EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 7B

The following notes are useful for this discussion: Note 10, Note 11

1. Eigenvalue Placement in Discrete Time

Consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \tag{1}$$

(a) **Is the system given in eq. (1) stable?**

Solution: For notation's sake, let's write the system in the familiar form

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i] \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3}$$

We have to calculate the eigenvalues of matrix A. Doing so, we find:

$$\det(\lambda I - A) = 0 \implies \lambda_1 = 1, \lambda_2 = -2 \tag{4}$$

Since the magnitudes of the eigenvalues λ_1 and λ_2 are greater than 1, the system is unstable.

(b) **Derive a state space representation of the resulting closed loop system**. Use state feedback of the form:

$$u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \tag{5}$$

Hint: If you're having trouble parsing the expression for $u[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?)

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] \tag{6}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \right) \tag{7}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \tag{8}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \right) \vec{x}[i] \tag{9}$$

$$= \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{\mathrm{CL}}} \vec{x}[i]. \tag{10}$$

(c) **Find the appropriate state feedback constants, f_1, f_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.**

Solution: From the previous part we have computed the closed loop system as

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{\mathrm{CL}}} \vec{x}[i] \tag{11}$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) \tag{12}$$

$$= \det\left(\begin{bmatrix} f_1 - \lambda & 1+f_2 \\ 2 & -1-\lambda \end{bmatrix}\right) \tag{13}$$

$$= \lambda^2 + (1 - f_1)\lambda + (-f_1 - 2f_2 - 2) \tag{14}$$

We want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$. This means that we should choose the constants f_1 and f_2 so that the characteristic equation is

$$0 = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = \lambda^2 - \frac{1}{4}. \tag{15}$$

Thus, we can match the coefficients of λ in the charateristic polynomial, which indicates we should choose f_1 and f_2 satisfying the following system of equations:

$$0 = 1 - f_1 \tag{16}$$

$$-\frac{1}{4} = -f_1 - 2f_2 - 2 \tag{17}$$

We can solve this two variable, two equation system and find that $f_1 = 1$, $f_2 = -\frac{11}{8}$.
Alternatively, we know what the eigenvalues are; we can plug in each λ into characteristic polynomial, and doing so will yield the same system of equations in f_1, f_2.

(d) **Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?**

Solution: Yes, the closed loop system has eigenvalues $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$, which both have magnitude less than one. Therefore, the closed loop system is stable.

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (18). As before, we use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system.

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] + \vec{w}[i] \tag{18}$$

What would the desired eigenvalues now be? Can you move all the eigenvalues to where you want? In particular, can you make this system stable given the form of the input?

Solution:

$$\vec{x}[i+1] = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \tag{19}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \tag{20}$$

Finding the eigenvalues λ:

$$0 = \det\left(\begin{bmatrix} f_1 - \lambda & f_2 + 1 \\ f_1 + 2 & f_2 - 1 - \lambda \end{bmatrix} \right) \tag{21}$$

$$= (f_1 - \lambda)(f_2 - 1 - \lambda) - (f_1 + 2)(f_2 + 1) \tag{22}$$

$$= f_1(f_2 - 1) - f_1\lambda - \lambda(f_2 - 1) + \lambda^2 - (f_1 f_2 + f_1 + 2f_2 + 2) \tag{23}$$

$$= f_1 f_2 - f_1 - f_1\lambda - \lambda f_2 + \lambda + \lambda^2 - f_1 f_2 - f_1 - 2f_2 - 2 \tag{24}$$

$$= \lambda^2 + (1 - f_1 - f_2)\lambda - 2(1 + f_1 + f_2) \tag{25}$$

$$= (\lambda + 2)\big(\lambda - (1 + f_1 + f_2)\big) \tag{26}$$

We can see that the eigenvalue at $\lambda = -2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input.

(f) **[Practice]** Can you place the eigenvalues at complex conjugates, such that $\lambda_1 = a + \mathrm{j}b, \lambda_2 = a - \mathrm{j}b$ using only real feedback gains f_1, f_2? How about placing them at any arbitrary complex numbers, such that $\lambda_1 = a + \mathrm{j}b, \lambda_2 = c + \mathrm{j}d$?

Solution: Recall what the eigenvalues end up equalling; they are the solutions to a characteristic polynomial derived by forming the determinant of a real-valued matrix. Here, we can place the eigenvalues at complex conjugates because we know that the quadratic formula allows for complex solutions to real-coefficient polynomials. To be able to place the eigenvalues, we know from eq. (14) that we need to be able to tune the coefficients of the characteristic polynomial, which involve only the real feedback coefficients.

The condition, given real feedback coefficients, is that *the eigenvalues must be complex conjugates*. To place the eigenvalues at arbitrary complex numbers that are not necessarily complex conjugates, this is no longer possible.

2. Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \tag{27}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

(a) **Is the system controllable?**

Solution:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \tag{29}$$

Since the controllability matrix $\mathcal{C}$ only has rank 2, the system is not controllable. We would need it to be rank 3 here to span the full space $\mathbb{R}^3$.

(b) **Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some ℓ? For what input sequence $u[i]$ up to $i = \ell - 1$?**

Solution: No, if we write out our expressions for $x[i]$ we can see:

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \tag{31}$$

$$\vec{x}[2] = \begin{bmatrix} 4 \\ 2u[0] - 6 \\ 2u[1] - 3 \end{bmatrix} \tag{32}$$

$$\vec{x}[i] = \begin{bmatrix} 2^i x_0[0] \\ -3x_0[i-1] + x_2[i-1] \\ x_1[i-1] + 2u[i-1] \end{bmatrix} \tag{33}$$

Note that in this expression for $x[i]$, $x_0[i] = 2^i$ is decoupled from all other states and inputs. From this expression we also see that there is no choice of inputs us to get to $x_0[\ell] = -2$. Therefore, we will never be able to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for any ℓ.

(c) **Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some ℓ? For what input sequence $u[i]$ up to $i = \ell - 1$?**

Hint: look at the intermediate results of the previous subpart, where you wrote down what $x[0], x[1]$, etc. were. Apply these new values to those expressions.

Solution: If we look at our expressions for $x[i]$ starting from $i = 1$, we see that we can set our inputs to reach the desired state in a single timestep ($\ell = 1$) by setting $u[0] = -1$.

$$\vec{x}[1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[0] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \tag{34}$$

Thus we see that a system being uncontrollable does not mean we are unable to reach anything at all, but just that the range that can be reached is limited.

(d) **Find the set of all $\vec{x}[2]$, given that you are free to choose the $u[0]$ and $u[1]$ of your choice.**

Solution:

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \tag{35}$$

$$\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] \tag{36}$$

$$= \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix} \tag{37}$$

Since we can set $u[0]$ and $u[1]$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ after two timesteps. This means that we can reach any value for $\vec{x}[2]$; contrast this with how the first component of the state vector is fixed at 4 after two timesteps, and cannot be changed by the inputs.

Contributors:

- Ioannis Konstantakopoulos.
- John Maidens.
- Anant Sahai.
- Regina Eckert.
- Druv Pai.
- Neelesh Ramachandran.
- Titan Yuan.
- Kareem Ahmad.

