification of the quotient $\frac{y(t)}{x_d(t)} = \frac{y(t)}{x_0 e^{\alpha t}}$; in particular, we consider the function $z(t) := \frac{y(t)}{e^{\alpha t}}$.

Show that $z(t) = x_0$ for all t, and explain why this means that $y(t) = x_d(t)$ for all t.

(HINT: Show first that $z(0) = x_0$ and then that $\frac{d}{dt}z(t) = 0$. Argue that these two facts imply that $z(t) = x_0$ for all t. Then show that this implies $y(t) = x_d(t)$ for all t.)

(HINT: Remember that we said y is any solution to eq. (1) and eq. (2), so we only know these properties of y. If you need something about y to be true, see if you can show it from eq. (1) and eq. (2).)

(HINT: When taking $\frac{d}{dt}z(t)$, remember to use the quotient rule, along with what we know about y.)

Solution: The solution goes in four stages, as per the hint.

Step 1. We show that $z(0) = x_0$. Indeed, using eq. (2),

$$z(0) = \frac{y(0)}{e^{\alpha \cdot 0}} = \frac{x_0}{e^0} = \frac{x_0}{1} = x_0. \tag{11}$$

Step 2. We show that $\frac{d}{dt}z(t) = 0$. Indeed, using the quotient rule from calculus and eq. (1),

$$\begin{aligned}
\frac{d}{dt}z(t) &= \frac{d}{dt}\frac{y(t)}{e^{\alpha t}} &(12)\\
&= \frac{e^{\alpha t}\left(\frac{d}{dt}y(t)\right) - y(t)\left(\frac{d}{dt}e^{\alpha t}\right)}{e^{2\alpha t}} &(13)\\
&= \frac{e^{\alpha t}(\alpha y(t)) - y(t)(\alpha e^{\alpha t})}{e^{2\alpha t}} &(14)\\
&= \frac{\alpha e^{\alpha t}y(t) - \alpha e^{\alpha t}y(t)}{e^{2\alpha t}} &(15)\\
&= \frac{0}{e^{2\alpha t}} &(16)\\
&= 0. &(17)
\end{aligned}$$

Step 3. We show that $z(t) = x_0$ for all t. Indeed, since $\frac{d}{dt}z(t) = 0$, we know that $z(t)$ is a constant. Since $z(0) = x_0$, this gives that $z(t)$ is the constant value x_0, and hence $z(t) = x_0$ for all t.

Step 4. We show that $y(t) = x_d(t)$ for all t. Indeed, since $z(t) = x_0$ and $z(t) = \frac{y(t)}{e^{\alpha t}}$, we have $x_0 = \frac{y(t)}{e^{\alpha t}}$. We multiply both sides by $e^{\alpha t}$ to get $y(t) = x_0 e^{\alpha t}$. But this is just $x_d(t)$, so $y(t) = x_d(t)$ for all t.

2. Simple Scalar Differential Equations Driven by an Input

In this question, we will show the existence and uniqueness of solutions to differential equations with inputs. In particular, we consider the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{18}$$

$$x(0) = x_0 \tag{19}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a known function of time. Feel free to assume u is "nice" in the sense that it is integrable, continuous, and differentiable with bounded derivative – basically, let u be nice enough that all the usual calculus theorems work.

(a) We will first demonstrate the existence of a solution to eqs. (18) and (19).

Define $x_d\colon \mathbb{R} \to \mathbb{R}$ by

$$x_d(t) := \mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau \tag{20}$$

Show that x_d satisfies eqs. (18) and (19).

(HINT: When showing that x_d satisfies eq. (18), one possible approach to calculate the derivative of the integral term is to use the fundamental theorem of calculus and the product rule.)

Solution: We first show that x_d satisfies eq. (18). Using the fundamental theorem of calculus and the product rule, we can calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}x_d(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\right) \tag{21}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}x_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau \tag{22}$$

$$= x_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) \tag{23}$$

$$= x_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) \tag{24}$$

$$= x_0\left(\lambda\mathrm{e}^{\lambda t}\right) + \left\{\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t}\right)\left(\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) + \left(\mathrm{e}^{\lambda t}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right)\right\} \tag{25}$$

$$= \lambda x_0\mathrm{e}^{\lambda t} + \lambda\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau + \mathrm{e}^{\lambda t}\left(\mathrm{e}^{-\lambda\tau}bu(\tau)\Big|_{\tau=t}\right) \tag{26}$$

$$= \lambda\left(x_0\mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) + \mathrm{e}^{\lambda t}\left(\mathrm{e}^{-\lambda t}bu(t)\right) \tag{27}$$

$$= \lambda x_d(t) + bu(t) \tag{28}$$

so x_d satisfies eq. (18).

Alternatively, we could have used the Leibniz rule (not in scope) to get the terms in the square brackets:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_d(t) = \lambda\mathrm{e}^{\lambda t}x_0 + \left[\mathrm{e}^{\lambda(t-t)}bu(t)\cdot 1 - \mathrm{e}^{\lambda(t-0)}\cdot 0 + \lambda\int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\right] \tag{29}$$

$$= \lambda\left[\mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\right] + bu(t) \tag{30}$$

$$= \lambda x_d(t) + bu(t) \tag{31}$$

which again shows that x_d satisfies eq. (18).

Now we show that x_d satisfies eq. (19). Indeed,

$$x_d(0) = \left(\mathrm{e}^{\lambda t} x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)} b u(\tau)\,\mathrm{d}\tau\right)\Bigg|_{t=0} = \underbrace{\mathrm{e}^{\lambda \cdot 0}}_{=1} x_0 + \underbrace{\int_0^0 \mathrm{e}^{\lambda(0-\tau)} b u(\tau)\,\mathrm{d}\tau}_{=0} = x_0. \tag{32}$$

Thus x_d satisfies eq. (19).

(b) Now, we will show that x_d is the unique solution to eqs. (18) and (19).

Suppose that $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eqs. (18) and (19). **Show that $y(t) = x_d(t)$ for all t.**

(HINT: This time, show that $z(t) := y(t) - x_d(t) = 0$ for all t. Do this by showing that $z(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \lambda z(t)$, then use the uniqueness theorem for homogeneous first-order linear differential equations from the last problem. Note that the specific form of $x_d(t)$ in eq. (20) is irrelevant for the solution and should not be used.)

Solution: Again, the solution is in some parts.

Step 1. We show that $z(0) = 0$. Indeed,

$$z(0) = y(0) - x_d(0) = x_0 - x_0 = 0. \tag{33}$$

Step 2. We show that $\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \lambda z(t)$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \frac{\mathrm{d}}{\mathrm{d}t}(y(t) - x_d(t)) \tag{34}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} y(t) - \frac{\mathrm{d}}{\mathrm{d}t} x_d(t) \tag{35}$$

$$= (\lambda y(t) + b u(t)) - (\lambda x_d(t) + b u(t)) \tag{36}$$

$$= \lambda y(t) - \lambda x_d(t) \tag{37}$$

$$= \lambda z(t). \tag{38}$$

Step 3. We show that $z(t) = 0$ for all t. Indeed, we know that $z(t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \lambda z(t) \tag{39}$$

$$z(0) = 0. \tag{40}$$

This is a first-order linear differential equation, so we know from the previous homework that its unique solution is

$$z(t) = z(0) \cdot \mathrm{e}^{\lambda t} = 0 \cdot \mathrm{e}^{\lambda t} = 0. \tag{41}$$

This is what was claimed, so we are done.

(c) Extend the solution in eq. (20) to the diagonal, vector differential equation case. **Namely, show that the solution for**

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = \Lambda \vec{x}(t) + \vec{b} u(t) \tag{42}$$

is given by

$$\vec{x}(t) = \mathrm{e}^{\Lambda t} \vec{x}(0) + \int_0^t \mathrm{e}^{\Lambda(t-\tau)} \vec{b} u(\tau)\,\mathrm{d}\tau \tag{43}$$

where $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^n$, $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix, $\vec{b} \in \mathbb{R}^n$, and $u(t) : \mathbb{R} \to \mathbb{R}$. For notational convenience, we define $e^{\Lambda x} = \begin{bmatrix} e^{\lambda_1 x} & & & \\ & e^{\lambda_2 x} & & \\ & & \ddots & \\ & & & e^{\lambda_n x} \end{bmatrix}$ where $\Lambda := \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$.

(HINT: You may use the fact that, if $\vec{z}(t) : \mathbb{R} \to \mathbb{R}^n$ and $\vec{z}(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$, it is the case that $\int_a^b \vec{z}(t)\,dt = \begin{bmatrix} \int_a^b z_1(t)\,dt \\ \int_a^b z_2(t)\,dt \\ \vdots \\ \int_a^b z_n(t)\,dt \end{bmatrix}$.)

(HINT: Consider breaking down the diagonal vector differential equation case into n different scalar cases. How can you combine the results from eq. (20) with this?)

Solution: The kth row of this differential equation will be

$$\frac{d}{dt}x_k(t) = \lambda_k x_k(t) + b_k u(t) \tag{44}$$

which has the solution

$$x_k(t) = e^{\lambda_k t}x_k(0) + \int_0^t e^{\lambda_k(t-\tau)} b_k u(\tau)\,d\tau \tag{45}$$

This yields the vector solution as

$$\vec{x}(t) = \begin{bmatrix} e^{\lambda_1 t}x_1(0) \\ e^{\lambda_2 t}x_2(0) \\ \vdots \\ e^{\lambda_n t}x_n(0) \end{bmatrix} + \begin{bmatrix} \int_0^t e^{\lambda_1(t-\tau)} b_1 u(\tau)\,d\tau \\ \int_0^t e^{\lambda_2(t-\tau)} b_2 u(\tau)\,d\tau \\ \vdots \\ \int_0^t e^{\lambda_n(t-\tau)} b_n u(\tau)\,d\tau \end{bmatrix} \tag{46}$$

We can simplify the first term to be

$$\begin{bmatrix} e^{\lambda_1 t}x_1(0) \\ e^{\lambda_2 t}x_2(0) \\ \vdots \\ e^{\lambda_n t}x_n(0) \end{bmatrix} = e^{\Lambda t}\vec{x}(0) \tag{47}$$

and, using the hint, we can simplify the second term as follows:

$$\begin{bmatrix} \int_0^t e^{\lambda_1(t-\tau)} b_1 u(\tau)\,d\tau \\ \int_0^t e^{\lambda_2(t-\tau)} b_2 u(\tau)\,d\tau \\ \vdots \\ \int_0^t e^{\lambda_n(t-\tau)} b_n u(\tau)\,d\tau \end{bmatrix} = \int_0^t \begin{bmatrix} e^{\lambda_1(t-\tau)} b_1 u(\tau) \\ e^{\lambda_2(t-\tau)} b_2 u(\tau) \\ \vdots \\ e^{\lambda_n(t-\tau)} b_n u(\tau) \end{bmatrix} d\tau \tag{48}$$

$$= \int_0^t \begin{bmatrix} e^{\lambda_1(t-\tau)} b_1 \\ e^{\lambda_2(t-\tau)} b_2 \\ \vdots \\ e^{\lambda_n(t-\tau)} b_n \end{bmatrix} u(\tau)\,d\tau \tag{49}$$

$$= \int_0^t e^{\Lambda(t-\tau)}\vec{b}u(\tau)\,d\tau \tag{50}$$

Altogether, this yields,

$$\vec{x}(t) = e^{\Lambda t}\vec{x}(0) + \int_0^t e^{\Lambda(t-\tau)}\vec{b}u(\tau)\,d\tau \tag{51}$$

(d) **Extend the result from the previous part for an arbitrary vector differenial equation given by**

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{52}$$

where A is a *diagonalizable* matrix (but not necessarily a *diagonal* matrix). You may assume that A can be diagonalized as $A = V\Lambda V^{-1}$. For notational convenience, you may want to define $\widetilde{\vec{b}} = V^{-1}\vec{b}$.

Solution: Define $\vec{x}_\lambda(t) = V^{-1}\vec{x}(t)$. We know that the differential equation governing $\vec{x}_\lambda(t)$ will be

$$\frac{d}{dt}\vec{x}_\lambda(t) = \Lambda\vec{x}_\lambda(t) + V^{-1}\vec{b}u(t) \tag{53}$$

$$= \Lambda\vec{x}_\lambda(t) + \widetilde{\vec{b}}u(t) \tag{54}$$

The solution to this, using the previous part, is

$$\vec{x}_\lambda(t) = e^{\Lambda t}\vec{x}_\lambda(0) + \int_0^t e^{\Lambda(t-\tau)}\widetilde{\vec{b}}u(\tau)\,d\tau \tag{55}$$

Hence, the solution for $\vec{x}(t)$ will be

$$\vec{x}(t) = Ve^{\Lambda t}\vec{x}_\lambda(0) + V\int_0^t e^{\Lambda(t-\tau)}\widetilde{\vec{b}}u(\tau)\,d\tau \tag{56}$$

$$= Ve^{\Lambda t}V^{-1}\vec{x}(0) + V\int_0^t e^{\Lambda(t-\tau)}\widetilde{\vec{b}}u(\tau)\,d\tau \tag{57}$$

where we substituted $\vec{x}_\lambda(0) = V^{-1}\vec{x}(0)$.

3. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

Solution:

$$AV = A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \tag{58}$$

$$= \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots, A\vec{v}_n \end{bmatrix} \tag{59}$$

$$= \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} \tag{60}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{61}$$

$$= V\Lambda \tag{62}$$

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{63}$$

(HINT: What condition on a matrix's columns means that it would be invertible? It is fine to cite the appropriate result from 16A.)

Solution: Columns of V are eigenvectors of A which are known to be linearly independent. Since V has linearly independent columns, it has full column rank, and therefore, is invertible.

$$AV = V\Lambda \tag{64}$$

$$AVV^{-1} = V\Lambda V^{-1} \tag{65}$$

$$A = V\Lambda V^{-1} \tag{66}$$

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

Solution: We take $A = V\Lambda V^{-1}$ and apply invertible operations to both sides of the equality:

$$A = V\Lambda V^{-1} \tag{67}$$

$$V^{-1}A = V^{-1}V\Lambda V^{-1} \tag{68}$$

$$V^{-1}AV = V^{-1}V\Lambda V^{-1}V \tag{69}$$

$$V^{-1}AV = I\Lambda I \tag{70}$$

$$V^{-1}AV = \Lambda. \tag{71}$$

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: Recall the definition of an eigenvector (i.e., $A\vec{v} = \lambda\vec{v}$). Then, recall what $U^{-1}U$ is. Lastly, consider how matrix multiplication works column-wise.)

Solution: We start with a calculation which is essentially the reverse of the calculation in part (b):

$$A = UDU^{-1} \tag{72}$$
$$AU = UDU^{-1}U \tag{73}$$
$$AU = UD. \tag{74}$$

Now let's expand the definitions of U as a square matrix and D as a diagonal matrix:

$$AU = A\begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix} \tag{75}$$
$$= \begin{bmatrix}A\vec{u}_1 & \dots & A\vec{u}_n\end{bmatrix} \tag{76}$$
$$UD = \begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix}\begin{bmatrix}d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix} \tag{77}$$
$$= \begin{bmatrix}d_1\vec{u}_1 & \dots & d_n\vec{u}_n\end{bmatrix}. \tag{78}$$

Comparing columns, we see that $A\vec{u}_i = d_i\vec{u}_i$. This is exactly the eigenvector-eigenvalue equation! In particular, this says that $\vec{u}_i$ is an eigenvector of A, with eigenvalue d_i.

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A \cdot A}_{k \text{ times}} \tag{79}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Conclude that A^k is diagonalizable.

Solution: Consider the i^{th} eigenvector of A, $\vec{v}_i$ and the corresponding eigenvalue λ_i.

$$A^k\vec{v}_i = A^{k-1} \cdot A\vec{v}_i \tag{80}$$
$$= A^{k-1}\lambda_i\vec{v}_i \tag{81}$$
$$= \lambda_i A^{k-2} \cdot A\vec{v}_i \tag{82}$$
$$= \lambda_i^2 A^{k-3} \cdot A\vec{v}_i \tag{83}$$

$$\vdots \tag{84}$$

$$= \lambda_i^k \vec{v}_i \tag{85}$$

Thus by definition, v_i is an eigenvector of A^k with corresponding eigenvalue λ_i^k.

Alternate solution: Since A is diagonalizable, we can express A as

$$A = V\Lambda V^{-1} \tag{86}$$

Substituting A as shown in Equation 86 in 79, we get

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A \cdot A}_{k \text{ times}} \tag{87}$$

$$= \underbrace{V\Lambda V^{-1} \cdot V\Lambda V^{-1} \cdot \cdots \cdot V\Lambda V^{-1} \cdot V\Lambda V^{-1}}_{k \text{ times}} \tag{88}$$

$$= \underbrace{V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1} \cdot \cdots \cdot V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1}}_{k \text{ times}} \tag{89}$$

$$= V \underbrace{\Lambda \cdot \Lambda \cdot \cdots \cdot \Lambda \cdot \Lambda}_{k \text{ times}} V^{-1} \tag{90}$$

$$= V\Lambda^k V^{-1} \tag{91}$$

Since Λ is a diagonal matrix,

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \tag{92}$$

Thus, A^k is clearly diagonalizable, where the eigenvectors of A^k are just the eigenvectors of A, and the eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k$.

4. Vector Differential Equations

Note: it's recommended to finish the previous question (Eigenvectors and Diagonalization) before this problem.

Consider a system of ordinary differential equations that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{93}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time t, and $A \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix with constant coefficients. We call eq. (93) a vector differential equation.

(a) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (93).

Consider a second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0, \tag{94}$$

where $a, b \in \mathbb{R}$.

Write this differential equation in the form of (eq. (93)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (94) without having to take a second derivative, and instead just taking the first derivative of something.)

Solution: If we set $x_1(t) = y(t)$, $x_2(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}$, then we have

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2(t) \tag{95}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -a\frac{\mathrm{d}y(t)}{\mathrm{d}t} - by(t) = -ax_2(t) - bx_1(t) \tag{96}$$

We can write this in the form of eq. (93) as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{97}$$

(b) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{98}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now let $a = -1$ and $b = -2$ in eq. (94), i.e.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = 0, \tag{99}$$

Solve eq. (99) with the initial conditions $y(0) = 1$, $\frac{dy}{dt}(0) = 1$, using the general form in eq. (98). *(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (HINT: Given your specific choice of x_1 and x_2 in part (a), how many unknowns are there really?)*

Solution: We have

$$\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \tag{100}$$

First, we calculate the eigenvalues of this matrix. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \tag{101}$$

Thus the eigenvalues are $\lambda_1 = -1, \lambda_2 = 2$. Since they are distinct, we can proceed with this method.

We know the solution for $x_1(t), x_2(t)$ is of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{-t} + c_1 e^{2t} \\ c_2 e^{-t} + c_3 e^{2t} \end{bmatrix} \tag{102}$$

At $t = 0$, we have $y(0) = 1$ and $\frac{dy}{dt}(0) = 1$. Using our differential equation (eq. (99)), we can get $\frac{d^2y}{dt^2}(0) = \frac{dy}{dt}(0) + 2y(0) = 3$. Plugging these in,

$$x_1(0) = y(0) = 1 = c_0 + c_1 \tag{103}$$

$$x_2(0) = \frac{dy}{dt}(0) = 1 = c_2 + c_3 \tag{104}$$

$$\frac{dx_1}{dt}(0) = \frac{dy}{dt}(0) = 1 = -c_0 + 2c_1 \tag{105}$$

$$\frac{dx_2}{dt}(0) = \frac{d^2y}{dt^2}(0) = 3 = -c_2 + 2c_3 \tag{106}$$

This gives $c_0 = \frac{1}{3}, c_1 = \frac{2}{3}, c_2 = -\frac{1}{3}, c_3 = \frac{4}{3}$. Alternatively, you could've seen that $c_2 = -c_0$ and $c_3 = 2c_1$ since $x_2(t)$ is the derivative of $x_1(t)$ which makes it solvable with just the first 2 equations. Thus we have

$$x_1(t) = y(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \tag{107}$$

$$x_2(t) = \frac{dy(t)}{dt} = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} \tag{108}$$

5. System Identification

You are given a discrete-time system as a black box. You don't know the specifics of the system but you know that it takes one scalar input and has two states that you can observe. You assume that the system is linear and of the form

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{w}[i], \tag{109}$$

where $\vec{w}[i]$ is an external small unknown disturbance, $u[i]$ is a scalar input, and

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad x[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}. \tag{110}$$

You want to identify the system parameters (a_1, a_2, a_3, a_4, b_1 and b_2) from measured data. However, you can only interact with the system via a black box model, i.e., you can see the states $\vec{x}[t]$ and set the inputs $u[i]$ that allow the system to move to the next state.

(a) You observe that the system has state $\vec{x}[i] = \begin{bmatrix} x_1[i] & x_2[i] \end{bmatrix}^\top$ at time i. You pass input $u[i]$ into the black box and observe the next state of the system: $\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] & x_2[i+1] \end{bmatrix}^\top$.
Write scalar equations for the new states, $x_1[i+1]$ and $x_2[i+1]$. Write these equations in terms of the a_i, b_i, the states $x_1[i]$, $x_2[i]$ and the input $u[i]$. Here, assume that $\vec{w}[i] = \vec{0}$ (i.e., the model is perfect).

Solution:

$$x_1[i+1] = a_1 x_1[i] + a_2 x_2[i] + b_1 u[i] \tag{111}$$
$$x_2[i+1] = a_3 x_1[i] + a_4 x_2[i] + b_2 u[i]. \tag{112}$$

(b) Now we want to identify the system parameters. We observe the system at the start state $\vec{x}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$. We can then input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$. We can continue this for a sequence of ℓ inputs.

Let us define an ℓ-length trajectory to be an initial condition $\vec{x}[0]$, an input sequence $u[0], \ldots, u[\ell-1]$, and the corresponding states that are produced by the system $x[1], \ldots, x[\ell]$. **Assuming that the model is perfect ($\vec{w}[i] = \vec{0}$), what is the minimum value of ℓ you need to identify the system parameters?**

Solution: There are 6 unknowns so we need 6 equations to properly identify the system. Each additional timestep gives two new equations. To form the 6 equations we need to give the black box $\ell = 3$ inputs. Namely, given inputs $u[0], u[1]$, and $u[2]$, we can see the state at times $t = 0, 1, 2, 3$ to give us our six equations.

Notice that the initial condition on its own gives us no equations because the unknowns we are interested in do not impact the initial condition. They govern the evolution of the system, and hence the states at times $1, 2, 3$ each give us two equations.

Note that having 6 equations is a necessary, but not sufficient, condition for us to be able to invert the system to uniquely determine the system parameters. For example, if $A = I$ and $u[0] = \cdots = u[\ell-1] = 0$, then we would only have two independent equations.

(c) We now remove our assumption that $\vec{w} = 0$. We assume it is small, so the model is approximately correct and we have

$$\vec{x}[i+1] \approx A\vec{x}[i] + Bu[i]. \tag{113}$$

Say we feed in a total of 4 inputs $u[0], \ldots, u[3]$, and observe the states $\vec{x}[0], \ldots, \vec{x}[4]$. To identify the system we need to set up an approximate (because of potential, small, disturbances) matrix equation

$$DP \approx S \tag{114}$$

using the observed values above and the unknown parameters we want to find. Let our parameter vector be

$$P := \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \\ b_1 & b_2 \end{bmatrix} \tag{115}$$

Find the corresponding D and S to do system identification. Write both out explicitly.

Solution:

Using eq. (112), we get

$$\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ x_1[2] & x_2[2] & u[2] \\ x_1[3] & x_2[3] & u[3] \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \\ b_1 & b_2 \end{bmatrix} \approx \begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ x_1[3] & x_2[3] \\ x_1[4] & x_2[4] \end{bmatrix} \tag{116}$$

so

$$D = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ x_1[2] & x_2[2] & u[2] \\ x_1[3] & x_2[3] & u[3] \end{bmatrix}, \text{ and } S = \begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ x_1[3] & x_2[3] \\ x_1[4] & x_2[4] \end{bmatrix}. \tag{117}$$

(d) Now that we have set up $DP \approx S$, we can estimate a_0, a_1, a_2, a_3, b_0, and b_1. **Give an expression for the estimates of $\vec{p}_1$ and $\vec{p}_2$ (which are denoted $\hat{\vec{p}}_1$ and $\hat{\vec{p}}_2$ respectively) in terms of D and S.** Denote the columns of S as $\vec{s}_1$ and $\vec{s}_2$, so we have $S = [\vec{s}_1\ \vec{s}_2]$. Assume that the columns of D are linearly independent. *(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.) (HINT: Can we split $DP = S$ into separate equations for p_1 and p_2?)*

Solution:

Notice that eq. (116) can be split into two matrix equations, one for each of p_1 and p_2:

$$D\vec{p}_1 \approx \vec{s}_1 \tag{118}$$
$$D\vec{p}_2 \approx \vec{s}_2. \tag{119}$$

Since D isn't square, it isn't invertible. However, we can still find $\vec{p}_1$ and $\vec{p}_2$ that best satisfy the equation via least-squares, which gives the solution

$$\hat{\vec{p}}_1 = (D^\top D)^{-1} D^\top \vec{s}_1 \tag{120}$$

$$\hat{\vec{p}}_2 = (D^\top D)^{-1} D^\top \vec{s}_2. \tag{121}$$

Here, $D^\top D$ is invertible (i.e. the solution is well-defined) because the columns of D are linearly independent. This was proved in 16A, but for completeness we include it here.

Assume that the columns of D are linearly independent. Let $\vec{v} \in \mathbb{R}^3$ such that $(D^\top D)\vec{v} = 0$. Then $0 = \vec{v}^\top D^\top D\vec{v} = (D\vec{v})^\top (D\vec{v}) = \|D\vec{v}\|_2^2$, so $D\vec{v} = 0$. Since D has linearly independent columns, then $\vec{v} = 0$. This means that the nullspace of $D^\top D$ is $\{0\}$, so $D^\top D$ must have full rank and is invertible.

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