Homework 6

This homework is due on Saturday, March 2, 2024, at 11:59PM.

1. Least Squares

(a) Consider the system of equations $\vec{a}x \approx \vec{b}$ where $\vec{a}, \vec{b} \in \mathbb{R}^2$ and $x \in \mathbb{R}$.

i. Which of the following will result in the $\vec{v} \in \text{Span}(\vec{a})$ that is closest to $\vec{b}$ in Euclidean distance?
(HINT: It might be helpful to draw the vectors.)

(A) Projecting $\vec{b}$ onto $\vec{a}$

(B) Projecting $\vec{a}$ onto $\vec{b}$

(C) Subtracting $\vec{b}$ from $\vec{a}$

(D) Subtracting $\vec{a}$ from $\vec{b}$

(E) None of the above

ii. The vector $\vec{v}$ can also be determined by minimizing the length of the error vector, represented as

(A) $\vec{v} = \text{argmin}_{\vec{b}} \left\|\vec{a} - \vec{b}\right\|$

(B) $\vec{v} = \text{argmin}_{\vec{v}} \|\vec{a} - \vec{v}\|$

(C) $\vec{v} = \text{argmin}_{\vec{b}} \left\|\vec{b} - \vec{v}\right\|$

(D) $\vec{v} = \text{argmin}_{\vec{v}} \left\|\vec{b} - \vec{v}\right\|$

(b) For each choice of A and b below, determine whether or not the $\hat{x}$ that minimizes the least squares error vector is unique.

i. $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(A) Yes

(B) No

ii. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

(A) Yes

(B) No

(c) For the following three questions, consider the system of $A\vec{x} \approx \vec{b}$ with $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

i. Can we apply the least squares formula?

(A) Yes

(B) No

ii. What is the determinant of $A^\top A$?

iii. Does $A\vec{x} = \vec{b}$ have zero, one, or more than one solution for $\vec{x}$?

(A) No solutions

(B) One unique solution

(C) More than one solution

(d) Find the best approximation $x = \hat{x}$ to this system of equations:

$$a_1 x = b_1 \tag{1}$$

$$a_2 x = b_2 \tag{2}$$

i. Write the problem into $A\vec{x} \approx \vec{b}$ format and solve for $\hat{x}$ using least squares. Choose the correct $\hat{x}$.

(A) $\hat{x} = \dfrac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$

(B) $\hat{x} = \dfrac{a_1 b_1 - a_2 b_2}{a_1^2 + a_2^2}$

(C) $\hat{x} = \dfrac{a_1 b_2 + a_2 b_1}{a_1^2 + a_2^2}$

(D) $\hat{x} = \dfrac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2}$

(E) None of the above

ii. Suppose the inner product is defined instead as a non-Euclidean $\langle x, y \rangle = x^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} y$. Which of the following expressions must be true with respect to the minimized least squares error vector, $\vec{e}$?

(A) $\vec{e}^\top A = \vec{0}$

(B) $A^\top \vec{e} = \vec{0}$

(C) $A^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$

(D) $\left(A^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A \right)^{-1} \vec{e} = \vec{0}$

(E) None of the above

2. Eigenstuff

(a) You are provided the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$, and matrix $\mathbf{B} = \begin{bmatrix} 1-\alpha & 0.4 & 0.7 \\ 0 & 0.6-\alpha & 0.2 \\ 0 & 0 & 0.1-\alpha \end{bmatrix}$ where $\alpha \in \mathbb{R}$. If there exists a vector $\vec{x} \in \mathbb{R}^3$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, which of the following are true? (**Select all that apply.**)

(A) $\text{rank}(\mathbf{A}) = 3$

(B) $\vec{x}$ is in the null space of $\mathbf{B}$

(C) $\vec{x}$ is in an eigenspace of $\mathbf{B}$

(D) $\vec{x}$ is in an eigenspace of $\mathbf{A}$

(b) You are given that one of the eigenvalues of $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$ is $\lambda = 1$. Determine one possible eigenvector $\vec{v}$, corresponding to eigenvalue $\lambda = 1$.

(A) $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(B) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(C) $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

(D) $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

3. Orthogonal Space

Let $\vec{v}$ be a vector in $\mathbb{R}^2$, where $\mathbb{R}^2$ has an inner product. We define W to be the set of all vectors orthogonal to $\vec{v}$, i.e.

$$W = \{\vec{w} \mid \langle \vec{v}, \vec{w} \rangle = 0\} \tag{3}$$

(a) In the paragraph below, select the best choice for each blank to **complete the proof showing that** W **is a subspace**:

First, we need to show that the set contains the zero vector. We see that $\langle \vec{v}, \vec{0} \rangle = 0$, so this condition is fulfilled. Next, we need to show that the set (1)___________. Suppose we have $\vec{x}, \vec{y} \in W$, then (2)___________, so this condition is fulfilled. Finally, we need to show that the set (3)___________. Suppose we have $\alpha \in \mathbb{R}$ and $\vec{x} \in W$, then (4)___________, so this condition is fulfilled. Therefore the set is a valid subspace.

(1) (A) is closed under scalar multiplication
(B) is closed under vector addition
(C) is homogeneous
(D) is non-empty
(E) fulfills superposition

(2) (A) $\langle \vec{v}^\top \vec{x}, \vec{v}^\top \vec{y} \rangle = 0$
(B) $\langle \vec{v}, \vec{x} \rangle = \langle \vec{v}, \vec{y} \rangle$
(C) $\langle \vec{v} + \vec{x}, \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$
(D) $\langle \vec{v}, \vec{x} + \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$

(3) (A) is closed under scalar multiplication
(B) is closed under vector addition
(C) is homogeneous
(D) is non-empty
(E) fulfills superposition

(4) (A) $\langle \vec{v}, \alpha\vec{x} \rangle = \alpha\langle \vec{v}, \vec{x} \rangle = 0$
(B) $\langle \alpha\vec{v}, \alpha\vec{x} \rangle = \alpha\langle \vec{v}, \vec{x} \rangle = 0$
(C) $\langle \alpha\vec{v}^\top \vec{x}, \vec{0} \rangle = \alpha\langle \vec{v}^\top \vec{x}, \vec{0} \rangle = 0$
(D) $\alpha\langle \vec{v}, \vec{x} \rangle = \alpha \cdot 0$

(b) Now suppose the inner product is defined as $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top Q \vec{y}$ for $Q \in \mathbb{R}^{2\times 2}$.

i. If $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and we still define subspace W to be the set of all vectors that are orthogonal to $\vec{v}$ from part (a), which of the following options is a basis for W if the matrix $Q = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$?

(A) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(C) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(E) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

ii. What are the necessary properties for a valid, real inner product? **(Select all that apply.)**

(A) positive definiteness

(B) closed under scalar multiplication

(C) closed under vector addition

(D) quadratic

(E) linear

(F) non-empty

(G) symmetric

(H) contains the zero vector

iii. Which of the following choices of matrix Q results in a valid inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top Q \vec{y}$? (**Select all that apply.**)

(A) $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(E) None of the above

4. Tracking Terry

Terry is a mischievous child, and his mother is interested in tracking him.

(a) Terry texts his current location as a vector $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but there is a problem! These coordinates are *not* in the standard basis, but rather in the basis $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. That is to say that the first number 2 above is how many multiples of $\vec{v}_1$ to use and the second number 3 is how many multiples of $\vec{v}_2$ to use in computing his actual location. Here, both $\vec{v}_1$ and $\vec{v}_2$ are vectors in the standard basis.

Let Terry's location in the standard basis be $\vec{x}$. Write $\vec{x}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

(b) Terry's friend tells you that Terry's location in the standard basis is $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Using this along with the previous info that Terry's location in the V basis is $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, **is it possible to determine the basis vectors $\vec{v}_1, \vec{v}_2$ Terry is using. If it is impossible to do so, explain why.**
(HINT: How many unknowns do you have? How many equations?)

(c) Terry's basis vectors $\vec{v}_1, \vec{v}_2$ get leaked to his mom on accident, so she knows they are

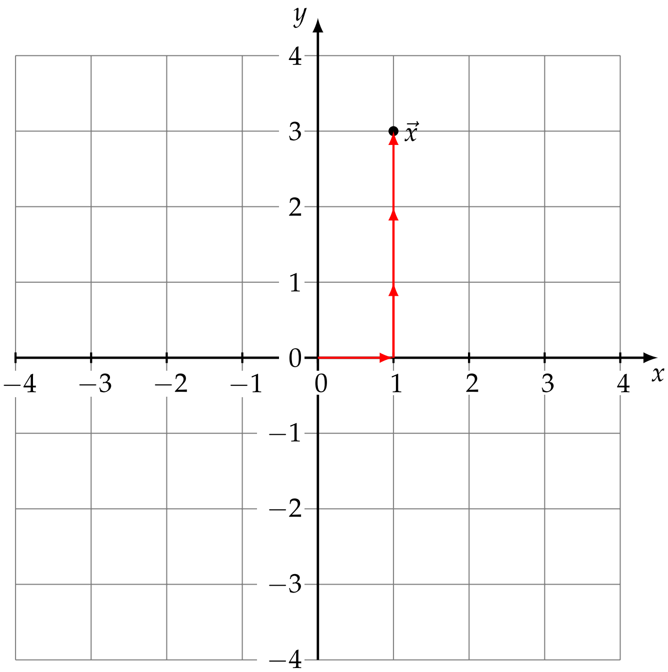
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}. \tag{4}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

In order to do this, he needs a way to convert coordinates from the V basis to the P basis. Thus, **find the matrix T such that if $\vec{x}_v$ is a location expressed in V coordinates and $\vec{x}_p$ is the same location expressed in P coordinates, then $\vec{x}_p = T\vec{x}_v$.**

(d) Terry now wants to make a map and route to where he currently is, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. **For both the P and V bases from part 4.c, illustrate the sum of scaled basis vectors that are necessary to go from the origin to $\vec{x}$.** An example is shown below when using the standard basis. This illustrates that the same location can be represented by many different coordinate systems/bases.


y
4
3
2
1
0
−1
−2
−3
−4
x
−4
−3
−2
−1
0
1
2
3
4
$\vec{x}$

5. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{6}$$

(HINT: What condition on a matrix's columns means that it would be invertible? It is fine to cite the appropriate result from 16A.)

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: Recall the definition of an eigenvector (i.e., $A\vec{v} = \lambda\vec{v}$). Then, recall what $U^{-1}U$ is. Lastly, consider how matrix multiplication works column-wise.)

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A \cdot A}_{k \text{ times}} \tag{7}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Conclude that A^k is diagonalizable.

6. Vector Differential Equations

Note: it's recommended to finish the previous question (Eigenvectors and Diagonalization) before this problem.

Consider a system of ordinary differential equations that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{8}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time t, and $A \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix with constant coefficients. We call eq. (8) a vector differential equation.

(a) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (8).

Consider a second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0, \tag{9}$$

where $a, b \in \mathbb{R}$.

Write this differential equation in the form of (eq. (8)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (9) without having to take a second derivative, and instead just taking the first derivative of something.)

(b) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{10}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by repeating the same steps in the previous question, and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now let $a = -1$ and $b = -2$ in eq. (9), i.e.

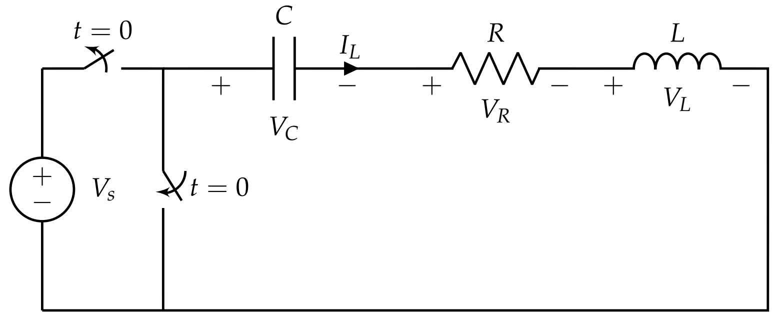
$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = 0, \tag{11}$$

Solve eq. (11) with the initial conditions $y(0) = 1$, $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$, using the general form in eq. (10).

(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (HINT: Given your specific choice of x_1 and x_2 in part (a), how many unknowns are there really?)

7. RLC Responses

Consider the following circuit:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 6 combined will try to show you the various RLC system responses and how they relate to changing circuit properties.

(a) We first need to construct our state space system. Our state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, **show that the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$ is**

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{12}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \tag{13}$$

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$.** This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

(c) **Show that, for the 2×2 matrix A, the two eigenvalues of A are**

$$\lambda_1 = -\frac{1}{2}\frac{R}{L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{14}$$
$$\lambda_2 = -\frac{1}{2}\frac{R}{L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{15}$$

(HINT: The quadratic formula will be involved.)

(d) **Under what condition on the circuit parameters R, L, C will A have two distinct real eigenvalues?**

(e) **Under what condition on the circuit parameters R, L, C will A have two imaginary eigenvalues? What will the eigenvalues be in this case?**

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, **show that the corresponding eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ are**

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \quad \text{and} \quad \vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}. \tag{16}$$

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{z}(t)$. **Show that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t)$ with**

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{17}$$

(HINT: Write out the original differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, then use the given change of coordinates to write everything in terms of $\vec{z}(t)$.)

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