Homework 6

This homework is due on Friday, October 6, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, October 13, 2022, at 11:59PM.

1. **Study Group Reassignment**

   We hope your study groups from the beginning of the semester have been going well! If you did not fill out the original matching form and would now like to join a group, or if your current study group is not meeting your needs, you can request a new study group via this form. Requests for new study groups are due Friday at 11:59 PM.

2. Inner Products

(a) For the following inner product defined on $\mathbb{R}^2$, which inner product properties hold?

$$\langle \vec{x},\ \vec{y} \rangle = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^{\top} (3\vec{x} + 3\vec{y}) \tag{1}$$

i. Symmetry (True/False)
ii. Linearity (True/False)
iii. Positive-Definiteness (True/False)

Explain your answers.

(b) Consider the following valid inner product over the vector space of 2x2 real matrices $\mathbb{R}^{2\times 2}$, defined as

$$\langle A,\ B \rangle = \left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} \tag{2}$$

for any $A, B \in \mathbb{R}^{2\times 2}$. Given this inner product definiton, what is $\left\| \begin{bmatrix} 2 & 5 \\ 6 & 2 \end{bmatrix} \right\|^2$?

3. Least Squares with Shazam

(a) The application Shazam is able to detect what song is playing by means of an *acoustic footprint*. This is a small set of information that identifies the song. Shazam then checks that footprint in its database, to check for another song that has that footprint. Here is the footprint we obtained via sampling: (we are representing the footprint as a vector)

$$\vec{x}_{\text{sample}} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{3}$$

Say Shazam has narrowed it down to the following three songs with the corresponding footprints:

"Electric Love - Børns": $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

"She's Electric - Oasis": $\vec{x}_2 = \begin{bmatrix} 2 \\ -2 \\ -8 \\ 7 \end{bmatrix}$

"Electric Feel - MGMT": $\vec{x}_3 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 2 \end{bmatrix}$

Shazam is going to determine which song it is by projecting the footprint of our sample onto each of the song candidates, and ranking the songs based on the normalized inner product of $\vec{x}_{\text{sample}}$ onto the footprints. **Based on this information, which song is playing?**

(b) Shazam wants to partner with Spotify. For its Discover Weekly algorithms, Spotify would like to know what characteristics of a song make it attractive to a first-time listener. Shazam provides the number of Shazams for a set of new songs to Spotify, who combines it with their data on the songs. Here is the table of data that Spotify assembles:

Shazam Popularity	Tempo	Danceability	Acoustic-ness
1109	100	0.8	0.6
5501	90	0.5	0.2
2031	68	0.4	0.7
13045	120	0.9	0.2

Spotify would now like to use the data it has to predict what the number of shazams for a new song, whose characteristics (Tempo, Danceability, Acoustic-ness) are represented as $\vec{a}_n$. **Which is the correct formula for how Spotify would use Least Squares to calculate this?** Let M be the matrix of Tempo, Danceabilty and Acoustic-ness, and $\vec{b}_n$ be the number of shazams they get:

$$M = \begin{bmatrix} 100 & 90 & 68 & 120 \\ 0.8 & 0.5 & 0.4 & 0.9 \\ 0.6 & 0.2 & 0.7 & 0.2 \end{bmatrix} \tag{4}$$

and

$$\vec{b}_n = \begin{bmatrix} 1109 \\ 5501 \\ 2031 \\ 13045 \end{bmatrix} \tag{5}$$

Options:

i. $(MM^\top)^{-1}M\vec{b}_n$

ii. $\vec{a}_n{}^\top(MM^\top)^{-1}M\vec{b}_n$

iii. $\vec{a}_n{}^\top(\vec{b}_n^\top M^\top)^\top$

iv. $\vec{b}_n{}^\top M\vec{a}_n$

v. $\vec{b}_n{}^\top M\vec{a}_n$

(c) Say Spotify gets some new data to incorporate into its data set, the energy of the song. Here is the table with the added data:

Shazam Popularity	Tempo	Danceability	Acoustic-ness	Energy
1109	100	0.8	0.6	0.70
5501	90	0.5	0.2	0.35
2031	68	0.4	0.7	0.55
13045	120	0.9	0.2	0.55

Will it still be possible to run least squares with all of this data?

(d) What is the maximum number of features we could have per song, assuming we keep the number of songs the same?

Options:

i. 4

ii. 3

iii. 2

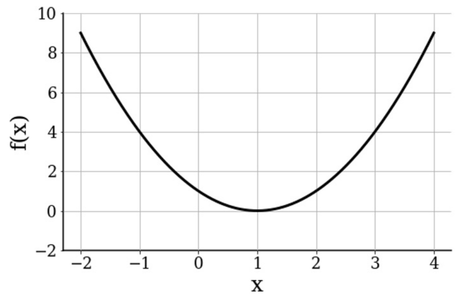
iv. 5

v. As many as we want

4. Orthogonality

$f(x)$ and $g(x)$ are both polynomials with degree at most 2. $f(x) = x^2 - 2x + 1$. We define inner product between two polynomials as $\langle f(x),\ g(x) \rangle = f(0)g(0) + f(1)g(1) + f(2)g(2)$.

If $g(x)$ is orthogonal to $f(x)$, which is possible equation for $g(x)$?



(a) $-x^2 + 2x - 1$

(b) $x^2 + x - 1$

(c) $x - 1$

(d) x

5. A Quirky Quantum Question

(a) In quantum mechanics, states of particles are represented by vectors in a vector space. In this problem, we'll say that all states exist in $\mathbb{R}^2$.

A particular matrix, $\widehat{\mathbf{H}}$ (called the Hamiltonian operator), has the unique property that its eigenvalues represent a particle's allowed energy values. Quantum mechanics tells us that if the values of $\widehat{\mathbf{H}}$ are real, it must be symmetric – that is, it can be written as

$$\widehat{\mathbf{H}} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \tag{6}$$

Assume that we know $a > 0$ and $b > 0$. What further condition on a and b forces the allowed energy values (the eigenvalues) to always be nonnegative?

(b) Miki experimentally determines that particles associated with the $\widehat{\mathbf{H}}$ matrix from Question 1 have allowed energy values $\lambda_1 = \frac{5}{2}$ and $\lambda_2 = \frac{9}{2}$. Find a and b.

(c) Now, given a new matrix $\widehat{\mathbf{H}} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, let the eigenvalues be $\lambda_1 < \lambda_2$ and the normalized eigenvectors be $\vec{v}_{\lambda_1}$ and $\vec{v}_{\lambda_2}$ (corresponding to eigenvalues λ_1 and λ_2, respectively, and scaled to a magnitude of 1), span $\mathbb{R}^2$. If a particle is in some state $\vec{v}_s \in \mathbb{R}^2$, then it can be expressed as $\vec{v}_s = \alpha\vec{v}_{\lambda_1} + \beta\vec{v}_{\lambda_2}$, where α and β are real constants. If $v_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, what are possible magnitudes of α? (In quantum mechanics, α^2 represents the probability that measuring the particle's energy will yield λ_1.)

6. Discrete System Induction

Consider the discrete system given by

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] \tag{7}$$

(a) Write $\vec{x}[1]$ in terms of $\vec{x}[0]$ and $\vec{u}[0]$. Then, write $\vec{x}[2]$ in terms of $\vec{x}[0]$, $\vec{u}[1]$, and $\vec{u}[0]$. Lastly, write $\vec{x}[3]$ in terms of $\vec{x}[0]$, $\vec{u}[2]$, $\vec{u}[1]$, and $\vec{u}[0]$.

(b) We can generalize this to write $\vec{x}[i]$ in terms of $\vec{x}[0]$ and $\vec{u}[i-1], \ldots, \vec{u}[0]$ as follows:

$$\vec{x}[i] = A^i\vec{x}[0] + \sum_{j=0}^{i-1} A^{i-1-j}B\vec{u}[j] \tag{8}$$

Verify that this equation holds for $i = 1$. *This is equivalent to testing your base case in induction.*

(c) **Show that eq. (8) holds for $\vec{x}[i+1]$.** That is, write $\vec{x}[i+1]$ in terms of $\vec{x}[i]$ and plug in eq. (8) for $\vec{x}[i]$. Show that this simplifies to eq. (8) where we now replace i with $i+1$. *This is equivalent to testing your inductive hypothesis.*