

# EECS 16B    Designing Information Devices and Systems II

## Fall 2021    Discussion Worksheet    Discussion 12A

The following notes are useful for this discussion: [Note 19](#)

### 1. Jacobians and Linear Approximation

Recall that for a scalar-valued function  $f(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$  with vector-valued arguments, we can linearize the function at  $(\vec{x}_*, \vec{y}_*)$

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_*, \vec{y}_*) + (D_{\vec{x}}f)\Big|_{(\vec{x}_*, \vec{y}_*)} \cdot (\vec{x} - \vec{x}_*) + (D_{\vec{y}}f)\Big|_{(\vec{x}_*, \vec{y}_*)} \cdot (\vec{y} - \vec{y}_*). \quad (1)$$

where

$$D_{\vec{x}}f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \quad (2)$$

$$D_{\vec{y}}f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \quad (3)$$

- (a) When the function  $\vec{f}(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^m$  takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in  $\vec{f}$  independently as a separate function  $f_i$ , and linearize each of them:

$$\vec{f}(\vec{x}, \vec{y}) = \begin{bmatrix} f_1(\vec{x}, \vec{y}) \\ f_2(\vec{x}, \vec{y}) \\ \vdots \\ f_m(\vec{x}, \vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_*, \vec{y}_*) + D_{\vec{x}}f_1 \cdot (\vec{x} - \vec{x}_*) + D_{\vec{y}}f_1 \cdot (\vec{y} - \vec{y}_*) \\ f_2(\vec{x}_*, \vec{y}_*) + D_{\vec{x}}f_2 \cdot (\vec{x} - \vec{x}_*) + D_{\vec{y}}f_2 \cdot (\vec{y} - \vec{y}_*) \\ \vdots \\ f_m(\vec{x}_*, \vec{y}_*) + D_{\vec{x}}f_m \cdot (\vec{x} - \vec{x}_*) + D_{\vec{y}}f_m \cdot (\vec{y} - \vec{y}_*) \end{bmatrix} \quad (4)$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} D_{\vec{x}}f_1 \\ D_{\vec{x}}f_2 \\ \vdots \\ D_{\vec{x}}f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \quad (5)$$

and similarly

$$D_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \quad (6)$$

Then, the linearization becomes

$$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\vec{x}_*, \vec{y}_*) + (D_{\vec{x}}\vec{f})\Big|_{(\vec{x}_*, \vec{y}_*)} \cdot (\vec{x} - \vec{x}_*) + (D_{\vec{y}}\vec{f})\Big|_{(\vec{x}_*, \vec{y}_*)} \cdot (\vec{y} - \vec{y}_*). \quad (7)$$

Let  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  and  $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$ . Find  $D_{\vec{x}} \vec{f}$ , applying the definition above.

**Solution:** Here, we have

$$D_{\vec{x}} \vec{f} = \begin{bmatrix} 2x_1 x_2 & x_1^2 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}. \quad (8)$$

(b) Evaluate the approximation of  $\vec{f}$  using  $\vec{x}_* = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  at the point  $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$ , and compare with  $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$ .

Recall the definition that  $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$ .

**Solution:** Let  $\delta = 0.01$ . The true value is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} (2 + \delta)^2 (3 + \delta) \\ (2 + \delta)(3 + \delta)^2 \end{bmatrix} = \begin{bmatrix} 12 + 16\delta + 7\delta^2 + \delta^3 \\ 18 + 21\delta + 8\delta^2 + \delta^3 \end{bmatrix}. \quad (9)$$

On the other hand, our approximation is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) \approx \vec{f}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + \begin{bmatrix} 12 & 4 \\ 9 & 12 \end{bmatrix} \cdot \begin{bmatrix} \delta \\ \delta \end{bmatrix} = \begin{bmatrix} 12 + 16\delta \\ 18 + 21\delta \end{bmatrix}. \quad (10)$$

Again, our approximation essentially removes the higher order terms of  $\delta$ .

When we plug in  $\delta = 0.01$ , we have

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} 12.160701 \\ 18.210801 \end{bmatrix} \quad (11)$$

and our approximation is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) \approx \begin{bmatrix} 12.16 \\ 18.21 \end{bmatrix}. \quad (12)$$

(c) Let  $\vec{x}$  and  $\vec{y}$  be vectors with 2 rows, and let  $\vec{w}$  be another vector with 2 rows. Let  $\vec{f}(\vec{x}, \vec{y}) = \vec{x} \vec{y}^T \vec{w}$ . Find  $D_{\vec{x}} \vec{f}$  and  $D_{\vec{y}} \vec{f}$ .

**Solution:** Here, recall that

$$\vec{f} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 & y_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 w_1 + x_1 y_2 w_2 \\ x_2 y_1 w_1 + x_2 y_2 w_2 \end{bmatrix}. \quad (13)$$

Then,

$$D_{\vec{x}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} y_1 w_1 + y_2 w_2 & 0 \\ 0 & y_1 w_1 + y_2 w_2 \end{bmatrix} \quad (14)$$

and

$$D_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} x_1 w_1 & x_1 w_2 \\ x_2 w_1 & x_2 w_2 \end{bmatrix}. \quad (15)$$

We can also write

$$D_{\vec{x}}\vec{f} = \vec{y}^\top \vec{w} \cdot I \quad (16)$$

and

$$D_{\vec{y}}\vec{f} = \vec{x}\vec{w}^\top, \quad (17)$$

which can be derived by noticing that  $\vec{y}^\top \vec{w} = \vec{w}^\top \vec{y}$ .

- (d) Continuing the above part, **find the linear approximation of  $\vec{f}$  near  $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and with  $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .**

**Solution:** We have

$$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\vec{x}_*, \vec{y}_*) + D_{\vec{x}}\vec{f} \cdot (\vec{x} - \vec{x}_*) + D_{\vec{y}}\vec{f} \cdot (\vec{y} - \vec{y}_*) \quad (18)$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 - 1 \\ y_2 - 1 \end{bmatrix} \quad (19)$$

$$(20)$$

Let's do an approximation of  $\vec{f}\left(\begin{bmatrix} 1 + \delta_1 \\ 1 + \delta_2 \end{bmatrix}, \begin{bmatrix} 1 + \delta_3 \\ 1 + \delta_4 \end{bmatrix}\right)$ , then,

$$\vec{f}\left(\begin{bmatrix} 1 + \delta_1 \\ 1 + \delta_2 \end{bmatrix}, \begin{bmatrix} 1 + \delta_3 \\ 1 + \delta_4 \end{bmatrix}\right) \approx \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 3 + 3\delta_1 + 2\delta_3 + \delta_4 \\ 3 + 3\delta_2 + 2\delta_3 + \delta_4 \end{bmatrix}.$$

We can compare with the true value

$$\begin{aligned} \vec{f}\left(\begin{bmatrix} 1 + \delta_1 \\ 1 + \delta_2 \end{bmatrix}, \begin{bmatrix} 1 + \delta_3 \\ 1 + \delta_4 \end{bmatrix}\right) &= \begin{bmatrix} 1 + \delta_1 \\ 1 + \delta_2 \end{bmatrix} \begin{bmatrix} 1 + \delta_3 & 1 + \delta_4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \delta_1 \\ 1 + \delta_2 \end{bmatrix} (3 + 2\delta_3 + \delta_4) \\ &= \begin{bmatrix} 3 + 3\delta_1 + 2\delta_3 + \delta_4 + 2\delta_1\delta_3 + \delta_1\delta_4 \\ 3 + 3\delta_2 + 2\delta_3 + \delta_4 + 2\delta_2\delta_3 + \delta_2\delta_4 \end{bmatrix}, \end{aligned} \quad (21)$$

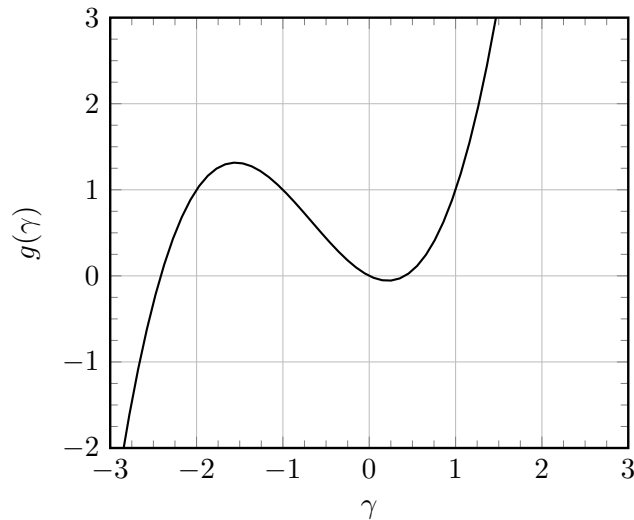
and we see that our approximation removes the second order  $\delta$  terms  $\delta_1\delta_3$ ,  $\delta_1\delta_4$ ,  $\delta_2\delta_3$  and  $\delta_2\delta_4$ .

## 2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{d}{dt} \begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix} = \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} -2\beta(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix} = \vec{f}(\vec{x}(t), u(t)) \quad (22)$$

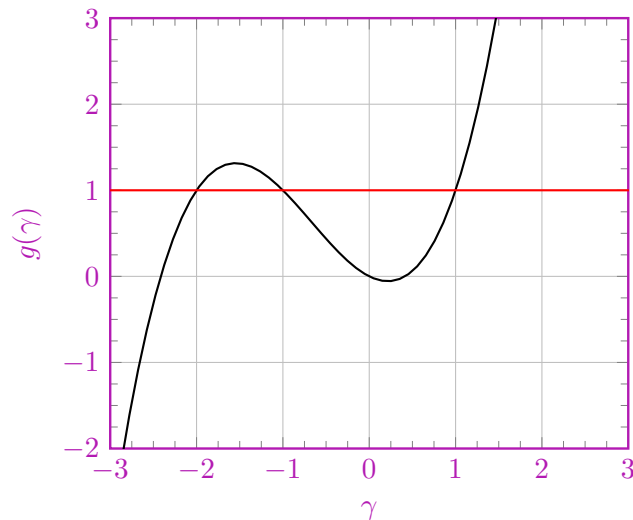
where  $\vec{x}(t) = \begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix}$  and  $g(\cdot)$  is a nonlinear function with the following graph:



The  $g(\cdot)$  is the only nonlinearity in this system. We want to linearize this entire system around a operating point/equilibrium. Any point  $x_*$  is an operating point if  $\frac{d}{dt} \vec{x}(t) = \vec{0}$ .

(a) **If we have fixed  $u_*(t) = -1$ , what values of  $\gamma$  and  $\beta$  will ensure  $\frac{d}{dt} \vec{x}(t) = \vec{0}$ ?**

**Solution:** To find the equilibrium point, we'll start by finding the values for which  $g(\gamma) + u^* = g(\gamma) - 1 = 0$ . In other words, we need to find values of  $\gamma$  such that  $g(\gamma) = 1$ . Although we don't have an equation for  $g(\gamma)$ , we can still find these points *graphically*, by using our graph. If we add a horizontal line at  $g(\gamma) = 1$ , we get the following:



Having done this, it looks like we'll have  $f_2(\vec{x}, u^*) = g(\gamma) - u^* = 0$  for  $\gamma = -2, \gamma = -1$ , and  $\gamma = 1$ . Now we just need to find an  $\beta$  that sets  $f_1(\vec{x}, u^*) = -2\beta + \gamma = 0$  for each of these. Setting  $\beta = \frac{1}{2} \cdot \gamma$  will do this.

With that, we have our three equilibrium points, namely

$$\vec{x}_1^* = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \quad \vec{x}_2^* = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix} \quad \vec{x}_3^* = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}. \quad (23)$$

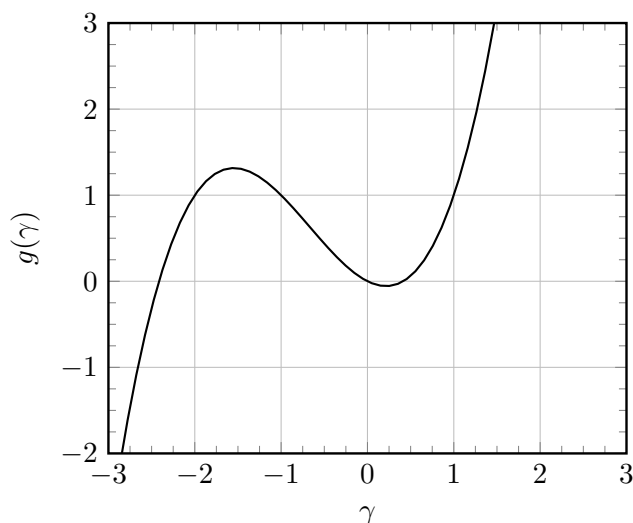
- (b) Now that you have the three operating points, **linearize the system about the operating point**  $(\vec{x}_3^*, u_*)$  **that has the largest value for**  $\gamma$ . Specifically, what we want is as follows. Let  $\delta \vec{x}_i(t) = \vec{x}(t) - \vec{x}_i^*$  for  $i = 1, 2, 3$ , and  $\delta u(t) = u(t) - u_*$ . We can in principle write the *linearized system* for each operating point in the following form:

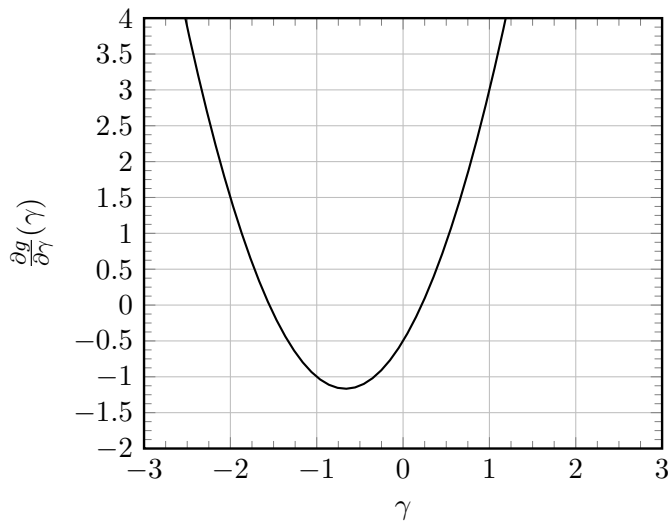
$$\text{(linearization about } (\vec{x}_i^*, u_*)) \quad \frac{d}{dt} \delta \vec{x}_i(t) = A_i \delta \vec{x}_i(t) + B_i \delta u(t) + \vec{w}_i(t) \quad (24)$$

where  $\vec{w}_i(t)$  is a disturbance that also includes the approximation error due to linearization.

For this part, **find**  $A_i$  **and**  $B_i$ .

We have provided below the function  $g(\gamma)$  and its derivative  $\frac{\partial g}{\partial \gamma}$ .





**Solution:** To linearize the system, we need to compute the two Jacobians

$$D_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \beta} & \frac{\partial f_1}{\partial \gamma} \\ \frac{\partial f_2}{\partial \beta} & \frac{\partial f_2}{\partial \gamma} \end{bmatrix} \quad (25)$$

$$D_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}, \quad (26)$$

and evaluate them at the operating points that we found in the previous part. The Jacobian matrices evaluated at the operating points will be the  $A_i$  and  $B_i$  matrices.

If we work out the partial derivatives, we get

$$\frac{\partial f_1}{\partial \beta} = \frac{\partial}{\partial \beta}(-2\beta + \gamma) = -2 \quad (27)$$

$$\frac{\partial f_1}{\partial \gamma} = \frac{\partial}{\partial \gamma}(-2\beta + \gamma) = 1 \quad (28)$$

$$\frac{\partial f_2}{\partial \beta} = \frac{\partial}{\partial \beta}(g(\gamma) + u) = 0 \quad (29)$$

$$\frac{\partial f_2}{\partial \gamma} = \frac{\partial}{\partial \gamma}(g(\gamma) + u) = \frac{\partial g}{\partial \gamma} \quad (30)$$

$$\frac{\partial f_1}{\partial u} = \frac{\partial}{\partial u}(-2\beta + \gamma) = 0 \quad (31)$$

$$\frac{\partial f_2}{\partial u} = \frac{\partial}{\partial u}(g(\gamma) + u) = 1, \quad (32)$$

$$(33)$$

which gives

$$D_{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & \frac{\partial g}{\partial \gamma} \end{bmatrix} \quad (34)$$

$$D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (35)$$

It turns out that the only part of  $D_{\bar{x}}$  and  $D_u$  that depends on the operating point is  $\partial g / \partial \gamma$ , and we can read these off of the given graph. The relevant values are

$$\left. \frac{\partial g}{\partial \gamma} \right|_{\gamma=-2} = 1.5 \quad (36)$$

$$\left. \frac{\partial g}{\partial \gamma} \right|_{\gamma=-1} = -1 \quad (37)$$

$$\left. \frac{\partial g}{\partial \gamma} \right|_{\gamma=2} = 3, \quad (38)$$

which correspond to  $\bar{x}_1^*$ ,  $\bar{x}_2^*$ , and  $\bar{x}_3^*$ , respectively. Finally, this gives

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (39)$$

$$A_2 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (40)$$

$$A_3 = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (41)$$

(42)

(c) **Which of the operating points are *stable*? Which are *unstable*?**

**Solution:** To assess the stability or instability of each operating point, we need to find the eigenvalues of each linearization. Since  $A_1$ ,  $A_2$ , and  $A_3$  are all *upper triangular*, their eigenvalues are just the two entries along their diagonals. So, the linearization will be stable if both diagonal entries are negative (remember, these are *continuous-time* systems), and unstable if they aren't both negative. This means that:

- $\bar{x}_1^*$  is *unstable*, since the eigenvalues of  $A_1$  are  $-2$  and  $1.5$ ;
- $\bar{x}_2^*$  is *stable*, since the eigenvalues of  $A_2$  are  $-2$  and  $-1$ ;
- $\bar{x}_3^*$  is *unstable*, since the eigenvalues of  $A_3$  are  $-2$  and  $3$ .

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