Homework 7

This homework is due on Saturday, March 9, 2024, at 11:59PM.

1. Similar Matrices

In linear algebra, if A and B are two $n \times n$ matrices, we say that A and B are *similar* if there exists an invertible matrix P such that $B = P^{-1}AP$ (or equivalently, $A = PBP^{-1}$). In this problem, we will explore the properties of similar matrices.

(a) If A and B are similar matrices, show that $A^k = PB^kP^{-1}$ for any positive integer k.

(b) Show that if A and B are similar, then they have the same determinant (i.e., $\det(A) = \det(B)$). *(HINT: The determinant of a product of matrices is the product of the determinants of the matrices.)*

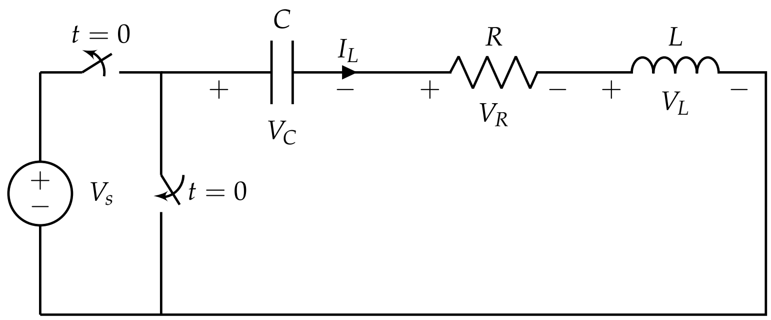
(c) Prove that if A and B are similar, they have the same characteristic polynomial (i.e., $\det(A - \lambda I) = \det(B - \lambda I)$), and therefore the same eigenvalues.

(d) If A is similar to a diagonal matrix D that consists of the eigenvalues of A (i.e., $A = PDP^{-1}$), show that P is the matrix whose columns are the eigenvectors of A.

2. An Alternative "Second Order" Perspective on Solving the RLC Circuit

In Homework 6, we solved an RLC circuit by setting state variables $x_1(t) = V_C(t)$ and $x_2(t) = I_L(t)$, and using these to build a linear first-order vector differential equation. In this problem, we will see how to solve the same system by picking *different* state variables $x_1(t) = V_C(t)$ and $x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$, getting a linear *second order scalar* differential equation, and solving that differential equation.

Consider the following series RLC circuit:



As before, assume that the system has reached steady-state for $t < 0$. At time $t = 0$, the switches change state, disconnecting the voltage source and replacing it with a short.

Suppose that we now insisted on expressing all of our components in terms of one waveform $V_C(t)$ instead of two of them (voltage across the capacitor and current through the inductor).

For this problem, use R for the resistor, L for the inductor, and C for the capacitor in all expressions.

(a) **Write the current $I_L(t)$ through the inductor in terms of the voltage $V_C(t)$ across the capacitor.**

(b) **Write the voltage drop across the inductor, $V_L(t)$, in terms of the second derivative of $V_C(t)$.**

(c) **Show that a differential equation governing $V_C(t)$ is**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_C(t) + \frac{R}{L}\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) + \frac{1}{LC}V_C(t) = 0. \tag{1}$$

(d) Let us define our state variables as $x_1(t) = V_C(t)$ and $x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$. Construct a vector differential equation for this system, i.e., **find A such that**

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{2}$$

Then, show that the two eigenvalues of A are

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}, \qquad \lambda_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}. \tag{3}$$

(e) Assuming we have R, L, C such that A has distinct eigenvalues, it turns out that the solution to the vector differential equation you found in the previous part will have the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix} \tag{4}$$

where λ_1, λ_2 are the eigenvalues of A, and c_1, c_2, c_3, c_4 are constants.

Show that

$$c_3 = \lambda_1 c_1, \qquad c_4 = \lambda_2 c_2. \tag{5}$$

Then use the initial conditions of the RLC circuit to show that

$$c_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} V_s, \qquad c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} V_s. \tag{6}$$

(HINT: First, differentiate $x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ to get a form for $x_2(t)$, and match coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ to get the desired expressions for c_3 and c_4. Next, use the initial conditions for RLC to compute $V_C(0)$ and $\frac{d}{dt} V_C(t)\Big|_{t=0}$, and derive a system of two equations with c_1 and c_2.)

We have found $\lambda_1, \lambda_2, c_1, c_2$, so by substituting into eq. (4) we have solved for $x_1(t) = V_C(t)$!

3. Vector Differential Equations Driven by an Input

In this question, we will show the existence and uniqueness of solutions to systems of differential equations with inputs. In particular, we previously considered the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{7}$$

$$x(0) = x_0 \tag{8}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a known function of time, and we showed that there exists a unique solution to this differential equation, namely

$$x(t) := \mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau \tag{9}$$

(a) Extend the solution in eq. (9) to the diagonal, vector differential equation case. **Namely, show that the solution for**

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \Lambda\vec{x}(t) + \vec{b}u(t) \tag{10}$$

is given by

$$\vec{x}(t) = \mathrm{e}^{\Lambda t}\vec{x}(0) + \int_0^t \mathrm{e}^{\Lambda(t-\tau)}\vec{b}u(\tau)\,\mathrm{d}\tau \tag{11}$$

where $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^n$, $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix, $\vec{b} \in \mathbb{R}^n$, and $u(t) : \mathbb{R} \to \mathbb{R}$. As a reminder, using the Taylor series definition of a matrix exponential, $\mathrm{e}^{\Lambda x} = \begin{bmatrix} \mathrm{e}^{\lambda_1 x} & & & \\ & \mathrm{e}^{\lambda_2 x} & & \\ & & \ddots & \\ & & & \mathrm{e}^{\lambda_n x} \end{bmatrix}$ where

$$\Lambda := \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

(HINT: You may use the fact that, if $\vec{z}(t) : \mathbb{R} \to \mathbb{R}^n$ and $\vec{z}(t) := \begin{bmatrix} z_1(t) & z_2(t) & \dots & z_n(t) \end{bmatrix}^T$, it is the case that $\int_a^b \vec{z}(t)\,\mathrm{d}t = \begin{bmatrix} \int_a^b z_1(t)\,\mathrm{d}t & \int_a^b z_2(t)\,\mathrm{d}t & \dots & \int_a^b z_n(t)\,\mathrm{d}t \end{bmatrix}^T$.)

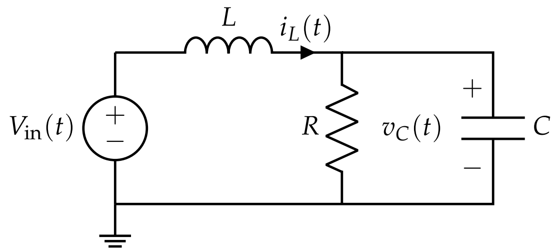
(b) **Extend the result from the previous part for an arbitrary vector differenial equation given by**

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{12}$$

where A is a *diagonalizable* matrix (but not necessarily a *diagonal* matrix). You may assume that A can be diagonalized as $A = V\Lambda V^{-1}$. Express your answer in terms of e^{At}.

4. RLC Vector Differential Equation

In this problem, you will approach solving the following RLC circuit using the vector differential equation method.



(a) **Derive a set of differential equations, one for $I_L(t)$ and another for $V_C(t)$.**

(b) Suppose our vector differential equation was written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\vec{x}(t) + \begin{bmatrix} e \\ f \end{bmatrix} V_{\text{in}}(t) \tag{13}$$

The state vector is defined as $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$. **Find the values of a, b, c, d, e, and f in terms of R, L, C.**

(c) Suppose you are now told that the system is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -6 & 8 \\ -1 & 0 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{\text{in}}(t) \tag{14}$$

Decouple your state variables by choosing an appropriate transformation $\vec{x}(t) = V\vec{z}(t)$. Rewrite the system in the new basis, showing the state matrix is now diagonal.

(d) Suppose you are given that the initial conditions for our states in the standard basis are $V_C(0)$ and $I_L(0)$. **Write $z_1(0)$ and $z_2(0)$ (the initial conditions in the diagonalized basis) in terms of $V_C(0)$ and $I_L(0)$.**

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