# EECS 16B Designing Information Devices and Systems II
# Fall 2021 Discussion Worksheet Discussion 6A

The following notes are useful for this discussion: Note 9, Discussion 2B, Homework 04.

## 1. Translating System of Differential Equations from Continuous Time to Discrete Time

Working through this question will help you better understand differential equations with inputs, and the sampling of a continuous-time system of differential equations into a discrete-time view. These concepts are important for control, since it is often easier to think about doing what we want in discrete-time. This question should initially feel similar to dis02B, and in later subparts, we extend our analysis to the case of a vector differential equation.

(a) Consider the scalar system below:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t). \tag{1}$$

Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width $\Delta$. This is the same case we considered in dis02B. In other words:

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in (i\Delta, (i+1)\Delta]. \tag{2}$$

Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let's revisit the solution for eq. (1), when we're given the initial conditions at $t_0$, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = \mathrm{e}^{\lambda\Delta(t)}x(t_0) + \int_{t_0}^{t} u(\theta)e^{\lambda(t-\theta)}d\theta \tag{4}$$

where $\Delta(t) = t - t_0$. **Given that we start at $t = i\Delta$, where $x(t) = x_d[i]$, and satisfy eq. (1) where do we end up at $x_d[i+1]$?**

**Solution:** For $t \in (i\Delta, (i+1)\Delta]$, the differential equation takes the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t) = \lambda x(t) + u_d[i] \tag{5}$$

where we know the inital conditions that $x(i\Delta) = x_d[i]$. We can solve this equation for $x(t)$ using the integral equation from eq. (4) and the fact that $u(t)$ is piecewise constant. In particular, we get the following form

$$x(t) = \mathrm{e}^{\lambda(t-i\Delta)}x_d[i] + \int_{i\Delta}^{t} u(\theta)e^{\lambda(t-\theta)}d\theta \tag{6}$$

Plugging in the timestep of interest, we set $t = (i+1)\Delta$, to evaluate $x_d[i+1]$ as

$$x_d[i+1] = \mathrm{e}^{\lambda\Delta}x_d[i] + \int_{i\Delta}^{\Delta(i+1)} u_d[i]e^{\lambda((i+1)\Delta-\theta)}d\theta \tag{7}$$

$$= \mathrm{e}^{\lambda\Delta}x_d[i] + u_d[i]\frac{\mathrm{e}^{\lambda\Delta} - \mathrm{e}^0}{\lambda} \tag{8}$$

$$= \mathrm{e}^{\lambda\Delta}x_d[i] + u_d[i]\frac{\mathrm{e}^{\lambda\Delta} - 1}{\lambda} \tag{9}$$

which gives us the solution for $x_d[i+1]$.

**Alternative solution**: We can guess that the form of the solution will be:

$$x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta \tag{10}$$

Why is it in terms of $t-i\Delta$? Given the value $x_d[i] = x(i\Delta)$, we want to model the growth of $x$ between $i\Delta$ and $t$, independently of the specific values of $i\Delta$ and $t$. We only care about their difference, given that we are in a specific interval.

To fit $x(t)$ to eq. (5), we equate the LHS of eq. (5) to the RHS. The LHS is:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) = \lambda\alpha e^{\lambda(t-i\Delta)} \tag{11}$$

so equating the LHS with the RHS gives:

$$\lambda\alpha e^{\lambda(t-i\Delta)} = \lambda x(t) + u_d[i] \tag{12}$$

$$= \lambda\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) + u_d[i] \tag{13}$$

$$= \lambda\alpha e^{\lambda(t-i\Delta)} + \lambda\beta + u_d[i] \tag{14}$$

$$\implies 0 = \lambda\beta + u_d[i] \tag{15}$$

$$\implies \beta = -\frac{u_d[i]}{\lambda}. \tag{16}$$

Now we use the initial condition, $x(i\Delta) = x_d[i]$. Expanding $x(i\Delta)$ as per our guess,

$$x_d[i] = x(i\Delta) = \alpha e^{\lambda(i\Delta-i\Delta)} + \beta = \alpha + \beta \tag{17}$$

And using $\beta = -\frac{u_d[i]}{\lambda}$ we get

$$x_d[i] = \alpha + \frac{-u_d[i]}{\lambda} \tag{18}$$

$$\implies \alpha = x_d[i] + \frac{u_d[i]}{\lambda}. \tag{19}$$

Now we have the values of $\alpha$ and $\beta$, which is all we need to write $x(t)$ fully. So for $t \in (i\Delta, (i+1)\Delta]$ (which is the assumption we made for eq. (5) to hold),

$$x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta = \left(x_d[i] + \frac{u_d[i]}{\lambda}\right)e^{\lambda(t-i\Delta)} - \frac{u_d[i]}{\lambda} \tag{20}$$

$$= e^{\lambda(t-i\Delta)}x_d[i] + \frac{e^{\lambda(t-i\Delta)}-1}{\lambda}u_d[i] \tag{21}$$

The reason we simplify in this manner is because we want to split the value of $x(t)$ into the effect of the initial condition $x_d[i]$, and the input $u_d[i]$. Now we can see how each independent part affects $x(t)$. Since $x(t)$ is continuous across all $t$, $x_d[i+1] = x((i+1)\Delta)$. The continuity condition ensures that the function doesn't have bad behavior at only the points $i\Delta$ or $(i+1)\Delta$. Of course, these discontinuities don't happen in real systems, so our assumption makes sense. Thus

$$x_d[i+1] = x((i+1)\Delta) = e^{\lambda((i+1)\Delta-i\Delta)}x_d[i] + \frac{e^{\lambda((i+1)\Delta-i\Delta)}-1}{\lambda}u_d[i] \tag{22}$$

$$= e^{\lambda\Delta}x_d[i] + \frac{e^{\lambda\Delta}-1}{\lambda}u_d[i]. \tag{23}$$

This is the quantity we want.

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{24}$$

where $\vec{x}(t)$ is $n$-dimensional. Suppose further that the matrix $A$ has distinct and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$. *(Hint: What's the significance of this information?)*

**If we apply a piecewise constant control input $u_d[i]$ as in (2), and sample the system $\vec{x}(t)$ at time intervals $t = i\Delta$, what are the corresponding $A_d$ and $\vec{b}_d$ in:**

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{25}$$

*(Hint : Define terms* $\Lambda_e^{\Delta} = \begin{bmatrix} e^{\lambda_1\Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n\Delta} \end{bmatrix}$, $\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\lambda_n} \end{bmatrix}$*)*

**Solution:** First, following the hint, we notice that with a full set of distinct eigenvalues and corresponding eigenvectors, we can change coordinates so that $\vec{x}(t) = V\vec{y}(t)$ and $\vec{y}(t) = V^{-1}\vec{x}(t)$. Using this transformation we diagonalize the system of differential equations, i.e

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{26}$$

$$\implies \frac{\mathrm{d}V\vec{y}(t)}{\mathrm{d}t} = AV\vec{y}(t) + \vec{b}u(t) \tag{27}$$

$$\therefore \frac{\mathrm{d}\vec{y}(t)}{\mathrm{d}t} = V^{-1}AV\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{28}$$

Note that using the basis of eigenvectors $V$, we've diagonalized A to get $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix}$

$$\therefore \frac{\mathrm{d}\vec{y}(t)}{\mathrm{d}t} = \Lambda \vec{y}(t) + V^{-1}\vec{b}u(t) \tag{29}$$

We use subscripts $k$ to index into vectors, giving us an uncouple system such that:

$$\frac{\mathrm{d}y_k(t)}{\mathrm{d}t} = \lambda y_k(t) + \left(V^{-1}b\right)_k u_d[i] \tag{30}$$

for which, we can solve using eq. (9) to get

$$y_k((i+1)\Delta) = e^{\lambda_k \Delta} y_k(i\Delta) + \left(\frac{e^{\lambda_k \Delta} - 1}{\lambda_k}\right)\left(V^{-1}b\right)_k u_d[i] \tag{31}$$

As we are in a diagonal basis now, we can compose these individual scalar elements, and write:

$$\vec{y}((i+1)\Delta) = \left(\begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n \Delta} \end{bmatrix}\right) \vec{y}(i\Delta) + \left(\begin{bmatrix} \frac{e^{\lambda_1 \Delta}-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n \Delta}-1}{\lambda_n} \end{bmatrix}\right) V^{-1}\vec{b}u_d[i] \tag{32}$$

Now, we name some of the terms above, for notational convenience going forward:

$$\Lambda_e^\Delta = \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n \Delta} \end{bmatrix} \qquad \Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{\lambda_n} \end{bmatrix} \qquad \vec{\tilde{u}}_d[i] = V^{-1}\vec{b}u_d[i] \tag{33}$$

So, with this new notation, we can write the second matrix in eq. (32) as:[1]

$$\begin{bmatrix} \frac{e^{\lambda_1 \Delta}-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n \Delta}-1}{\lambda_n} \end{bmatrix} = \begin{bmatrix} \frac{e^{\lambda_1 \Delta}}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n \Delta}}{\lambda_n} \end{bmatrix} + \begin{bmatrix} \frac{-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{-1}{\lambda_n} \end{bmatrix} \tag{34}$$

[1] In a matrix product, if both matrices are diagonal, the product is commutative.

$$= \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n \Delta} \end{bmatrix} - \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{\lambda_n} \end{bmatrix} \tag{35}$$

$$= \Lambda^{-1}\Lambda_e^{\Delta} - \Lambda^{-1} I \tag{36}$$

$$= \Lambda^{-1}\left(\Lambda_e^{\Delta} - I\right) \tag{37}$$

This gives us

$$\vec{y}((i+1)\Delta) = \Lambda_e^{\Delta}\vec{y}(i\Delta) + \Lambda^{-1}(\Lambda_e^{\Delta} - I)\vec{\tilde{u}}_d[i] \tag{38}$$

Using this form in the simplification, we find that:

$$\vec{x}_d[i+1] = V\vec{y}_d[i+1] \tag{39}$$

$$= V\left(\Lambda_e^{\Delta}\vec{y}_d[i] + \Lambda^{-1}\left(\Lambda_e^{\Delta} - I\right)\vec{\tilde{u}}_d[i]\right) \tag{40}$$

$$= \left(V\Lambda_e^{\Delta}V^{-1}\right)\vec{x}_d[i] + \left(V\Lambda^{-1}\left(\Lambda_e^{\Delta} - I\right)\right)\vec{\tilde{u}}_d[i] \tag{41}$$

Now, recall that our original goal was to write out $A_d$ and $\vec{b}_d$, and we can do that now with our expression. Resubstituting $\vec{\tilde{u}}_d[i] = V^{-1}\vec{b}u_d[i]$ we have:

$$\vec{x}_d[i+1] = \left(V\Lambda_e^{\Delta}V^{-1}\right)\vec{x}_d[i] + \left(V\Lambda^{-1}\left(\Lambda_e^{\Delta} - I\right)\right)V^{-1}\vec{b}u_d[i] \tag{42}$$

Therefore,

$$A_d = V\Lambda_e^{\Delta}V^{-1} \qquad \vec{b}_d = V\Lambda^{-1}\left(\Lambda_e^{\Delta} - I\right)V^{-1}\vec{b} \tag{43}$$

where $\Lambda_e^{\Delta} = \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n \Delta} \end{bmatrix}$ and $\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{\lambda_n} \end{bmatrix}$

(c) In the previous part, we had a matrix $A$ which was diagonalizable using a eigenbasis. You might recall from Homework 4, that for critically damped systems we had $A = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}$ (a non-diagonalizable matrix). Assuming the input $u(t) = 0$, consider the system of differential equations given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{44}$$

**Assuming that we know the solution at $t = i\Delta$, where $x(i\Delta) = x_d[i]$, find $A_d$ such that we have**

**a solution in the discrete time system for eq. (44)**

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] \tag{45}$$

*(Hint: From 1(a) we know for $t \geq t_0$*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t) \tag{46}$$

*with initial conditions $x(t) = x(t_0)$ for $t = t_0$, has solution of the form)*

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{\lambda(t-\theta)}u(\theta)d\theta \tag{47}$$

**Solution:** Starting with $x_2(t)$ we note that for the interval $(i\Delta, (i+1)\Delta]$ we know how to solve

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \lambda x_2(t) \tag{48}$$

In particular, using eq. (47) we have:

$$x_2(t) = e^{\lambda(t-i\Delta)}x_2[i] \tag{49}$$

$$\therefore x_2[i+1] = e^{\lambda\Delta}x_2[i] \tag{50}$$

Using this to find the solution for $x_1(t)$, which satisfies the following diff. eq

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \lambda x_1(t) + \beta x_2(t) \tag{51}$$

$$= \lambda x_1(t) + \beta e^{\lambda(t-i\Delta)}x_2[i] \tag{52}$$

We can solve the above equation by using eq. (47) by setting $u(\theta) = \mathrm{e}^{\lambda(\theta-i\Delta)}x_2[i]$

$$x_1(t) = e^{\lambda(t-i\Delta)}x_1[i] + \beta x_2[i] \int_{i\Delta}^{t} e^{\lambda(t-\theta)}e^{\lambda(\theta-i\Delta)}d\theta \tag{53}$$

Evaluating $x_1[i+1]$, we substitute $t = (i+1)\Delta$ to get

$$x_1[i+1] = e^{\lambda\Delta}x_1[i] + \beta x_2[i] \int_{i\Delta}^{(i+1)\Delta} e^{\lambda\Delta}d\theta \tag{54}$$

$$= e^{\lambda\Delta}x_1[i] + e^{\lambda\Delta}\beta\Delta x_2[i] \tag{55}$$

Putting in the matrix-vector form, we finally recover

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} e^{\lambda\Delta}x_1[i] + e^{\lambda\Delta}\beta\Delta x_2[i] \\ e^{\lambda\Delta}x_2[i] \end{bmatrix} \tag{56}$$

$$= \begin{bmatrix} e^{\lambda\Delta} & e^{\lambda\Delta}\beta\Delta \\ 0 & e^{\lambda\Delta} \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} \tag{57}$$

$$\therefore \vec{x}_d[i+1] = A_d\vec{x}_d[i] \tag{58}$$

where $A_d = \begin{bmatrix} e^{\lambda\Delta} & e^{\lambda\Delta}\beta\Delta \\ 0 & e^{\lambda\Delta} \end{bmatrix}$

(d) *(Practice)* In this subpart we generalize the above procedure, by making $u(t) \neq 0$. Consider the following system of differential equations:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{59}$$

where $A = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. **Given $\vec{x}_d[i]$, find $A_d$ and $\vec{b}_d$ such that**

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{60}$$

**Solution:** We follow a similar strategy as before, but now we have to solve the following equation:

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \lambda x_2(t) + b_2 u_d[i] \tag{61}$$

From eq. (47) we have:

$$x_2(t) = e^{\lambda(t-i\Delta)}x_2[i] + b_2\int_{i\Delta}^{t} e^{\lambda(t-\theta)}u_d[i]d\theta \tag{62}$$

$$= e^{\lambda(t-i\Delta)}x_2[i] + b_2 u_d[i]\frac{e^{\lambda(t-i\Delta)}-1}{\lambda} \tag{63}$$

$$= \mathrm{e}^{\lambda(t-i\Delta)}\left(x_2[i] + \frac{b_2 u_d[i]}{\lambda}\right) - \frac{b_2 u_d[i]}{\lambda} \tag{64}$$

For $t = (i+1)\Delta$, we get

$$x_2[i+1] = e^{\lambda\Delta}x_2[i] + b_2 u_d[i]\frac{e^{\lambda\Delta}-1}{\lambda} \tag{65}$$

Now, for the state variable $x_1(t)$, we have the differential equation:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \lambda x_1(t) + \beta x_2(t) + b_1 u_d[i] \tag{66}$$

$$= \lambda x_1(t) + \beta e^{\lambda(t-i\Delta)}\left(x_2[i] + \frac{b_2 u_d[i]}{\lambda}\right) + b_1 u_d[i] - \frac{\beta b_2 u_d[i]}{\lambda} \tag{67}$$

$$= \lambda x_1(t) + \beta e^{\lambda(t-i\Delta)}p + q \tag{68}$$

where for simplification we use $p = \beta x_2[i] + \frac{\beta b_2 u_d[i]}{\lambda}$ and $q = b_1 u_d[i] - \frac{\beta b_2 u_d[i]}{\lambda}$. Solving for $x_1(t)$

$$x_1(t) = e^{\lambda(t-i\Delta)}x_1[i] + \int_{i\Delta}^{t} e^{\lambda(t-\theta)}\left(pe^{\lambda(\theta-i\Delta)} + q\right)d\theta \tag{69}$$

$$= e^{\lambda(t-i\Delta)}x_1[i] + \int_{i\Delta}^{t} pe^{\lambda(t-i\Delta)}d\theta + \int_{i\Delta}^{t} qe^{\lambda(t-\theta)}d\theta \tag{70}$$

$$= e^{\lambda(t-i\Delta)}x_1[i] + pe^{\lambda(t-i\Delta)}(t-i\Delta) + q\frac{\mathrm{e}^{\lambda(t-i\Delta)}-1}{\lambda} \tag{71}$$

To compute $t = (i+1)\Delta$, we have

$$\begin{aligned}
x_1[i+1] &= \mathrm{e}^{\lambda\Delta} x_1[i] + p\mathrm{e}^{\lambda\Delta}\Delta + q\frac{e^{\lambda\Delta}-1}{\lambda} &(72)\\
&= \mathrm{e}^{\lambda\Delta} x_1[i] + \beta\mathrm{e}^{\lambda\Delta}\Delta\left(x_2[i] + \frac{b_2 u_d[i]}{\lambda}\right) + \frac{e^{\lambda\Delta}-1}{\lambda}\left(b_1 u_d[i] - \frac{\beta b_2 u_d[i]}{\lambda}\right) &(73)\\
&= \mathrm{e}^{\lambda\Delta} x_1[i] + \beta\mathrm{e}^{\lambda\Delta}\Delta x_2[i] + \frac{e^{\lambda\Delta}-1}{\lambda} b_1 u_d[i] + \left(\frac{\Delta e^{\lambda\Delta}}{\lambda} - \frac{e^{\lambda\Delta}-1}{\lambda^2}\right)\beta b_2 u_d[i] &(74)
\end{aligned}$$

In the matrix vector form, we recover the following form:

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} e^{\lambda\Delta} & \beta\Delta e^{\lambda\Delta} \\ 0 & e^{\lambda\Delta} \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + \begin{bmatrix} \frac{e^{\lambda\Delta}-1}{\lambda} & \beta\left(\frac{\Delta e^{\lambda\Delta}}{\lambda} - \frac{e^{\lambda\Delta}-1}{\lambda^2}\right) \\ 0 & \frac{e^{\lambda\Delta}-1}{\lambda} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_d[i] \tag{75}$$

Transforming to the matrix-vector form, we compare to

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + \vec{b}_d u_d[i] \tag{76}$$

which gives $A_d = \begin{bmatrix} e^{\lambda\Delta} & \beta\Delta e^{\lambda\Delta} \\ 0 & e^{\lambda\Delta} \end{bmatrix}$ and $\vec{b_d} = B_d\vec{b}$ where $B_d = \begin{bmatrix} \frac{e^{\lambda\Delta}-1}{\lambda} & \beta\left(\frac{\Delta e^{\lambda\Delta}}{\lambda} - \frac{e^{\lambda\Delta}-1}{\lambda^2}\right) \\ 0 & \frac{e^{\lambda\Delta}-1}{\lambda} \end{bmatrix}$

(e) Consider the discrete-time system

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + \vec{b}_d u_d[i] \tag{77}$$

Suppose that $\vec{x}_d[0] = \vec{x}_0$. **Unroll the implicit recursion to write $\vec{x}_d[i+1]$ as a sum that involves $\vec{x}_0$ and the $u_d[j]$ for $j = 0, 1, \ldots, i$.** You don't need to worry about what $A_d$ and $\vec{b}_d$ actually are in terms of the original parameters.

*(Hint: If we have a scalar difference equation, how would you solve the recurrence?)*

**Solution:**

Here, we derive the unrolled recursion and make a guess at the form of the solution in summation notation. Let's look at the pattern starting with $\vec{x}_d[1]$, given that $\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i]$,

$$\begin{aligned}
\vec{x}_d[1] &= A_d\vec{x}_d[0] + \vec{b}_d u_d[0] &(78)\\
\vec{x}_d[2] &= A_d\vec{x}_d[1] + \vec{b}_d u_d[1] &(79)\\
&= A_d(A\vec{x}_d[0] + \vec{b}_d u_d[0]) + \vec{b}_d u_d[1] &(80)\\
&= A_d^2\vec{x}_d[0] + A_d\vec{b}_d u_d[0] + \vec{b}_d u_d[1] &(81)\\
\vec{x}_d[3] &= A_d\vec{x}_d[2] + \vec{b}_d u_d[2] &(82)\\
&= A_d\left(A_d^2\vec{x}_d[0] + A_d\vec{b}_d u_d[0] + \vec{b}_d u_d[1]\right) + \vec{b}_d u_d[2] &(83)\\
&= A_d^3\vec{x}_d[0] + A_d^2\vec{b}_d u_d[0] + A_d\vec{b}_d u_d[1] + \vec{b}_d u_d[2] &(84)
\end{aligned}$$

So, given this pattern, if we guess:

$$\vec{x}_d[i] = A_d^i \vec{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) \vec{b}_d \tag{85}$$

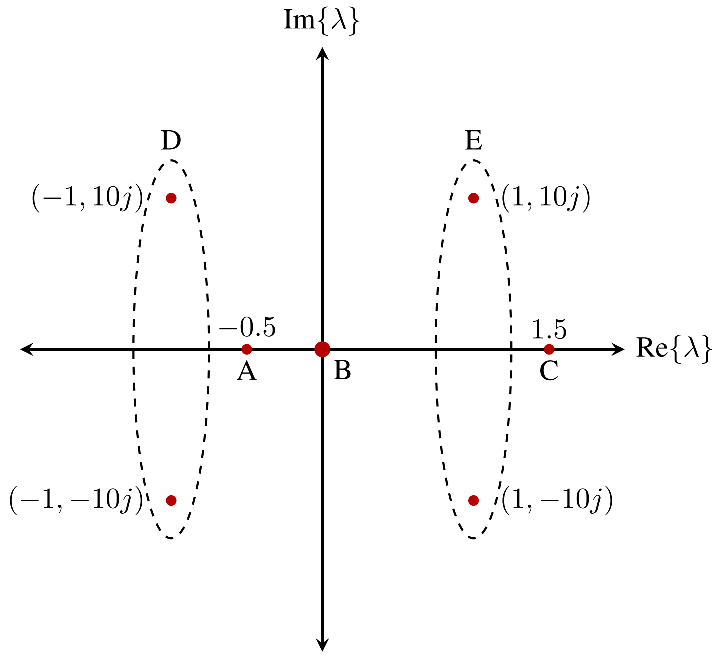
Then, let's see what we get for $\vec{x}_d[i+1]$, and make sure our guess is correct:

$$\begin{aligned}
\vec{x}_d[i+1] &= A_d \vec{x}_d[i] + \vec{b}_d u_d[i] && (86)\\
&= A_d \left( A_d^i \vec{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A^{i-1-j} \right) \vec{b}_d \right) + \vec{b}_d u_d[i] && (87)\\
&= A_d^{i+1} \vec{x}_d[0] + \left( \left( \sum_{j=0}^{i-1} u_d[j] A^{i-j} \right) + u_d[i] \right) \vec{b}_d && (88)\\
&= A_d^{i+1} \vec{x}_d[0] + \left( \sum_{j=0}^{i} u_d[j] A^{i-j} \right) \vec{b}_d && (89)
\end{aligned}$$

This satisfies (85), for $i+1$ and hence our guess was correct!

**2. Continuous-time System Responses**

We have a differential equation $\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t)$, where $A$ is a real matrix and has eigenvalues $\lambda$. For systems (A, B, C) it is a scalar differential equation, whereas for D, E which have more than 1 eigenvalue, this equation is a vector differential equation. **For each set of $\lambda$ values plotted on the real-imaginary complex plane, sketch $x_1(t)$ with an initial condition of $x_1(0) = 1$. Do we have sufficient information to exactly plot $x_1(t)$ for each vector differential equation? If not, sketch a couple of possible solutions.**. In the scalar case, $x_1(t) \equiv x(t)$.




**Solution:** We recall that if the imaginary component of an eigenvalue is nonzero, the system will experience oscillations in its settling response. Since we have a real matrix $A$, any eigenvalue with nonzero imaginary components must appear in complex conjugate pairs, which explains why D and E are grouped by pairs of eigenvalues. As we've seen previously in underdamped example from RLC circuit (in Homework 4), real differential equation systems can have complex eigenvalues. From phasor analysis, we recall that solutions have the form

$$x(t) = \frac{x_0 e^{\lambda_r t} \mathrm{e}^{\mathrm{j}\phi}}{2} e^{\mathrm{j}\omega t} + \frac{x_0 e^{\lambda_r t} \mathrm{e}^{-\mathrm{j}\phi}}{2} e^{-\mathrm{j}\omega t} \tag{90}$$

$$= x_0 e^{\lambda_r t} cos(\omega t + \phi) \tag{91}$$

with phasor $\widetilde{x} = \frac{x_0 e^{\lambda_r t} \mathrm{e}^{\mathrm{j}\phi}}{2}$, where $\lambda_r$ is $\mathrm{Re}\{\lambda\}$. Note that the above equation has two degrees of freedoms, $x_0$ (amplitude) and $\phi$ (phase). However we're only given one initial condition, i.e we know $x_1(0) = 1$. However, plugging $t = 0$ in eq. (91) we get $x(0) = x_0 cos(\phi) = 1$. Here don't have sufficient information for recovering the exact values of both $x_0, \phi$. For instance, we could have ($x_0 = 1, \phi = 0$), or ($x_0 = \sqrt{2}, \phi = \frac{\pi}{4}$) as valid solutions.

We analyze each case sequentially to determine what the system's response might look like:

(A): We see a decay since the real component of eigenvalue is negative, but here, since the imaginary component is zero, there will be no oscillations (there is a direct exponential decay from 1 to 0).

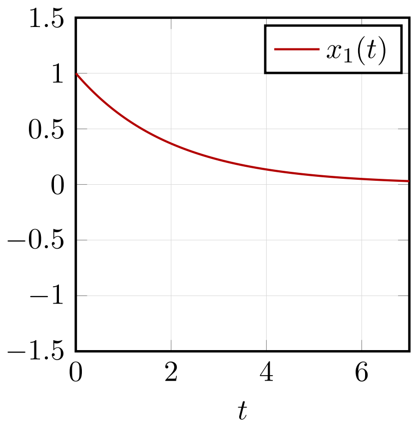
(B): When the eigenvalue has a real component exactly on zero, then the (ideal) system here will neither grow nor decay over time; it will remain at the initial condition, which is 1. Since any disturbance to the system might impact the system's behavior, we can say this system is marginally stable.

(C): In this case the real component is positive, so the system will grow over time. The real part is larger in magnitude, so the growth will proceed at a greater rate. Since there is no imaginary component here, the system will not oscillate, and instead grows directly from the initial condition towards $\infty$.

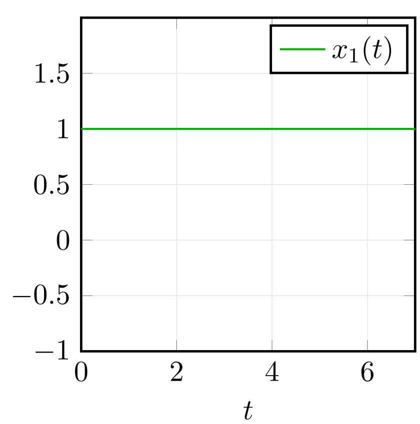
(D): These eigenvalues have negative real components, indicating that the system is stable, because the effect of the initial condition will decay over time. However, the imaginary component is nonzero, and these imaginary components are connected to sinusoids in ways that we have seen. This indicates that the response will experience oscillations around the value 0, as it decays from 1 to 0.

(E): The real component here is positive, which leads to exponential growth of the initial condition over time. Since the imaginary components of this complex conjugate pair of eigenvalues is nonzero, the system will oscillate as it grows.

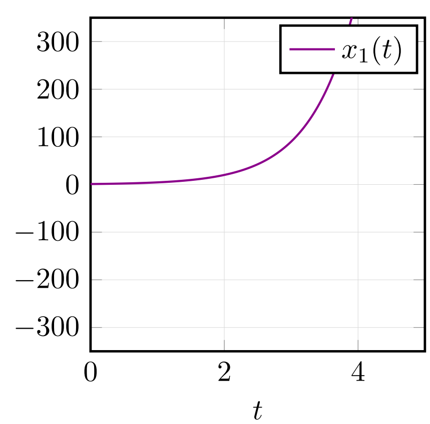
The plots below demonstrate the qualitative behavior described above (and include the imaginary component as well for systems D/E).



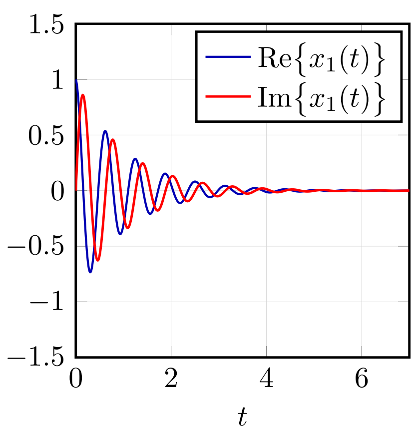

**(a)** System A ($\lambda = -0.5$)



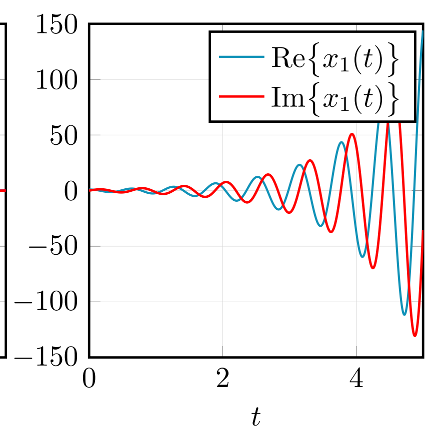

**(b)** System B ($\lambda = 0$)




**(c)** System C ($\lambda = 1.5$)




**(d)** System D ($\lambda = -1 \pm 10j$)




**(e)** System E ($\lambda = 1 \pm 10j$)

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