Homework 7

This homework is due on Saturday, March 9, 2024, at 11:59PM.

1. Similar Matrices

In linear algebra, if A and B are two $n \times n$ matrices, we say that A and B are *similar* if there exists an invertible matrix P such that $B = P^{-1}AP$ (or equivalently, $A = PBP^{-1}$). In this problem, we will explore the properties of similar matrices.

(a) If A and B are similar matrices, show that $A^k = PB^kP^{-1}$ for any positive integer k.

Solution:

$$\begin{aligned} A^k &= (PBP^{-1})^k \\ &= PBP^{-1} \cdot PBP^{-1} \cdots PBP^{-1} \\ &= P \cdot B(P^{-1}P) \cdot B \cdot (P^{-1}P) \cdots B \cdot P^{-1} \\ &= PB^kP^{-1} \end{aligned}$$

(b) Show that if A and B are similar, then they have the same determinant (i.e., $\det(A) = \det(B)$). *(HINT: The determinant of a product of matrices is the product of the determinants of the matrices.)*

Solution: If $B = P^{-1}AP$, then

$$\begin{aligned} \det(B) &= \det\left(P^{-1}AP\right) \\ &= \det\left(P^{-1}\right)\det(A)\det(P) \\ &= \det\left(P^{-1}\right)\det(P)\det(A) \\ &= \det\left(P^{-1}P\right)\det(A) \\ &= \det(I)\det(A) \\ &= \det(A) \end{aligned}$$

(c) Prove that if A and B are similar, they have the same characteristic polynomial (i.e., $\det(A - \lambda I) = \det(B - \lambda I)$), and therefore the same eigenvalues.

Solution: If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda)P$$

Then,

$$\begin{aligned} \det(B - \lambda I) &= \det\left(P^{-1}(A - \lambda)P\right) \\ &= \det\left(P^{-1}\right) \cdot \det(A - \lambda I) \cdot \det(P) \\ &= \det\left(P^{-1}\right)\det(P) \cdot \det(A - \lambda I) \cdot \end{aligned}$$

$$
\begin{aligned}
&= \det\left(P^{-1}P\right) \cdot \det(A - \lambda I) \cdot \\
&= \det(I) \cdot \det(A - \lambda I) \\
&= \det(A - \lambda I)
\end{aligned}
$$

(d) If A is similar to a diagonal matrix D that consists of the eigenvalues of A (i.e., $A = PDP^{-1}$), show that P is the matrix whose columns are the eigenvectors of A.

Solution: If $A = PDP^{-1}$, then $AP = PD$. Suppose the columns of P are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. By equating the columns of $AP = PD$, we get

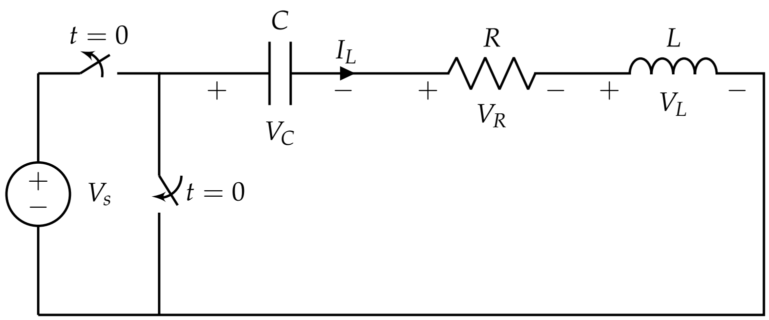
$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

Since P is invertible, its columns are nonzero and linearly independent. Thus, the columns of P are the eigenvectors of A.

2. An Alternative "Second Order" Perspective on Solving the RLC Circuit

In Homework 6, we solved an RLC circuit by setting state variables $x_1(t) = V_C(t)$ and $x_2(t) = I_L(t)$, and using these to build a linear first-order vector differential equation. In this problem, we will see how to solve the same system by picking *different* state variables $x_1(t) = V_C(t)$ and $x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$, getting a linear *second order scalar* differential equation, and solving that differential equation.

Consider the following series RLC circuit:



As before, assume that the system has reached steady-state for $t < 0$. At time $t = 0$, the switches change state, disconnecting the voltage source and replacing it with a short.

Suppose that we now insisted on expressing all of our components in terms of one waveform $V_C(t)$ instead of two of them (voltage across the capacitor and current through the inductor).

For this problem, use R for the resistor, L for the inductor, and C for the capacitor in all expressions.

(a) **Write the current $I_L(t)$ through the inductor in terms of the voltage $V_C(t)$ across the capacitor.**

Solution: The current $I_L(t)$ through the inductor L must be the same as the current $I_C(t)$ through C, which is $C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$. Hence, we can write

$$I_L(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t). \tag{1}$$

(b) **Write the voltage drop across the inductor, $V_L(t)$, in terms of the second derivative of $V_C(t)$.**

Solution: The voltage drop is

$$V_L(t) = L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = LC\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}V_C(t)\right) = LC\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_C(t). \tag{2}$$

(c) **Show that a differential equation governing $V_C(t)$ is**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_C(t) + \frac{R}{L}\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) + \frac{1}{LC}V_C(t) = 0. \tag{3}$$

Solution: Note that the current passing through the resistor is

$$I_R(t) = -\frac{V_C(t) + V_L(t)}{R} = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t). \tag{4}$$

or equivalently,

$$V_L(t) + RC\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) + V_C(t) = 0. \tag{5}$$

Plugging in $V_L(t)$, we have

$$LC\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_C(t) + RC\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) + V_C(t) = 0. \tag{6}$$

Finally, dividing by LC,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_C(t) + \frac{R}{L}\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) + \frac{1}{LC}V_C(t) = 0. \tag{7}$$

(d) Let us define our state variables as $x_1(t) = V_C(t)$ and $x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$. Construct a vector differential equation for this system, i.e., **find A such that**

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{8}$$

Then, show that the two eigenvalues of A are

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}, \qquad \lambda_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}. \tag{9}$$

Solution: We have

$$x_1(t) = y(t) = V_C(t) \tag{10}$$

$$x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}V_C(t). \tag{11}$$

We can write a second order differential equation of the form

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0 \tag{12}$$

as a matrix differential equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{13}$$

Thus

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}. \tag{14}$$

The characteristic polynomial of A is

$$p_A(\lambda) := \det(A - \lambda I) \tag{15}$$

$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ -\frac{1}{LC} & -\frac{R}{L} - \lambda \end{bmatrix}\right) \tag{16}$$

$$= (-\lambda)\left(-\frac{R}{L} - \lambda\right) - 1 \cdot \left(-\frac{1}{LC}\right) \tag{17}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}. \tag{18}$$

The solutions to $p_A(\lambda) = 0$ are obtained by the quadratic formula to be

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \qquad \lambda_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}. \tag{19}$$

(e) Assuming we have R, L, C such that A has distinct eigenvalues, it turns out that the solution to the vector differential equation you found in the previous part will have the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix} \tag{20}$$

where λ_1, λ_2 are the eigenvalues of A, and c_1, c_2, c_3, c_4 are constants.

Show that

$$c_3 = \lambda_1 c_1, \qquad c_4 = \lambda_2 c_2. \tag{21}$$

Then use the initial conditions of the RLC circuit to show that

$$c_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} V_s, \qquad c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} V_s. \tag{22}$$

(HINT: First, differentiate $x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ to get a form for $x_2(t)$, and match coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ to get the desired expressions for c_3 and c_4. Next, use the initial conditions for RLC to compute $V_C(0)$ and $\frac{d}{dt} V_C(t)\Big|_{t=0}$, and derive a system of two equations with c_1 and c_2.)

Solution: By definition,

$$x_2(t) = \frac{d}{dt} V_C(t) = \frac{d}{dt} x_1(t) \tag{23}$$

so if

$$x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{24}$$

then

$$\begin{aligned} x_2(t) &= \frac{d}{dt} x_1(t) && (25) \\ &= \frac{d}{dt}\left(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right) && (26) \\ &= \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t}. && (27) \end{aligned}$$

But we know that

$$x_2(t) = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}. \tag{28}$$

Thus by pattern matching the coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, we get

$$c_3 = \lambda_1 c_1 \qquad c_4 = \lambda_2 c_2. \tag{29}$$

Now to solve for c_1 and c_2. Recall that in steady state, a capacitor looks like an open circuit, so $V_C(0) = V_s$. By definition, $V_C(t) = x_1(t)$, so $x_1(0) = V_s$. Plugging in, we have

$$V_s = x_1(0) = c_1 e^{\lambda_1 \cdot 0} + c_2 e^{\lambda_2 \cdot 0} = c_1 + c_2. \tag{30}$$

Now we have one equation in the variables c_1 and c_2. To solve the system we need two equations. This motivates looking at

$$x_2(0) = \lambda_1 c_1 e^{\lambda_1 \cdot 0} + \lambda_2 c_2 e^{\lambda_2 \cdot 0} = \lambda_1 c_1 + \lambda_2 c_2. \tag{31}$$

To find the physical value of $x_2(0) = \frac{\mathrm{d}}{\mathrm{d}t} V_C(t)\Big|_{t=0}$, note that in steady state there is no change in any state variable by definition, so $\frac{\mathrm{d}}{\mathrm{d}t} V_C(t)\Big|_{t=0} = 0$. (An alternate physically motivated argument is to note that inductor current in steady state is $I_L = 0$, and it cannot change infinitely fast, so at time 0 we have $I_L(0) = 0$. Since $I_L(t) \propto \frac{\mathrm{d}V_C(t)}{\mathrm{d}t}$, we also have $\frac{\mathrm{d}V_C(t)}{\mathrm{d}t}\Big|_{t=0} = 0$.) Hence $x_2(0) = 0$. This sets up the system of equations

$$c_1 + c_2 = V_s \tag{32}$$

$$\lambda_1 c_1 + \lambda_2 c_2 = 0. \tag{33}$$

There are several ways we can solve this system, and one way is to note that this is a matrix-vector equation of the form

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}. \tag{34}$$

To solve it, we can use the matrix inverse that was provided by the hint to get

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} V_s \\ 0 \end{bmatrix} \tag{35}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} V_s \\ 0 \end{bmatrix} \tag{36}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 \\ -\lambda_1 \end{bmatrix} V_s. \tag{37}$$

Thus we have

$$c_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} V_s, \qquad c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} V_s. \tag{38}$$

We have found $\lambda_1, \lambda_2, c_1$, and c_2, so by substituting into eq. (20) we have solved for $x_1(t) = V_C(t)$!

3. Vector Differential Equations Driven by an Input

In this question, we will demonstrate the existence and uniqueness of solutions to systems of differential equations with inputs. In particular, we previously considered the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{39}$$

$$x(0) = x_0 \tag{40}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a known function of time, and we showed that there exists a unique solution to this differential equation, namely

$$x(t) := \mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau \tag{41}$$

(a) Extend the solution in eq. (41) to the diagonal, vector differential equation case. **Namely, show that the solution for**

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \Lambda\vec{x}(t) + \vec{b}u(t) \tag{42}$$

is given by

$$\vec{x}(t) = \mathrm{e}^{\Lambda t}\vec{x}(0) + \int_0^t \mathrm{e}^{\Lambda(t-\tau)}\vec{b}u(\tau)\,\mathrm{d}\tau \tag{43}$$

where $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^n$, $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix, $\vec{b} \in \mathbb{R}^n$, and $u(t) : \mathbb{R} \to \mathbb{R}$. As a reminder, using the Taylor series definition of a matrix exponential, $\mathrm{e}^{\Lambda x} = \begin{bmatrix} \mathrm{e}^{\lambda_1 x} & & & \\ & \mathrm{e}^{\lambda_2 x} & & \\ & & \ddots & \\ & & & \mathrm{e}^{\lambda_n x} \end{bmatrix}$ where

$$\Lambda := \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

(HINT: You may use the fact that, if $\vec{z}(t) : \mathbb{R} \to \mathbb{R}^n$ and $\vec{z}(t) := \begin{bmatrix} z_1(t) & z_2(t) & \dots & z_n(t) \end{bmatrix}^T$, it is the case that $\int_a^b \vec{z}(t)\,\mathrm{d}t = \begin{bmatrix} \int_a^b z_1(t)\,\mathrm{d}t & \int_a^b z_2(t)\,\mathrm{d}t & \dots & \int_a^b z_n(t)\,\mathrm{d}t \end{bmatrix}^T$. In other words, our integral performs item-wise.)

Solution: The kth row of this differential equation will be

$$\frac{\mathrm{d}}{\mathrm{d}t}x_k(t) = \lambda_k x_k(t) + b_k u(t) \tag{44}$$

which has the solution

$$x_k(t) = \mathrm{e}^{\lambda_k t}x_k(0) + \int_0^t \mathrm{e}^{\lambda_k(t-\tau)}b_k u(\tau)\,\mathrm{d}\tau \tag{45}$$

This yields the vector solution as

$$\vec{x}(t) = \begin{bmatrix} \mathrm{e}^{\lambda_1 t}x_1(0) \\ \mathrm{e}^{\lambda_2 t}x_2(0) \\ \vdots \\ \mathrm{e}^{\lambda_n t}x_n(0) \end{bmatrix} + \begin{bmatrix} \int_0^t \mathrm{e}^{\lambda_1(t-\tau)}b_1 u(\tau)\,\mathrm{d}\tau \\ \int_0^t \mathrm{e}^{\lambda_2(t-\tau)}b_2 u(\tau)\,\mathrm{d}\tau \\ \vdots \\ \int_0^t \mathrm{e}^{\lambda_n(t-\tau)}b_n u(\tau)\,\mathrm{d}\tau \end{bmatrix} \tag{46}$$

We can simplify the first term to be

$$\begin{bmatrix} e^{\lambda_1 t} x_1(0) \\ e^{\lambda_2 t} x_2(0) \\ \vdots \\ e^{\lambda_n t} x_n(0) \end{bmatrix} = e^{\Lambda t}\vec{x}(0) \tag{47}$$

and, using the hint, we can simplify the second term as follows:

$$\begin{bmatrix} \int_0^t e^{\lambda_1(t-\tau)} b_1 u(\tau)\, d\tau \\ \int_0^t e^{\lambda_2(t-\tau)} b_2 u(\tau)\, d\tau \\ \vdots \\ \int_0^t e^{\lambda_n(t-\tau)} b_n u(\tau)\, d\tau \end{bmatrix} = \int_0^t \begin{bmatrix} e^{\lambda_1(t-\tau)} b_1 u(\tau) \\ e^{\lambda_2(t-\tau)} b_2 u(\tau) \\ \vdots \\ e^{\lambda_n(t-\tau)} b_n u(\tau) \end{bmatrix} d\tau \tag{48}$$

$$= \int_0^t \begin{bmatrix} e^{\lambda_1(t-\tau)} b_1 \\ e^{\lambda_2(t-\tau)} b_2 \\ \vdots \\ e^{\lambda_n(t-\tau)} b_n \end{bmatrix} u(\tau)\, d\tau \tag{49}$$

$$= \int_0^t e^{\Lambda(t-\tau)} \vec{b} u(\tau)\, d\tau \tag{50}$$

Altogether, this yields,

$$\vec{x}(t) = e^{\Lambda t}\vec{x}(0) + \int_0^t e^{\Lambda(t-\tau)} \vec{b} u(\tau)\, d\tau \tag{51}$$

(b) **Extend the result from the previous part for an arbitrary vector differenial equation given by**

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{52}$$

where A is a *diagonalizable* matrix (but not necessarily a *diagonal* matrix). You may assume that A can be diagonalized as $A = V\Lambda V^{-1}$. Express your answer in terms of e^{At}.

Solution: Define the change of basis $\vec{x}_\lambda(t) = V^{-1}\vec{x}(t)$. We know that the differential equation governing $\vec{x}_\lambda(t)$ will be

$$\frac{d}{dt}\vec{x}_\lambda(t) = \Lambda\vec{x}_\lambda(t) + V^{-1}\vec{b}u(t) \tag{53}$$

The solution to this, using the previous part, is

$$\vec{x}_\lambda(t) = e^{\Lambda t}\vec{x}_\lambda(0) + \int_0^t e^{\Lambda(t-\tau)} V^{-1}\vec{b} u(\tau)\, d\tau \tag{54}$$

Hence, the solution for $\vec{x}(t)$ will be

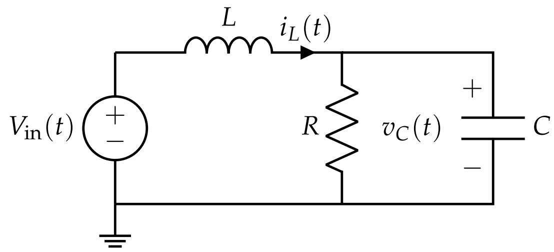
$$\vec{x}(t) = Ve^{\Lambda t}\vec{x}_\lambda(0) + V\int_0^t e^{\Lambda(t-\tau)} V^{-1}\vec{b} u(\tau)\, d\tau \tag{55}$$

$$= Ve^{\Lambda t}V^{-1}\vec{x}(0) + \int_0^t Ve^{\Lambda(t-\tau)} V^{-1}\vec{b} u(\tau)\, d\tau \tag{56}$$

$$= e^{At}\vec{x}(0) + \int_0^t e^{A(t-\tau)} \vec{b} u(\tau)\, d\tau\,. \tag{57}$$

4. RLC Vector Differential Equation

In this problem, you will approach solving the following RLC circuit using the vector differential equation method.



(a) **Derive a pair of differential equations based on the above circuit, one for $I_L(t)$ and another for $V_C(t)$.**

Solution: We begin by using our circuit analysis methods. Using KVL and replacing the inductor voltage, we obtain:

$$V_{\text{in}}(t) - V_L(t) - V_C(t) = 0 \tag{58}$$
$$V_L(t) = V_{\text{in}}(t) - V_C(t) \tag{59}$$
$$L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = V_{\text{in}}(t) - V_C(t) \tag{60}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = \frac{V_{\text{in}}(t)}{L} - \frac{V_C(t)}{L} \tag{61}$$

Following this, we need one more differential equation. This time we utilize KCL on the node connected to the inductor, resistor, and capacitor:

$$I_L(t) = I_R(t) + I_C(t) \tag{62}$$
$$I_C(t) = I_L(t) - I_R(t) \tag{63}$$
$$C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = I_L(t) - \frac{V_C(t)}{R} \tag{64}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{I_L(t)}{C} - \frac{V_C(t)}{RC} \tag{65}$$

Thus, we obtain our set of differential equations for this problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = \frac{V_{\text{in}}(t)}{L} - \frac{V_C(t)}{L} \tag{66}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{I_L(t)}{C} - \frac{V_C(t)}{RC} \tag{67}$$

(b) Suppose our vector differential equation was written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e \\ f \end{bmatrix} V_{\text{in}}(t) \tag{68}$$

The state vector is defined as $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$. **Find the values of a, b, c, d, e, and f in terms of R, L, C.**

Solution: Our first step is to obtain a vectorized version of the two differential equations we found in part (a). We do this by stacking complementary terms.

As suggested, we will take $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$. With this, we find that the system can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_{\text{in}} \tag{69}$$

Thus, $a = -\frac{1}{RC}$, $b = \frac{1}{C}$, $c = -\frac{1}{L}$, $d = 0$, $e = 0$, $f = \frac{1}{L}$.

(c) Suppose you are now told that the system is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -6 & 8 \\ -1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{\text{in}}(t) \tag{70}$$

Decouple your state variables by choosing an appropriate transformation $\vec{x}(t) = V\vec{z}(t)$. Rewrite the system in the new basis, showing the state matrix is now diagonal.

Solution: This problem part asks us to diagnolize the system (i.e. convert it to the V-basis). We can compute the eigenvalues of the state matrix as -4 and -2 with eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ respectively. The columns of V are simply the eigenvectors of the state matrix:

$$V = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \tag{71}$$

We know that the diagonlized system will look like the following (where $\vec{z}$ is our state vector in the diagnalized basis):

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = V^{-1}AV\vec{z}(t) + V^{-1}\vec{b}V_{\text{in}}(t) \tag{72}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t) + V^{-1}\vec{b}V_{\text{in}}(t) \tag{73}$$

where $A = \begin{bmatrix} -6 & 8 \\ -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ according to our problem.

There are a few ways to find V^{-1}. One is to use the determinant as follows:

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \tag{74}$$

$$V^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \tag{75}$$

$$V^{-1} = \begin{bmatrix} 0.5 & -1 \\ -0.5 & 2 \end{bmatrix} \tag{76}$$

Using the eigenvalues of the state matrix, we have everything for our answer:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t) + V^{-1}\vec{b}V_{\text{in}}(t) \tag{77}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}\vec{z}(t) + \begin{bmatrix} -1 \\ 2 \end{bmatrix}V_{\text{in}}(t) \tag{78}$$

(d) Suppose you are given that the initial conditions for our states in the standard basis are $V_C(0)$ and $I_L(0)$. **Write $z_1(0)$ and $z_2(0)$ (the initial conditions in the diagonalized basis) in terms of $V_C(0)$ and $I_L(0)$.**

Solution: To perform our change of basis on our original state vector $\vec{x}$, we had to apply the following transformation:

$$\vec{z}(t) = V^{-1}\vec{x}(t) \tag{79}$$

Therefore, to get our initial conditions in the diagonalized basis, we would perform the same transformation:

$$\vec{z}(t) = V^{-1}\vec{x}(t) \tag{80}$$

$$\vec{z}(t) = \begin{bmatrix} 0.5 & -1 \\ -0.5 & 2 \end{bmatrix}\begin{bmatrix} V_C(0) \\ I_L(0) \end{bmatrix} \tag{81}$$

$$\vec{z}(t) = \begin{bmatrix} 0.5V_C(0) - I_L(0) \\ -0.5V_C(0) + 2I_L(0) \end{bmatrix} \tag{82}$$

Thus, $z_1(0) = 0.5V_C(0) - I_L(0)$ and $z_2(0) = -0.5V_C(0) + 2I_L(0)$.

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