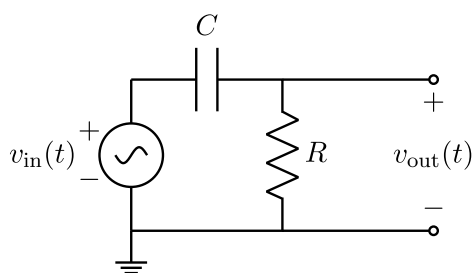
EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 5A

The following notes are useful for this discussion: Note 6

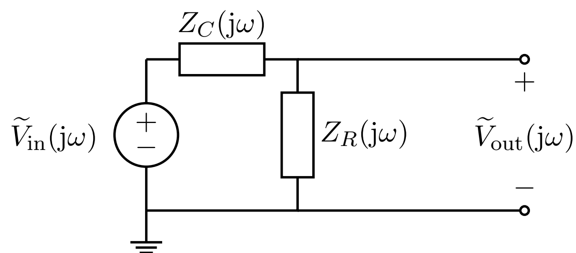
1. Transfer function practice

In this problem, you'll be deriving some transfer functions on your own. **For each circuit, determine the transfer function** $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega)}$**. How does each circuit respond as** $\omega \to 0$ **(low frequencies), as** $\omega \to \infty$ **(high frequencies) ? Is the circuit high-pass filter, low-pass filter, or band-pass filter?**

(a) **RC circuit**



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

Solution: We'll use the voltage divider formula to find $\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)$:

$$\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega) = \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega). \tag{1}$$

Recalling the expression for the impendances, we note that for the resistor $Z_R = R$, and for the capacitor $Z_C = \frac{1}{\mathrm{j}\omega C}$. Plugging in the impedances gives

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega)} = \frac{R}{R + \frac{1}{\mathrm{j}\omega C}} = \frac{\mathrm{j}\omega}{\mathrm{j}\omega + \frac{1}{RC}}. \tag{2}$$

At low frequencies, i.e with $\omega \ll \frac{1}{RC}$ we have

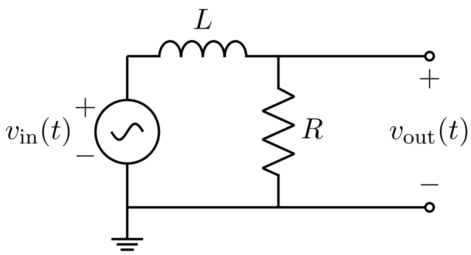
$$\lim_{\omega\to 0} H(\mathrm{j}\omega) = \lim_{\omega\to 0} \frac{\mathrm{j}\omega}{\mathrm{j}\omega + \frac{1}{RC}} = 0, \tag{3}$$

At high frequencies with $\omega \gg \frac{1}{RC}$ we have

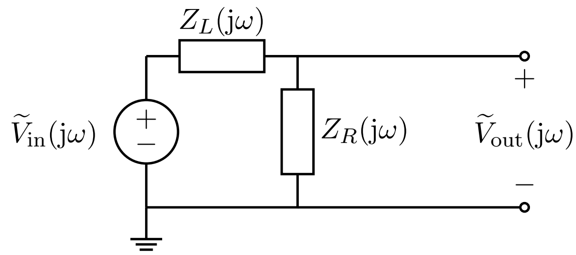
$$\begin{aligned}
\lim_{\omega\to\infty} H(\mathrm{j}\omega) &= \lim_{\omega\to\infty} \frac{\mathrm{j}\omega}{\mathrm{j}\omega + \frac{1}{RC}} &(4)\\
&= \lim_{\omega\to\infty} \frac{1}{1 + \frac{1}{\mathrm{j}\omega RC}} &(5)\\
&= 1. &(6)
\end{aligned}$$

So this circuit is a *high-pass filter.*

(b) **LR circuit**



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

Solution: The strategy is the same as the previous part, using the voltage divider formula, i.e. ,

$$\widetilde{V}_{\text{out}}(\text{j}\omega) = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{\text{in}}(\text{j}\omega).$$

A similar manipulation to the previous part gives

$$H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\text{j}\omega)}{\widetilde{V}_{\text{in}}(\text{j}\omega)} = \frac{R}{R + \text{j}\omega L} = \frac{\frac{R}{L}}{\frac{R}{L} + \text{j}\omega}. \tag{7}$$

At low frequencies, i.e with $\omega \ll \frac{R}{L}$ we have

$$\lim_{\omega\to 0} H(\text{j}\omega) = \lim_{\omega\to 0} \frac{\frac{R}{L}}{\frac{R}{L} + \text{j}\omega} = 1, \tag{8}$$

while at high frequencies with $\omega \gg \frac{R}{L}$, we have

$$\lim_{\omega\to\infty} H(\text{j}\omega) = \lim_{\omega\to\infty} \frac{\frac{R}{L}}{\frac{R}{L} + \text{j}\omega} = 0. \tag{9}$$

So this circuit is a *low-pass filter*. Notice that this circuit resembles the one in the previous part, except we have replaced the capacitor with an inductor. The effect of this change was to reverse the low-frequency and high-frequency behavior of the circuit! Another example of the complementarity of capacitors and inductors.

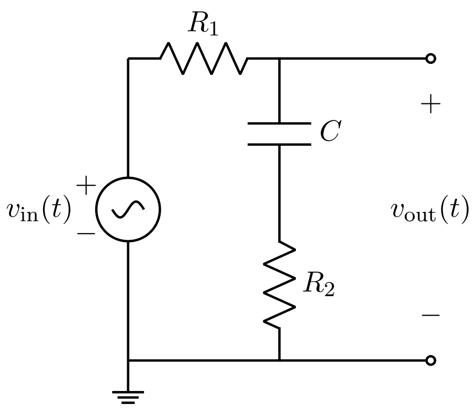
(c) **RCR circuit**

Solution: Even though there are three components instead of two, we can still use the voltage divider formula by treating R_2 and C as a single impedance given by $Z = Z_C + Z_{R_2}$, giving us $Z = R_2 + \frac{1}{\text{j}\omega C}$. This would give us

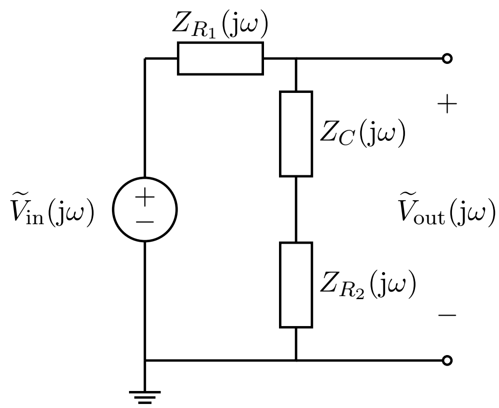
$$\widetilde{V}_{\text{out}}(\text{j}\omega) = \frac{Z}{Z_{R_1} + Z}\widetilde{V}_{\text{in}}(\text{j}\omega). \tag{10}$$

Then, the transfer function is

$$H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\text{j}\omega)}{\widetilde{V}_{\text{in}}(\text{j}\omega)} = \frac{R_2 + \frac{1}{\text{j}\omega C}}{R_1 + R_2 + \text{j}\frac{1}{\omega C}} = \frac{1 + \text{j}\omega R_2 C}{1 + \text{j}\omega C(R_1 + R_2)}. \tag{11}$$



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

At low frequencies, we have

$$\lim_{\omega \to 0} H(\mathrm{j}\omega) = 1, \tag{12}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} H(\mathrm{j}\omega) = \lim_{\omega \to \infty} \frac{R_2 + \frac{1}{\mathrm{j}\omega C}}{R_1 + R_2 + \frac{1}{\mathrm{j}\omega C}} = \frac{R_2}{R_1 + R_2}. \tag{13}$$

So at high frequencies, this circuit behaves like a regular voltage divider with just R_1 and R_2, as if the capacitor had vanished. This circuit is like a combination of a low-pass filter and a voltage divider: low frequency inputs are preserved, and high-frequency signals are diminished.

(d) **Assuming $v_{\text{in}}(t) = 12\sin(\omega_{\text{in}} t)$ compute the $v_{\text{out}}(t)$ using transfer functions computed in part (a).** For this part, we assume that $R = 1\,\mathrm{k}\Omega, L = 25\,\mu\mathrm{H}, C = 10\,\mu\mathrm{F}, \omega_{\text{in}} = 100\,\mathrm{Hz}$

Solution: Starting with $v_{\text{in}} = 12\sin(\omega t)$, to convert to the phasor domain we recall:

$$v_{\text{in}}(t) = 12\sin(\omega t) \tag{14}$$

$$v_0 \cos(\omega t + \phi) = \frac{v_0}{2}\left(e^{\mathrm{j}\omega t + \mathrm{j}\phi} + e^{-\mathrm{j}\omega t - \mathrm{j}\phi}\right) \tag{15}$$

$$= \frac{v_0 \mathrm{e}^{\mathrm{j}\phi}}{2}\mathrm{e}^{\mathrm{j}\omega t} + \frac{v_0 e^{-\mathrm{j}\phi}}{2}\mathrm{e}^{-\mathrm{j}\omega t} \tag{16}$$

where v_0 is the source voltage, ϕ is the phase shift, and t is the time. From the defintion of phasors, this gives us $\widetilde{V}_{\text{in}}(\mathrm{j}\omega) = \frac{v_0 \mathrm{e}^{\mathrm{j}\phi}}{2}$. Particularly for the sinusoidal signal, we have the phase shift $\phi = -\frac{\pi}{2}$.

$$\widetilde{V}_{\text{in}}(\mathrm{j}\omega) = 6\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}} \tag{17}$$

Using the input frequency ω_{in} we can now compute the transfer function in part (a) - (c).
For transfer function from (a), in terms of ω was $H(\mathrm{j}\omega) = \frac{\mathrm{j}\omega}{\mathrm{j}\omega + \frac{1}{RC}}$. Substituting for ω, R, C we have

$$H(\mathrm{j}\omega) = \frac{\mathrm{j}100}{\frac{1}{10^3 * 10^{-5}} + \mathrm{j}100} \tag{18}$$

$$= \frac{j}{1 + j} \tag{19}$$

$$\therefore |H(\mathrm{j}\omega)| = \frac{1}{\sqrt{2}}, \angle H(\mathrm{j}\omega) = \frac{\pi}{4} \tag{20}$$

From the definition of transfer funcitons, we conclude

$$\widetilde{V}_{\text{out}}(\mathrm{j}\omega) = H(\mathrm{j}\omega)\widetilde{V}_{\text{in}}(\mathrm{j}\omega) \tag{21}$$

$$= \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{j}\frac{\pi}{4}} * 6\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}} \tag{22}$$

$$= 3\sqrt{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}} \tag{23}$$

The last step is changing back to the time domain. Recall, that

$$v_{\text{out}}(t) = \widetilde{V}_{\text{out}}(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}\mathrm{e}^{-\mathrm{j}\omega t} \tag{24}$$

$$= 2|\widetilde{V}_{\text{out}}(\mathrm{j}\omega)|\cos\left(\omega t + \angle\widetilde{V}_{\text{out}}(\mathrm{j}\omega)\right) \tag{25}$$

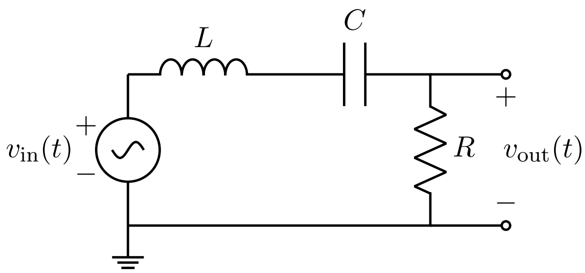
Substituting the values from eq. (23) we recover

$$v_{\text{out}}(t) = 6\sqrt{2}\cos\left(\omega_{\text{in}} t - \frac{\pi}{4}\right) \tag{26}$$

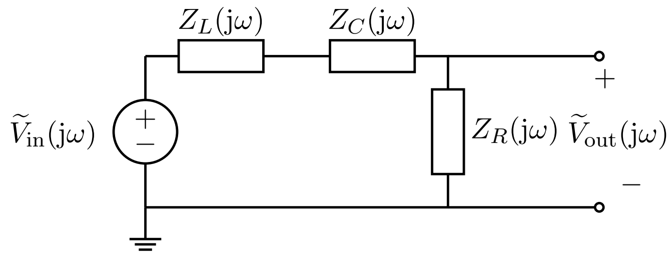
(e) **Visualizing Transfer functions**
In this part, we visualize the transfer function for different types of circuits in a Jupyter Notebook.

2. Band-pass filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

(a) **Write down the impedance of the series RLC combination in the form $Z_{RLC}(\mathrm{j}\omega) = A(\omega) + \mathrm{j}X(\omega)$, where $A(\omega)$ and $X(\omega)$ are real valued functions of ω.**

Solution: Since the capacitor, resistor and inductor are in series, the equivalent impedance is

$$Z_{RLC}(\mathrm{j}\omega) = Z_R(\mathrm{j}\omega) + Z_L(\mathrm{j}\omega) + Z_C(\mathrm{j}\omega) \tag{27}$$

$$= R + \mathrm{j}\omega L + \frac{1}{j\omega C} \tag{28}$$

$$= R + \mathrm{j}\left(\omega L - \frac{1}{\omega C}\right) \tag{29}$$

so by pattern matching to $Z_{RLC}(\mathrm{j}\omega) = A(\omega) + \mathrm{j}X(\omega)$,

$$A(\omega) = R \tag{30}$$

$$X(\omega) = \omega L - \frac{1}{\omega C} \tag{31}$$

(b) **Write down the transfer function $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)}$ for this circuit.**

Solution: Using the same voltage divider rule we've used in the past, $\widetilde{V}_{\text{out}}(\mathrm{j}\omega)$ is:

$$\widetilde{V}_{\text{out}}(\mathrm{j}\omega) = \widetilde{V}_{\text{in}}(\mathrm{j}\omega)\frac{Z_R}{Z_{RLC}} \tag{32}$$

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{R}{Z_{RLC}} \tag{33}$$

$$= \frac{R}{R + \mathrm{j}(\omega L - \frac{1}{\omega C})} \tag{34}$$

(c) **At what frequency ω_n does $X(\omega_n) = 0$?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other. This is called the *natural frequency*.)

Solution:

$$X(\omega_n) = \omega_n L - \frac{1}{\omega_n C} = 0 \tag{35}$$

Multiplying both sides by ω_n:

$$\omega_n^2 L - \frac{1}{C} = 0 \tag{36}$$

$$\omega_n = \frac{1}{\sqrt{LC}}. \tag{37}$$

(d) **What happens to the relative magnitude of the impedances of the capacitor and inductor as ω moves above and below ω_n? What is the value of the transfer function at this frequency ω_n?**

Solution: As the frequency ω increases above ω_n, the impedance of the inductor ($\mathrm{j}\omega L$, which is directly proportional to ω) increases in magnitude, while the impedance of the capacitor ($1/(j\omega C)$, inversely proportional to ω) decreases in magnitude. Since the two components are in series, the impedance of the inductor will dominate, so $X(\omega) = \omega L - \frac{1}{\omega C} \approx \omega L.$

For the same reason, as the frequency ω decreases below ω_n, the impedance of the inductor decreases in magnitude, while the impedance of the capacitor increases in magnitude. Thus the impedance of the capacitor will dominate, so $X(\omega) = \omega L - \frac{1}{\omega C} \approx -\frac{1}{\omega C}$,

At ω_n, $Z_{RLC} = R$, since the imaginary components cancel out perfectly. As a result $H(\mathrm{j}\omega_n) = \frac{R}{R} = 1.$

Contributors:

- Alex Devonport.
- Nathan Lambert.
- Kareem Ahmad.
- Kumar Krishna Agrawal.
- Sanjit Batra.
- Kris Pister.
- Pavan Bhargava.
- Anant Sahai.
- Druv Pai.

