EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 11B

The following notes are useful for this discussion: Note 19

1. Linear Approximation

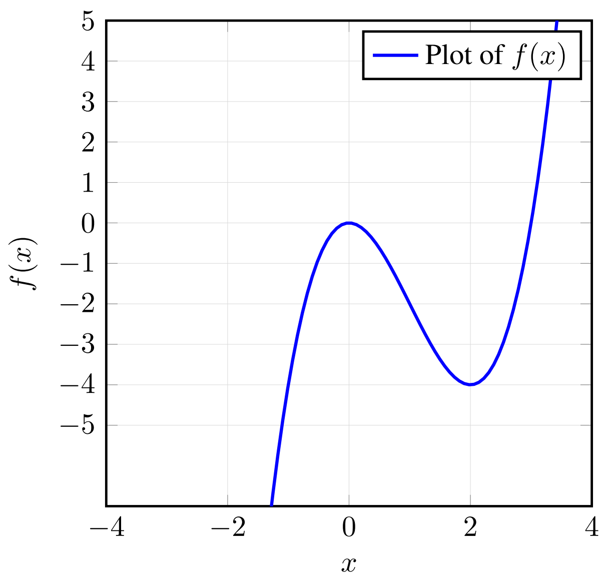
A common way to approximate a nonlinear function is to perform linearization near a point. In the case of a one-dimensional function $f(x)$, the linear approximation of $f(x)$ at a point $x_\star$ is given by

$$f(x) \approx f(x_\star) + f'(x_\star) \cdot (x - x_\star), \tag{1}$$

where $f'(x_\star) := \frac{\mathrm{d}f}{\mathrm{d}x}(x_\star)$ is the derivative of $f(x)$ at $x = x_\star$.

Keep in mind that wherever we see $x_\star$, this denotes a *constant value* or operating point.

(a) Suppose we have the single-variable function $f(x) = x^3 - 3x^2$. We can plot the function $f(x)$ as follows:



i. **Write the linear approximation of the function around an arbitrary point $x_\star$.**

Solution:

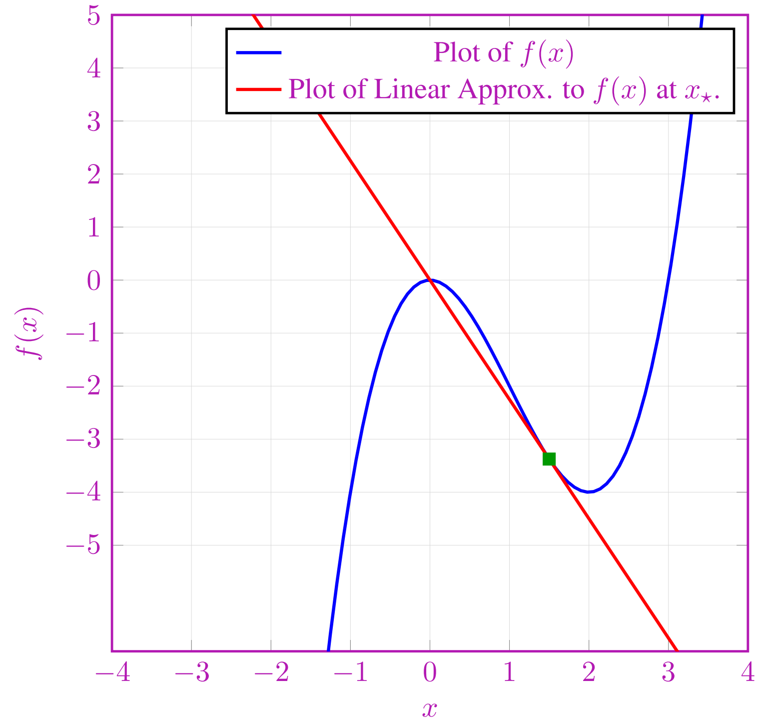
$$f(x) \approx f(x_\star) + f'(x_\star) \cdot (x - x_\star) \tag{2}$$
$$= f(x_\star) + (3x_\star^2 - 6x_\star) \cdot (x - x_\star) \tag{3}$$

ii. Using the expression above, **linearize the function around the point $x = 1.5$. Draw the linearization into the plot of part i).**

Solution:

$$f(x) \approx f(1.5) + \left(3 \cdot 1.5^2 - 6 \cdot 1.5\right) \cdot (x - 1.5) \tag{4}$$

$$\approx -3.375 + (-2.25) \cdot (x - 1.5) \tag{5}$$



Now that we have this specific point's linearization, we understand how the function behaves around the point. Let's use this linearization to evaluate the function's approximation at $x = 1.7$ (based on our approximation at $x = 1.5$, we want to see how a $\delta = +0.2$ shift in the x value changes the corresponding $f(x)$ value). How does this approximation compare to the exact value of the function at $x = 1.7$?

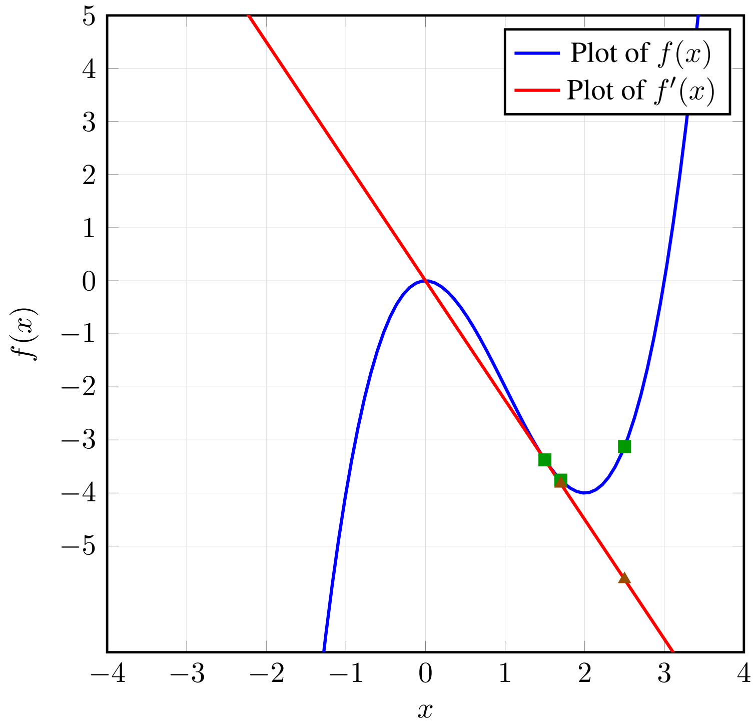
$$f(1.7) \approx -3.375 + (-2.25) \cdot (1.7 - 1.5) \tag{6}$$
$$\approx -3.375 - 0.45 \tag{7}$$
$$\approx -3.825 \tag{8}$$

Comparing to the exact value $f(1.7) = 1.7^3 - 3 \cdot 1.7^2 = -3.757$, we find that the difference is 0.068. Not too bad! What if we repeat with $\delta = 1$? To do so, we must use the approximation around $x = 1.5$ to compute $x = 2.5$, and compare to the exact value $f(2.5)$. How does our new approximation compare to the exact result?

$$f(2.5) \approx -3.375 + (-2.25) \cdot (2.5 - 1.5) \tag{9}$$
$$\approx -3.375 - 2.25 \tag{10}$$
$$\approx -5.625 \tag{11}$$

Comparing to the exact value $f(2.5) = 2.5^3 - 3 \cdot 2.5^2 = -3.125$, we find that the difference is much larger; the error jumped to 2.5! This is an error multiplication of $\frac{2.5}{0.068} \approx 37$, even though our δ only multiplied by 5. What happened?

Looking at the actual function, we see that the function has a significant curvature between our "anchor point" of $x_\star = 1.5$ and $x = 2.5$. Our linear model is unable to capture this curvature, and so we estimated $f(2.5)$ as if the function kept decreasing, as it did around $x = 1.5$ (where the slope was -2.25).



Now, we can extend this to higher dimensional functions. In the case of a two-dimensional function $f(x, y)$, the linear approximation of $f(x, y)$ at a point $(x_\star, y_\star)$ is given by

$$f(x, y) \approx f(x_\star, y_\star) + \frac{\partial f}{\partial x}(x_\star, y_\star) \cdot (x - x_\star) + \frac{\partial f}{\partial y}(x_\star, y_\star) \cdot (y - y_\star). \tag{12}$$

where $\frac{\partial f}{\partial x}(x_\star, y_\star)$ is the partial derivative of $f(x, y)$ with respect to x at the point $(x_\star, y_\star)$, and similarly for $\frac{\partial f}{\partial y}(x_\star, y_\star)$

(b) Now, let's see how we can find partial derivatives. When we are given a function $f(x, y)$, we calculate the partial derivative of f with respect to x by fixing y and taking the derivative with respect to x. **Given the function $f(x, y) = x^2 y$, find the partial derivatives $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$.**

Solution: We have

$$\frac{\partial f(x, y)}{\partial x} = 2xy \tag{13}$$

$$\frac{\partial f(x, y)}{\partial y} = x^2. \tag{14}$$

(c) **Write out the linear approximation of f near $(x_\star, y_\star)$.**

Solution: Based on the formula in eq. (12), we can write that:

$$f(x, y) \approx f(x_\star, y_\star) + 2x_\star y_\star \cdot (x - x_\star) + x_\star^2 \cdot (y - y_\star). \tag{15}$$

(d) We want to see if the approximation arising from linearization of this function is reasonable for a point close to our point of evaluation. **First, approximate $f(x,y)$ at the point $(2.01, 3.01)$ using $(x_\star, y_\star) = (2,3)$. Next, compare the result to $f(2.01, 3.01)$.**

Solution: Let $\delta = 0.01$. Then, the true value of $f(2.01, 3.01)$ is

$$f(2.01, 3.01) = (2+\delta)^2(3+\delta) = (4+4\delta+\delta^2)(3+\delta) = 12 + 16\delta + 7\delta^2 + \delta^3. \tag{16}$$

On the other hand, our approximation is

$$f(2.01, 3.01) \approx f(2,3) + 2 \cdot 2 \cdot 3 \cdot \delta + 2^2 \cdot \delta = 12 + 16\delta. \tag{17}$$

As we can see, our approximation removes the terms with δ^2 and δ^3. When δ is sufficiently small, these terms become very small, and hence our approximation is reasonable.

The actual numerical values are:

$$\begin{aligned} f(2,3) &= 12 \\ f(2.01, 3.01) &\approx 12.16 && \text{(using linearization)} \\ f(2.01, 3.01) &= 12.160701 && \text{(exact evaluation of} f\text{)} \end{aligned}$$

(e) We will now define the notion of a derivative as a function, and take a look at one possible representation of that function.

Given the representation of the derivative as a row-vector, describe a function that can take this representation, along with some column vector defining a change in direction, to return a scalar value (which is the change in the real-valued output). *Don't worry if this seems abstract for now, the next subpart will clarify.*

Solution: We can perform a matrix multiplication between the row vector and column vector to produce a 1×1 matrix, which we treat as a scalar. Specifically, $1 \times k \times k \times 1 = 1 \times 1$.

(f) Suppose we have now a scalar-valued function $f(\vec{x}, \vec{y})$, which takes in vector-valued arguments $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^k$ and outputs a scalar $\in \mathbb{R}$. That is, $f(\vec{x}, \vec{y})$ is $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$. For this new model involving a vector-valued function, how can we adapt our previous linearization method?

One way to linearize the function f is to do it for every single element in $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top$ and $\vec{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_k \end{bmatrix}^\top$. Then, when we are looking at x_i or y_j, we fix everything else as constant. This would give us the linear approximation

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_\star, \vec{y}_\star) + \sum_{i=1}^{n} \frac{\partial f(\vec{x}, \vec{y})}{\partial x_i}(x_i - x_{i,\star}) + \sum_{j=1}^{k} \frac{\partial f(\vec{x}, \vec{y})}{\partial y_j}(y_j - y_{j,\star}). \tag{18}$$

In order to simplify this equation, we can define the rows $D_{\vec{x}}$ and $D_{\vec{y}}$ as

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \tag{19}$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \dots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \tag{20}$$

Then, eq. (18) can be rewritten as

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{21}$$

Assume that $n = k$ and we define the function $f(\vec{x}, \vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x_i y_i$. Find $D_{\vec{x}} f$ and $D_{\vec{y}} f$.

[Practice] Next, suppose $g(\vec{x}, \vec{y}) = x_1 x_2^2 y_1 + x_1 y_2^3 + x_2 x_1 y_2 y_1 + \frac{x_1^2}{x_2^3 y_1}$. Find $D_{\vec{x}} g$ and $D_{\vec{y}} g$

Hint: it can help to look at eq. (12), and match the terms in eq. (18) to that formulation.

Solution: The derivative is a *row* vector (as denoted above), so if we apply the definition (and write out the given function explicitly as $x_1 y_1 + x_2 y_2 + \ldots + x_k y_k$), we have:

$$D_{\vec{x}} f = \vec{y}^\top \tag{22}$$

and

$$D_{\vec{y}} f = \vec{x}^\top. \tag{23}$$

For the second (more difficult) example, we can similarly compute:

$$\frac{\partial g(\vec{x}, \vec{y})}{\partial x_1} = x_2^2 y_1 + y_2^3 + x_2 y_1 y_2 + 2\frac{x_1}{x_2^3 y_1} \tag{24}$$

$$\frac{\partial g(\vec{x}, \vec{y})}{\partial x_2} = 2x_1 x_2 y_1 + x_1 y_1 y_2 - 3\frac{x_1^2}{x_2^4 y_1} \tag{25}$$

$$\frac{\partial g(\vec{x},\vec{y})}{\partial y_1} = x_1x_2^2 + x_1x_2y_2 - \frac{x_2^2}{x_2^3y_1^2} \tag{26}$$

$$\frac{\partial g(\vec{x},\vec{y})}{\partial y_2} = 3x_1y_2^2 + x_1x_2y_1 \tag{27}$$

Compiling these into derivative (row) vectors:

$$D_{\vec{x}}g = \begin{bmatrix} x_2^2y_1 + y_2^3 + x_2y_1y_2 + 2\frac{x_1}{x_2^3y_1} & 2x_1x_2y_1 + x_1y_1y_2 - 3\frac{x_1^2}{x_2^4y_1} \end{bmatrix} \tag{28}$$

$$D_{\vec{y}}g = \begin{bmatrix} x_1x_2^2 + x_1x_2y_2 - \frac{x_2^2}{x_2^3y_1^2} & 3x_1y_2^2 + x_1x_2y_1 \end{bmatrix} \tag{29}$$

(g) Following the above part, **find the linear approximation of $f(\vec{x}, \vec{y})$ near $\vec{x}_\star = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y}_\star = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.**

Recall that $f(\vec{x}, \vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x_i y_i$.

Solution: From the solution in the previous part, we can write

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star) \tag{30}$$
$$= \vec{x}_\star^\top \vec{y}_\star + \vec{y}_\star^\top (\vec{x} - \vec{x}_\star) + \vec{x}_\star^\top (\vec{y} - \vec{y}_\star). \tag{31}$$

Putting in $\vec{x}_\star = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y}_\star = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and let's find the approximation of $f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right)$, we have

$$f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right) \approx \vec{x}_\star^\top \vec{y}_\star + \vec{y}_\star^\top (\vec{x} - \vec{x}_\star) + \vec{x}_\star^\top (\vec{y} - \vec{y}_\star) \tag{32}$$
$$= 3 + \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix} \tag{33}$$
$$= 3 - \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4. \tag{34}$$

Let's compare this with the true value $f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right)$ We have:

$$f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right) = (1+\delta_1)(-1+\delta_3) + (2+\delta_2)(2+\delta_4) \tag{35}$$
$$= 3 - \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4 + \delta_1\delta_3 + \delta_2\delta_4. \tag{36}$$

As we can see, our approximation removes the second order δ terms $\delta_1\delta_3$ and $\delta_2\delta_4$, which is valid for small δ_i.

These linearizations are important for us because we can do many easy computations using linear functions.

Contributors:

- Neelesh Ramachandran.
- Kuan-Yun Lee.

