Homework 3

This homework is due on Friday, September 16, 2022 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, September 23, 2022 at 11:59PM.

1. **Hambley P4.61**

A DC source is connected to a series RLC circuit by a switch that closes at $t = 0$, as shown in Figure 1. The initial conditions are $i(0+) = 0$ and $v_C(0+) = 0$. Write the differential equation for $v_C(t)$. Solve for $v_C(t)$ given that $R = 80\,\Omega$.

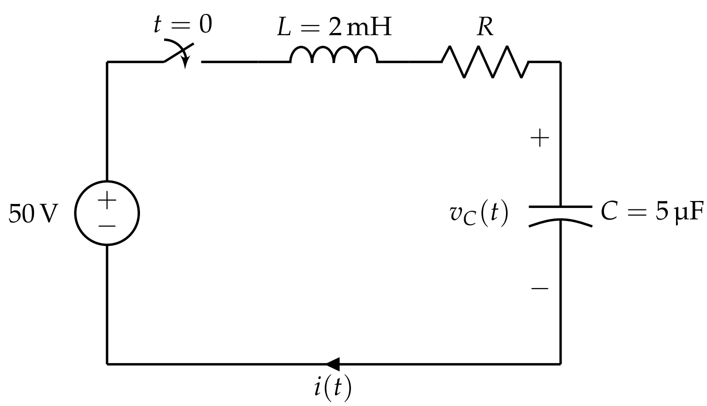


Figure 1: P4.61

Solution: We can apply KVL to the circuit to obtain:

$$50 = v_L(t) + v_R(t) + v_C(t) \tag{1}$$
$$= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + i(t)R + v_C(t) \tag{2}$$
$$= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) \tag{3}$$

Now, we have that $L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = L\frac{\mathrm{d}}{\mathrm{d}t}\left(C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}\right) = LC\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2}$. Plugging this in, we get

$$LC\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) = 50 \tag{4}$$
$$\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{LC}v_C(t) = \frac{50}{LC} \tag{5}$$

We have a second order differential equation, so our solution will be of the form $v_C(t) = v_{C_p}(t) + v_{C_C}(t)$, where $v_{C_p}(t)$ is the particular solution and $v_{C_C}(t)$ is the complementary solution. Here, we have a DC forcing function (i.e., $f(t)$ from eq 4.66 in the text is $\frac{50}{LC}$). Hence, the particular solution would be the solution if we replaced inductors with short circuits and capacitances with open circuits. This yields $v_{C_p}(t) = 50$. To find the damping ratio, we can pattern match $\alpha = \frac{R}{2L}$ and $\omega_0 = \sqrt{\frac{1}{LC}}$ from eq 4.67 in the text. Thus,

$$\xi = \frac{\alpha}{\omega_0} = \frac{\frac{R}{2L}}{\sqrt{\frac{1}{LC}}} = 2 \tag{6}$$

Since $\xi > 1$, the complementary solution will be of the form

$$v_{C_C}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{7}$$

where $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2679.49$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -37320.5$. Hence, the final solution is of the form

$$v_C(t) = v_{C_P}(t) + v_{C_C}(t) = 50 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{8}$$

To find K_1 and K_2, we can utilize the fact that $v_C(0) = 0$ and $\frac{dv_C(t)}{dt}|_{t=0} = \frac{i(0)}{C} = 0$. Plugging these in, we get the following system:

$$v_C(0) = 0 = 50 + K_1 + K_2 \tag{9}$$

$$\frac{dv_C(t)}{dt}|_{t=0} = 0 = s_1 K_1 + s_2 K_2 \tag{10}$$

Solving this system of equation yields $K_1 = -53.87$ and $K_2 = 3.87$. Hence, the final answer is

$$v_C(t) = 50 + (-53.87)e^{-2679.49t} + (3.87)e^{-37320.5t} \tag{11}$$

2. Hambley P4.64

Consider the circuit shown in Figure 2, with $R = 25\,\Omega$.

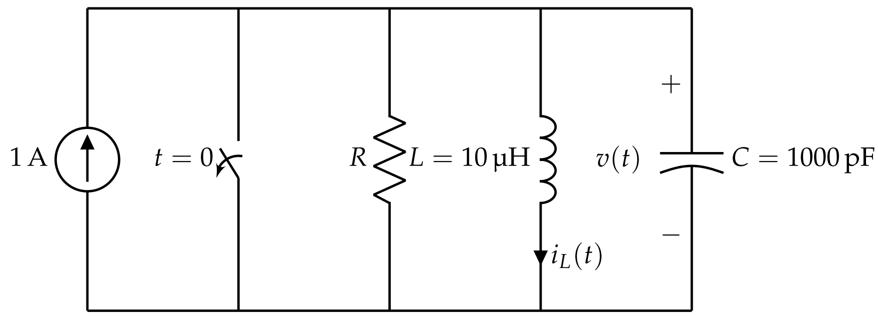


Figure 2: P4.64

(a) Compute the undamped resonant frequency, the damping coefficient, and the damping ratio.

Solution: From KCL, we have

$$1 = i_R(t) + i_L(t) + i_C(t) \tag{12}$$

$$= \frac{v(t)}{R} + i_L(t) + C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{13}$$

Taking derivatives on both sides, we get

$$C\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = 0 \tag{14}$$

We also know that $v(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$. Plugging this in, we get

$$C\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{L}v(t) = 0 \tag{15}$$

$$\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{LC}v(t) = 0 \tag{16}$$

Pattern matching to eq 4.67 in the text, we get $\alpha = \frac{1}{2RC} = 2 \times 10^7$ and $\omega_0 = \sqrt{\frac{1}{LC}} = 1 \times 10^7$. In this case, $\xi = \frac{\alpha}{\omega_0} = 2$ so it is an overdamped circuit (since $\xi > 1$).

(b) The initial conditions are $v(0+) = 0$ and $i_L(0+) = 0$. Show that this requires $v'(0+) = 10^9\,\frac{\mathrm{V}}{\mathrm{s}}$.

Solution: We still must satisfy KCL at $t = 0+$, so we have

$$1 = i_R(0+) + i_L(0+) + i_C(0+) \tag{17}$$

$$= \frac{v(0+)}{R} + i_L(0+) + Cv'(0+) \tag{18}$$

$$= Cv'(0+) \tag{19}$$

This leaves us with $v'(0+) = \frac{1}{C} = 10^9\,\frac{\mathrm{V}}{\mathrm{s}}$.

(c) Find the particular solution for $v(t)$.

Solution: To find the particular solution, we first notice that the forcing function is $f(t) = 0$ which is a constant. Hence, we can replace capacitors with open circuits and inductors with short circuits. If we were to do this, all of the current flows through the branch with the inductor and the particular solution is $v_P(t) = 0$.

(d) Find the general solution for $v(t)$, including the numerical values of all parameters.

Solution: Since $\zeta > 1$, the complementary solution will be of the form

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{20}$$

where $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2.68 \times 10^6$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -3.73 \times 10^7$. Since the particular solution $v_P(t) = 0$, we have that

$$v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{21}$$

Now, we will use our initial conditions of $v(0) = 0$ and $v'(0) = 10^9$. Plugging these in, we get the following system of equations:

$$v(0) = 0 = K_1 + K_2 \tag{22}$$

$$v'(0) = 10^9 = s_1 K_1 + s_2 K_2 \tag{23}$$

Solving the system of equations yields $K_1 = 28.89$ and $K_2 = -28.89$. Thus, the final answer is

$$v(t) = 28.89 e^{(-2.68 \times 10^6)t} - 28.89 e^{(-3.73 \times 10^7)t} \tag{24}$$

3. Hambley P5.15

Determine the rms value of $v(t) = A\cos(2\pi t) + B\sin(2\pi t)$.

Solution: The period here is $T = 1$, since $A\cos(2\pi(t+1)) + B\sin(2\pi(t+1)) = A\cos(2\pi t + 2\pi) + B\sin(2\pi t + 2\pi) = A\cos(2\pi t) + B\sin(2\pi t)$. Hence, we will compute the following integral:

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v(t)^2 \, \mathrm{d}t} \tag{25}$$

$$= \sqrt{\int_0^1 \left(A\cos(2\pi t) + B\sin(2\pi t)\right)^2 \mathrm{d}t} \tag{26}$$

$$= \sqrt{\int_0^1 \left(A^2\cos^2(2\pi t) + AB\cos(2\pi t)\sin(2\pi t) + B^2\sin^2(2\pi t)\right) \mathrm{d}t} \tag{27}$$

$$= \sqrt{\frac{A^2 + B^2}{2}} \tag{28}$$

4. Hambley P5.22

Suppose that $v_1(t) = 100\cos(\omega t)$ and $v_2(t) = 100\sin(\omega t)$. Use phasors to reduce the sum $v_s(t) = v_1(t) + v_2(t)$ to a single term of the form $V_m \cos(\omega t + \theta)$. Draw a phasor diagram, showing V_1, V_2, and V_s. State the phase relationships between each pair of these phasors.

Solution: Note that $v_2(t) = 100\cos\left(\omega t - \frac{\pi}{2}\right)$. As a result, we have that

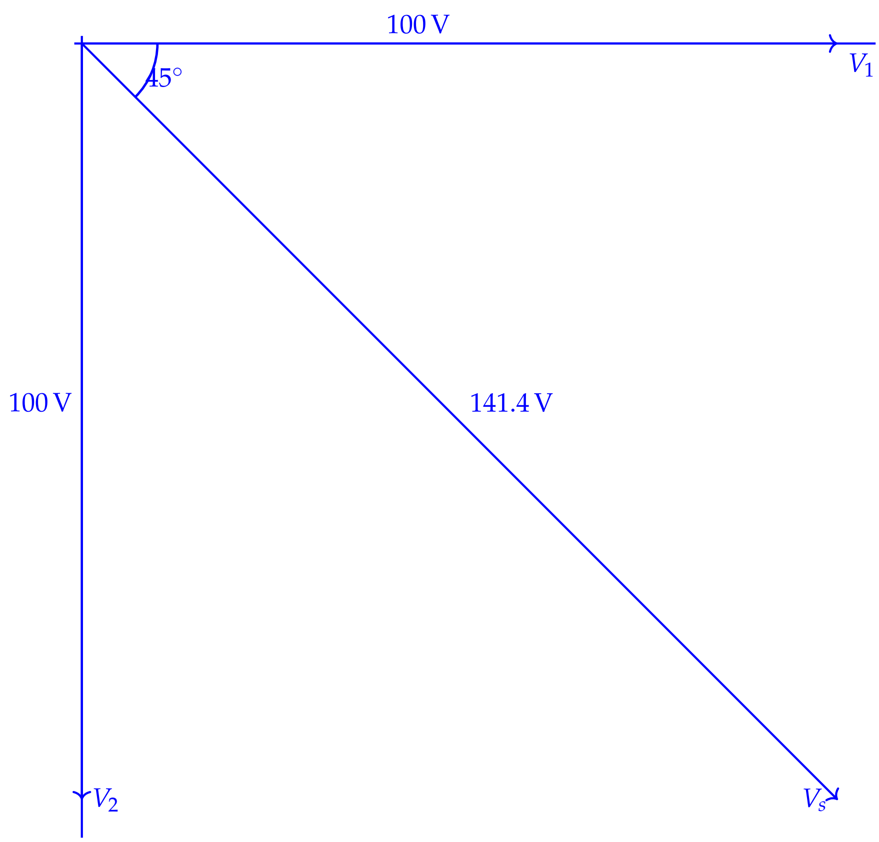
$$V_1 = 100 \tag{29}$$

$$V_2 = 100\angle(-90^\circ) = -\text{j}100 \tag{30}$$

which means $V_3 = 100 - \text{j}100$. Here, $\angle V_3 = \tan^{-1}(\frac{-100}{100}) = -45^\circ$ and $|V_3| = \sqrt{100^2 + 100^2} = 141.4$. Combined into phasor form, this yields $V_3 = 141.4\angle - 45^\circ$. Thus,

$$v_s(t) = 141.4\cos(\omega t - 45^\circ) \tag{31}$$

The phasor diagram is shown below:



In this case, V_2 lags V_1 by 90°, V_s lags V_1 by 45°, and V_s leads V_2 by 45°.

5. Hambley P5.23

Consider the phasors shown in Figure 3. The frequency of each signal is $f = 200\,\text{Hz}$. Write a time-domain expression for each voltage in the form $V_m \cos(\omega t + \theta)$. State the phase relationships between pairs of these phasors.

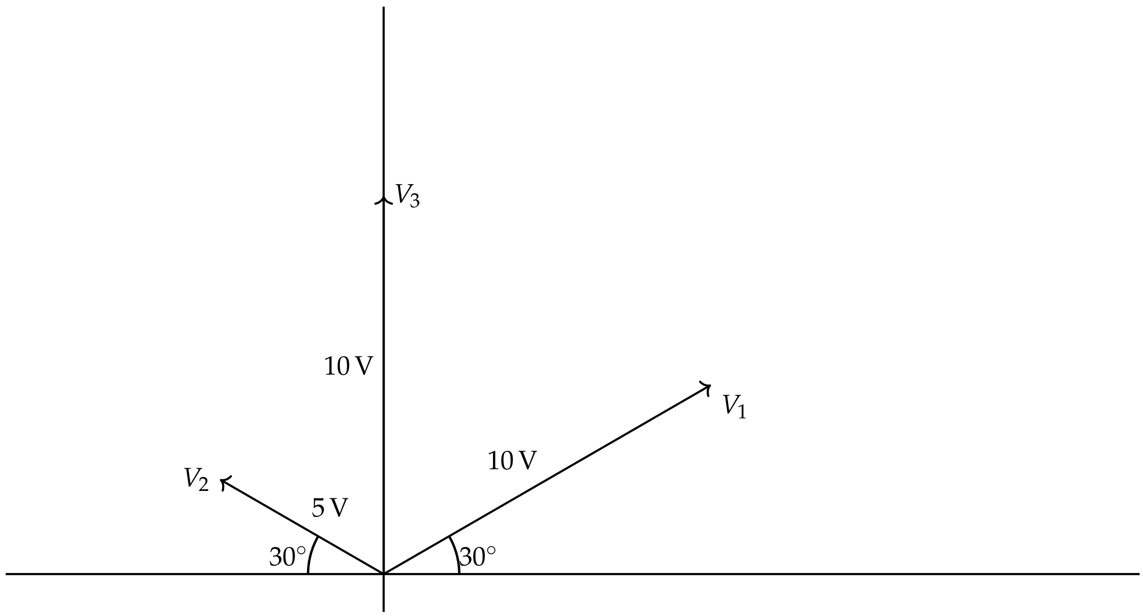


Figure 3: P5.23

Solution: We have that $\omega = 2\pi f = 400\pi$. Now, by reading the phase diagram, we have

$$v_1(t) = 10\cos(400\pi t + 30^\circ) \tag{32}$$

$$v_2(t) = 5\cos(400\pi t + 150^\circ) \tag{33}$$

$$v_3(t) = 10\cos(400\pi t + 90^\circ) \tag{34}$$

Here, $v_1(t)$ lags $v_2(t)$ by 120°, $v_1(t)$ lags $v_3(t)$ by 60°, and $v_2(t)$ leads $v_3(t)$ by 60°.

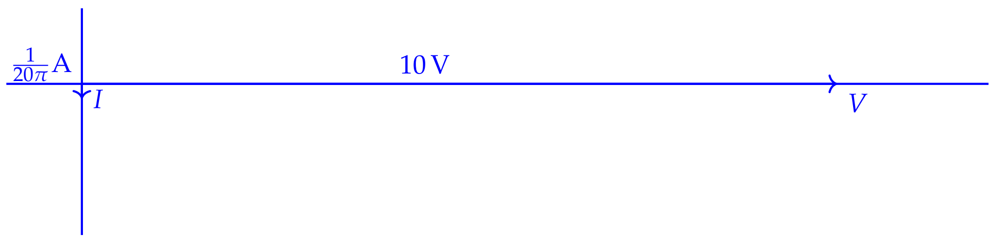
6. Hambley P5.32

A voltage $v_L(t) = 10\cos(2000\pi t)$ is applied to a 100 mH inductance. Find the complex impedance of the inductance. Find the phasor voltage and current, and construct a phasor diagram. Write the current as a function of time. Sketch the voltage and current to scale versus time. State the phase relationship between the current and voltage.

Solution: Note that $\omega = 2000\pi$. The complex impedance of the inductor is $Z_L = j\omega L = j200\pi$. The input phasor is $V_L = 10$, so the current is

$$I_L = \frac{V_L}{Z_L} = \frac{10}{j200\pi} = -j\frac{1}{20\pi} = \frac{1}{20\pi}\angle -90^\circ \tag{35}$$

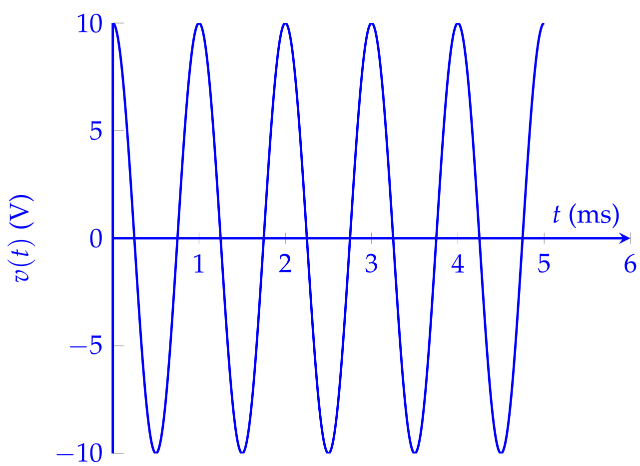
A phasor diagram is shown below:

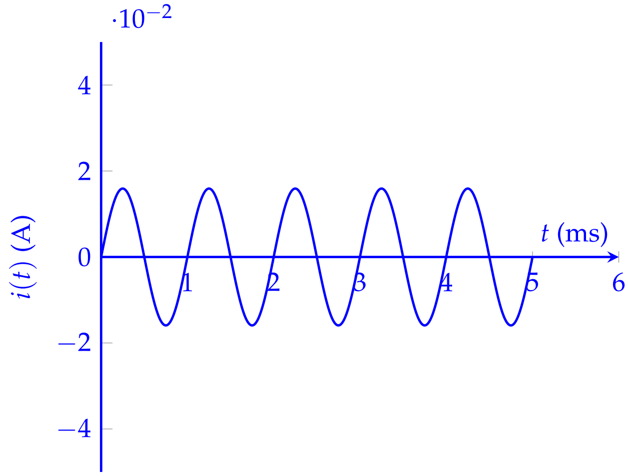


Converting this back into time domain, we get

$$i_L(t) = \frac{1}{20\pi}\cos(2000\pi t - 90^\circ) = \frac{1}{20\pi}\sin(2000\pi t) \tag{36}$$

The plots of $v_L(t)$ and $i_L(t)$ are shown below:





Here, $i_L(t)$ lags $v_L(t)$ by 90°.

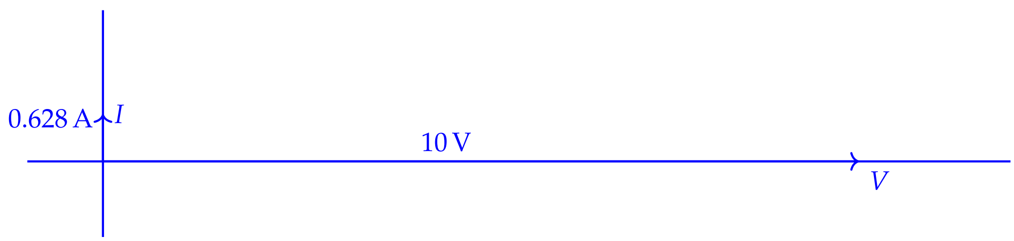
7. Hambley P5.33

A voltage $v_C(t) = 10\cos(2000\pi t)$ is applied to a $10\,\mu$F capacitance. Find the complex impedance of the capacitance. Find the phasor voltage and current, and construct a phasor diagram. Write the current as a function of time. Sketch the voltage and current to scale versus time. State the phase relationship between the current and voltage.

Solution: Here, $\omega = 2000\pi$. The complex impedance is $Z_C = -\frac{j}{\omega C} = -j15.92$. Since $V_C = 10$, we have that

$$I_C = \frac{V_C}{Z_C} = j0.63 = 0.63\angle 90^\circ \tag{37}$$

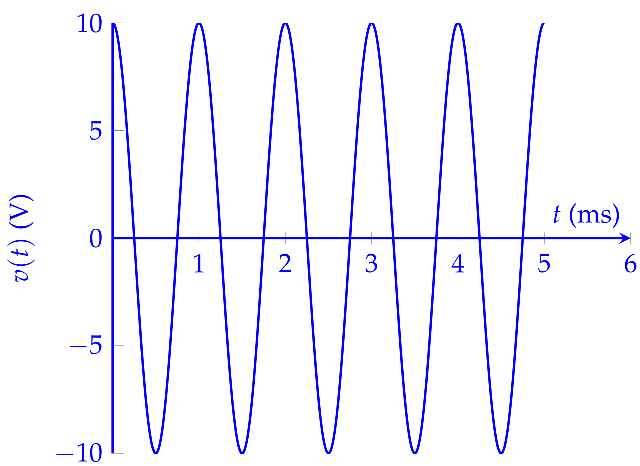
A phasor diagram is shown below:

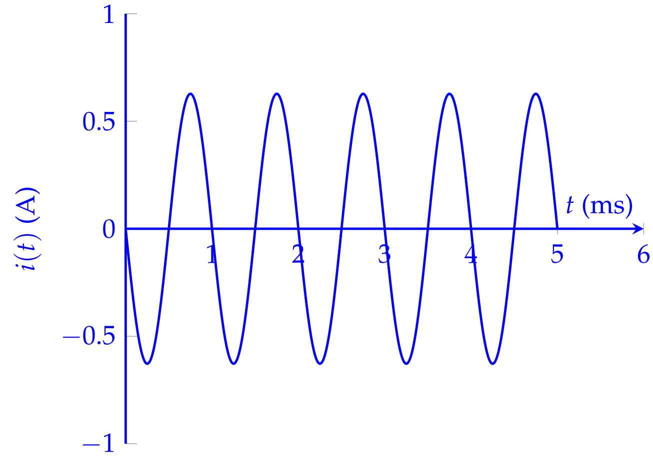


Converting the current phasor into time domain, we get

$$i_C(t) = 0.63\cos(2000\pi t + 90^\circ) = -0.63\sin(2000\pi t) \tag{38}$$

The plots of $v_C(t)$ and $i_C(t)$ are shown below:





Here, $i_C(t)$ leads $v_C(t)$ by 90°.

8. Hambley P5.38

Find the phasors for the current and the voltages for the circuit shown in Figure 4. Construct a phasor diagram showing V_s, I, V_R, and V_L. What is the phase relationship between V_s and I?

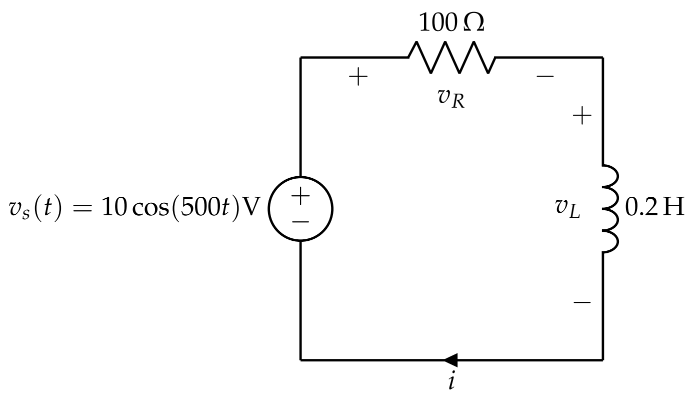


Figure 4: P5.38

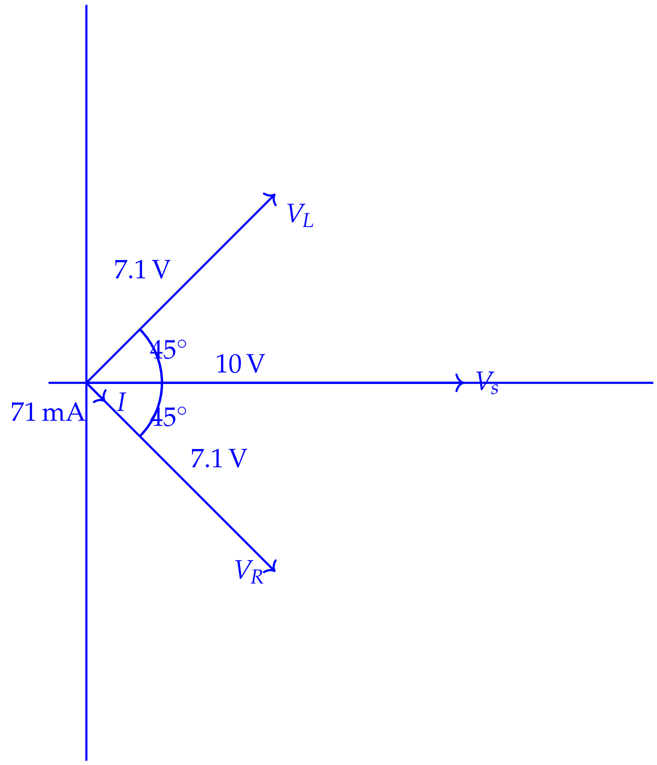
Solution: Here, $V_s = 10$ and $\omega = 500$. Using Ohm's law and plugging this in, we have

$$I = \frac{V_s}{R + j\omega L} \tag{39}$$
$$= \frac{10}{100 + j100} \tag{40}$$
$$= \frac{10}{\sqrt{100^2 + 100^2}\angle 45^\circ} \tag{41}$$
$$= \frac{10}{141.4\angle 45^\circ} \tag{42}$$
$$= 0.071\angle - 45^\circ \tag{43}$$

Now, to find V_R and V_L, we can again apply Ohm's law:

$$V_R = I \times R = 7.1\angle - 45^\circ \tag{44}$$
$$V_L = I \times j\omega L = 7.1(\angle 90^\circ)(\angle - 45^\circ) = 7.1\angle 45^\circ \tag{45}$$

A phase diagram is shown below:



From these calculations, we see that I lags V_s by 45°.

