1. Changing behavior through feedback

In this question, we discuss how feedback control can be used to change the effective behavior of a system.

(a) Consider the scalar system:

\[ x[i + 1] = 0.9x[i] + u[i] + w[i] \]

where \( u[i] \) is the control input we get to apply based on the current state and \( w[i] \) is the external disturbance, each at time \( i \).

Is the system stable? If \( |w[i]| \leq \epsilon \), what can you say about \( |x[i]| \) at all times \( i \) if you further assume that \( u[i] = 0 \) and the initial condition \( x[0] = 0 \)? How big can \( |x[i]| \) get?

Solution: The system is stable, as \( \lambda = 0.9 \rightarrow |\lambda| < 1 \). We can say that \( |x[i]| \) is bounded at all time if the disturbance is bounded. Unrolling the system’s recursion and extrapolating the general form,

\[
\begin{align*}
x[0] &= 0 \\
x[1] &= w[0] \\
x[2] &= 0.9w[0] + w[1] \\
x[3] &= 0.9^2w[0] + 0.9w[1] + w[2] \\
& \vdots \\
x[i] &= \sum_{k=0}^{i-1} 0.9^{i-k-1}w[k].
\end{align*}
\]

We can check that this form works by plugging it into our recursion:

\[
\begin{align*}
x[i + 1] &= 0.9x[i] + w[i] = 0.9 \left( \sum_{k=0}^{i-1} 0.9^{i-k-1}w[k] \right) + w[i] = \sum_{k=0}^{i-1} 0.9^{i-k}w[k] + w[i] = \sum_{k=0}^{i} 0.9^{i-k}w[k]
\end{align*}
\]

which is exactly what our formula predicts.

Thus

\[
|x[i]| = \left| \sum_{k=0}^{i-1} 0.9^{i-k-1}w[k] \right| \leq \sum_{k=0}^{i-1} 0.9^{i-k-1} |w[k]| = \sum_{k=0}^{i-1} 0.9^{i-k-1}\epsilon.
\]

In the limit as \( i \rightarrow \infty \), by the geometric series formula,
\[ |x[i]| \leq \frac{\epsilon}{1 - 0.9} = 10\epsilon \quad (10) \]

(b) Suppose that we decide to choose a control law \( u[i] = f x[i] \) to apply in feedback. \textbf{For what values of \( \lambda \) can you get the system to behave like:

\[ x[i + 1] = \lambda x[i] + w[i]? \quad (11) \]

\textbf{How would you pick \( f \)?}

(Note: In this case, \( w[i] \) can be thought of like another input to the system, except we can’t control it.)

\textbf{Solution:} We can control the system to have any value of \( \lambda \), as long as we’re not limited on the values of \( f \).

\[ x[i + 1] = 0.9x[i] + f x[i] + w[i] = \lambda x[i] + w[i]. \quad (12) \]

Fitting terms, \( f = \lambda - 0.9 \). Note we can get a \( \lambda > 1 \) if we so desire; there is nothing stopping us from putting arbitrarily big/small \( \lambda \) by the choice of \( f \).

(c) \textbf{For the previous part, which \( f \) would you choose to minimize how big \( |x[i]| \) can get?}

\textbf{Solution:} From eq. (11), in order to have the minimum bound on \( |x[i]| \), \( \lambda = 0 \). To get this \( \lambda \), \( f = -0.9 \). In the limit as \( i \to \infty \) in this case,

\[ |x[i]| \leq \frac{\epsilon}{1 - 0} = \epsilon \quad (13) \]

The minimum bound on \( |x(i)| = \epsilon \) is the same bound as on the disturbance.

(d) \textbf{What if instead of a 0.9, we had a 3 in the original eq. (1). What, if anything, would change?}

\textbf{Solution:} If our system were now,

\[ x[i + 1] = 3x[i] + u[i] + w[i], \quad (14) \]

the system would no longer be stable. However, we can still choose any eigenvalue \( \lambda \) using closed loop feedback. In this case, \( f = \lambda - 3 \).

(e) Now suppose that we have a vector-valued system with a vector-valued control:

\[ \vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \quad (15) \]

where we further assume that \( B \) is an invertible square matrix. Further, suppose we decide to apply linear feedback control using a square matrix \( F \) so we choose \( \vec{u}[i] = F\vec{x}[i] \).

\textbf{For what values of matrix \( G \) can you get the system to behave like:

\[ \vec{x}[i + 1] = G\vec{x}[i] + \vec{w}[i]? \quad (16) \]

\textbf{How would you pick \( F \) given knowledge of \( A, B \) and the desired goal dynamics \( G \)?}

\textbf{Solution:} Since in this case our input is the same rank as our output, we can arbitrarily choose the
matrix $G$. As long as $B$ is invertible (as given), we can define:

\[
\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]
\]

(17)

\[
\begin{align*}
&= A\vec{x}[i] + B\vec{x}[i] + \vec{w}[i] \\
&= (A + BF)\vec{x}[i] + \vec{w}[i] \\
&= G\vec{x}[i] + \vec{w}[i]
\end{align*}
\]

(18)

(19)

(20)

Therefore, matching terms,

\[
A + BF = G \implies F = B^{-1}(G - A).
\]

(21)
2. Controlling states by designing sequences of inputs

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider the following matrix chosen for ease of understanding what is going on:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\bar{b} = \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}
\]

Let’s assume we have a discrete-time system defined as follows:

\[
\bar{x}[i + 1] = A\bar{x}[i] + \bar{b}u[i].
\]

(a) We are given the initial condition \(\bar{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\). Let’s say we want to achieve \(\bar{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\) for some specific \(\ell \geq 0\). We don’t need to stay there, we just want to be in this state at that time. What is the smallest \(\ell\) such that this is possible? What is our choice of sequence of inputs \(u[i]\)?

**Solution:** To ease notation, let

\[
\bar{x}[i] = \begin{bmatrix}
x_1[i] \\
x_2[i] \\
x_3[i] \\
x_4[i] \\
\end{bmatrix}.
\]

Writing out expressions for \(x[i]\) we get:

\[
\bar{x}[1] = A\bar{x}[0] + \bar{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix},
\]

\[
\bar{x}[2] = \begin{bmatrix} x_3[0] \\ x_4[0] \\ u[0] \\ u[1] \end{bmatrix},
\]

\[
\bar{x}[3] = \begin{bmatrix} x_4[0] \\ u[0] \\ u[1] \\ u[2] \end{bmatrix},
\]

and if \(i \geq 4\),

\[
\bar{x}[i] = \begin{bmatrix} u[i - 4] \\ u[i - 3] \\ u[i - 2] \\ u[i - 1] \end{bmatrix}.
\]
Hence, the smallest $\ell$ is equal to 4, with $u[0] = [1], u[1] = [2], u[2] = [3], u[3] = [4]$.

(b) What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest $\ell$ and what is our choice of $u[i]$?

Solution: Looking over our expressions for $x[i]$ from the previous part, we see that the earliest $\ell$ whose expression can be set to the desired state is $\ell = 1$ requiring $u[0] = 4$.

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ u[0] \end{bmatrix}. \quad (28)$$

(c) If we start from $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, what is smallest $\ell$ such that $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, what is corresponding $u[i]$?

Solution: Looking over our expressions for $x[i]$, we see that the earliest $\ell$ whose expression can be set to the desired state in this case is $\ell = 4$ requiring $u[0] = 1, u[1] = 2, u[2] = 3, u[3] = 4$.

(d) If you would like to make sure that at time $\ell$ we are at $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ for the state, what controls could you use to get there? How big does $\ell$ have to be for this strategy to work?

Solution: As you might notice, using inputs $u[i] \in \{a, b, c, d\}$ (in that order), we can get to any desired state $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Hence with $\ell = 4$, we can guarantee that the $\vec{x}[\ell]$ is our desired state.

Contributors:

• Anant Sahai.
• Regina Eckert.
• Kumar Krishna Agrawal.
• Kuan-Yun Lee.
• Kareem Ahmad.