EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet **Discussion 4B**

The following notes are useful for this discussion: Note 5

1. Phasor Analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \widetilde{X}\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{X}}\mathrm{e}^{-\mathrm{j}\omega t}, \tag{1}$$

where $\widetilde{X}$ is a time-independent (possibly) complex quantity called the **phasor** representation of $x(t)$ (recall that $\overline{z}$ denotes the complex conjugate of z. The complex conjugate of a complex number $z = a + \mathrm{j}b$ is $\overline{z} = a - \mathrm{j}b$). Note that:

(a) $\widetilde{X}$ and $\overline{\widetilde{X}}$ are complex conjugates of each other.

(b) $\mathrm{e}^{\mathrm{j}\omega t}$ and $\mathrm{e}^{-\mathrm{j}\omega t}$ are complex conjugates of each other.

(c) $\widetilde{X}\,\mathrm{e}^{\mathrm{j}\omega t}$ and $\overline{\widetilde{X}}\,\mathrm{e}^{-\mathrm{j}\omega t}$ are also complex conjugates of each other.

Note: We define the phasor corresponding to $x(t)$ as the coefficient of $\mathrm{e}^{\mathrm{j}\omega t}$ in eq. (1). Other resources (such as some past iterations of this class) define it slightly differently; the definitions differ by a factor of $\frac{1}{2}$. Some reasons for competing definitions are discussed in Note 5. Although the definitions in general lead to the same answers, be careful to use our class' definition and not get tripped up. For example, if we ask about the magnitude of the phasor, you wouldn't want to be off by a constant!

The phasor analysis method consists of five steps. The steps below are phrased in terms of any general circuit, but our goal is to apply these steps to the circuit we're given. Specifically, consider the RC circuit in fig. 1.

The voltage source is given by the sinusoid

$$v_S(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

with $\omega = 1 \times 10^3\,\frac{\mathrm{rad}}{\mathrm{s}}$, $R = \sqrt{3}\,\mathrm{k}\Omega$, and $C = 1\,\mu\mathrm{F}$.

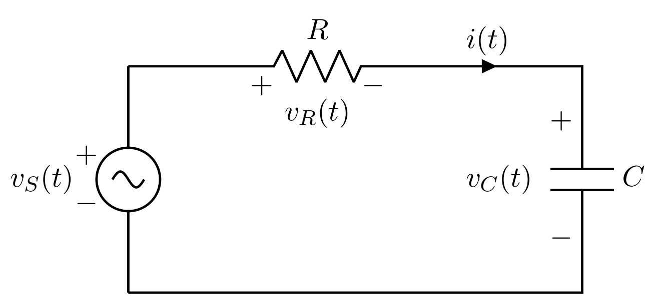

Figure 1

We seek to obtain a solution for $i(t)$ with the sinusoidal voltage source[1] $v_S(t)$.

(a) **Step 1: Write sources as exponentials: $\widetilde{X}\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{X}}\mathrm{e}^{-\mathrm{j}\omega t}$.**
All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. **Convert $v_S(t)$ into a exponential and write down its phasor representation $\widetilde{V}_S$.** Note that $v_S(t)$ is given in terms of a sine wave, not a cosine wave.

Solution:

$$\begin{aligned}
v_S(t) &= 12\sin\left(\omega t - \frac{\pi}{4}\right) \\
&= 12\left(\frac{\mathrm{e}^{\mathrm{j}(\omega t-\pi/4)}}{2\mathrm{j}} - \frac{\mathrm{e}^{-\mathrm{j}(\omega t-\pi/4)}}{2\mathrm{j}}\right) \\
&= 12\left(\frac{\mathrm{e}^{-\mathrm{j}(\pi/4)}}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}\omega t} - \frac{\mathrm{e}^{\mathrm{j}(\pi/4)}}{2\mathrm{j}}\mathrm{e}^{-\mathrm{j}\omega t}\right) \\
&= -\mathrm{j}\ 12\left(\frac{\mathrm{e}^{-\mathrm{j}(\pi/4)}}{2}\mathrm{e}^{\mathrm{j}\omega t} - \frac{\mathrm{e}^{\mathrm{j}(\pi/4)}}{2}\mathrm{e}^{-\mathrm{j}\omega t}\right) \\
&= 12\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\left(\frac{\mathrm{e}^{-\mathrm{j}(\pi/4)}}{2}\mathrm{e}^{\mathrm{j}\omega t} + \frac{\mathrm{e}^{\mathrm{j}\pi}\mathrm{e}^{\mathrm{j}(\pi/4)}}{2}\mathrm{e}^{-\mathrm{j}\omega t}\right) \\
&= 6\mathrm{e}^{-\mathrm{j}(3\pi/4)}\mathrm{e}^{\mathrm{j}\omega t} + 6\mathrm{e}^{\mathrm{j}(3\pi/4)}\mathrm{e}^{-\mathrm{j}\omega t} \\
&= 6\mathrm{e}^{-\mathrm{j}(3\pi/4)}\mathrm{e}^{\mathrm{j}\omega t} + \overline{6\mathrm{e}^{-\mathrm{j}(3\pi/4)}}\mathrm{e}^{-\mathrm{j}\omega t} \\
&= \widetilde{V}_S\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}}_s\mathrm{e}^{-\mathrm{j}\omega t}
\end{aligned}$$

From the problem statement and pattern matching, we can extract the phasor representation as the coefficient of $\mathrm{e}^{\mathrm{j}\omega t}$:

$$\widetilde{V}_S = 6\mathrm{e}^{-\mathrm{j}\frac{3\pi}{4}} \tag{3}$$

(b) **Step 2: Transform circuits to phasor domain.** The voltage source $v_S(t)$ is represented by its phasor $\widetilde{V}_S$. Similarly, $v_R(t)$ has phasor $\widetilde{V}_R$, and $v_C(t)$ has phasor $\widetilde{V}_C$.
The current $i(t)$ is related to its phasor counterpart $\widetilde{I}$ by

$$i(t) = \widetilde{I}\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{I}}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{4}$$

We redraw the circuit in phasor domain as in fig. 2. Recall that the impedances of the resistor, $Z_R(\mathrm{j}\omega)$, and capacitor, $Z_C(\mathrm{j}\omega)$, are as given below. We sometimes also refer to this as the "phasor representation" of R and C.

$$Z_R(\mathrm{j}\omega) = R \tag{5}$$

$$Z_C(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega C} \tag{6}$$

[1]The voltage source symbol here has a squiggly, not $+/-$. This is the symbol denoting a time-dependent, *sinusoidal* source; we have previously had input voltages dependent on time but in a piecewise-constant way (turns on at some time t). These do *not* have the mini-sine inside the source symbol.

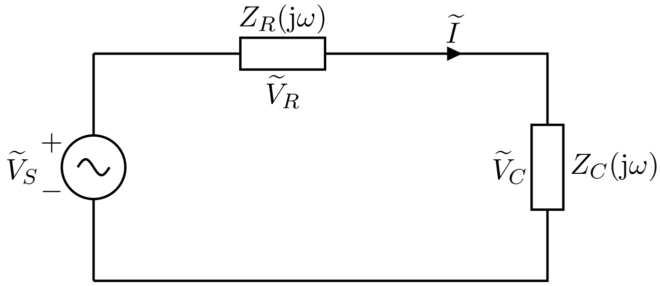

Figure 2: Circuit in "phasor domain"

Using the numbers given in the problem statement ($\omega = 1 \times 10^3\,\frac{\mathrm{rad}}{\mathrm{s}}$, $R = \sqrt{3}\,\mathrm{k}\Omega$, and $C = 1\,\mu\mathrm{F}$), **compute the numerical values of these impedances.**

Solution:

$$Z_R(\mathrm{j}\omega) = \sqrt{3} \cdot 10^3\ \Omega \tag{7}$$

$$Z_C(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega C} \tag{8}$$

$$= \frac{1}{\mathrm{j} \cdot (1 \cdot 10^3) \cdot (1 \cdot 10^{-6})} \tag{9}$$

$$= \frac{1000}{\mathrm{j}}\Omega \tag{10}$$

$$= -\mathrm{j}1000\Omega \tag{11}$$

(c) **[Practice]** As an intermediate step to use in the next subpart, **show that the fact that the first equation holds for all t implies the second equation:**

$$v_S(t) = v_R(t) + v_C(t) \tag{12}$$

$$\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C \tag{13}$$

Solution: This subpart shows that equations that derive from NVA hold in the phasor domain just as they hold in the time domain for circuits like this one. The details presented below are also in Note 5.

For this circuit, we can show that the following equation holds by applying NVA to the original circuit (you may recognize this as KVL):

$$v_S(t) = v_R(t) + v_C(t), \tag{14}$$

where we have denoted the voltage across the resistor as $v_R(t)$. If we expand all these quantities using phasors (using eq. (1)), we get

$$\underbrace{\widetilde{V}_S \mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}}_S \mathrm{e}^{-\mathrm{j}\omega t}}_{v_S(t)} = \underbrace{\widetilde{V}_R \mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}}_R \mathrm{e}^{-\mathrm{j}\omega t}}_{v_R(t)} + \underbrace{\widetilde{V}_C \mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}}_C \mathrm{e}^{-\mathrm{j}\omega t}}_{v_C(t)} \tag{15}$$

Collecting together all the $\mathrm{e}^{\mathrm{j}\omega t}$ terms and all the $\mathrm{e}^{-\mathrm{j}\omega t}$ terms, the above equation can be rewritten as

$$\left(\widetilde{V}_S - \widetilde{V}_R - \widetilde{V}_C\right)\mathrm{e}^{\mathrm{j}\omega t} + \left(\overline{\widetilde{V}_S - \widetilde{V}_R - \widetilde{V}_C}\right)\mathrm{e}^{-\mathrm{j}\omega t} = 0. \tag{16}$$

If $\omega \neq 0$,[2] it can be shown[3] that both of the coefficients of $e^{j\omega t}$ and $e^{-j\omega t}$ in the above equation must be equal to 0 for this equation to hold. That is:

$$\begin{aligned} \widetilde{V}_S - \widetilde{V}_R - \widetilde{V}_C &= 0, \quad \text{and} \\ \overline{\widetilde{V}_S - \widetilde{V}_R - \widetilde{V}_C} &= 0. \end{aligned} \tag{17}$$

Both of the equations above have the same meaning, i.e. , $\widetilde{V}_S - \widetilde{V}_R - \widetilde{V}_C = 0$ or

$$\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C. \tag{18}$$

This is exactly the same equation as given by $v_S(t) = v_R(t) + v_C(t)$, but using phasors. In the same way, you can show that KCL is also obeyed by phasors.

This conclusion implies that the standard rules for putting together circuit equations using NVA work identically with phasors as with time-varying notation. Now that we've shown that the phasor representations (i.e. , $\widetilde{I}$ and $\widetilde{V}$) of our circuit is equivalent to the time-varying representation (i.e. , $i(t)$ and $v(t)$), in the future we can write any NVA equations in phasor form directly.

(d) **Step 3: Cast the branch and element equations in the phasor domain.**

The previous subpart gave us a concrete relation we can use in the phasor domain to relate the voltages of the circuit elements. Specifically, **we know that $\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C$.**

Now, we must **substitute in the voltage phasors corresponding to these terms, using the element impedances given in Step 2.** At this point, feel free to leave the terms symbolic; in the next part, we will substitute in numbers.

Solution: The voltage-current relationships of elements should be in phasor form. The general formula (which holds for resistors, capacitors, and inductors) is as follows:

$$V_{\text{elem}} = I_{\text{elem}} \cdot Z_{\text{elem}}(j\omega). \tag{19}$$

We can therefore write that, for a resistor:

$$\widetilde{V}_R = \widetilde{I}_R Z_R(j\omega) \tag{20}$$

$$\widetilde{V}_R = \widetilde{I} R \tag{21}$$

For a capacitor,

$$\widetilde{V}_C = \widetilde{I}_C Z_C(j\omega) \tag{22}$$

$$\widetilde{V}_C = \widetilde{I}_C \frac{1}{j\omega C} \tag{23}$$

We can apply what we've found above to write the circuit in the phasor domain, starting from the given voltage equation (and noting that there's a single current $\widetilde{I}$ that flows through both R and C, so $\widetilde{I}_R \equiv \widetilde{I}_C \equiv \widetilde{I}$):

$$\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C \tag{24}$$

$$6e^{-j\frac{3\pi}{4}} = \widetilde{I}_R Z_R(j\omega) + \widetilde{I}_C Z_C(j\omega) \tag{25}$$

[2] Try working out the $\omega = 0$ case by yourself! It's even easier.

[3] This can be shown because the functions $e^{j\omega t}$ and $e^{-j\omega t}$ are linearly independent.

$$\implies 6e^{-j\frac{3\pi}{4}} = \widetilde{I}R + \widetilde{I}\frac{1}{j\omega C} \tag{26}$$

$$\implies 6e^{-j\frac{3\pi}{4}} = \widetilde{I}\left(R + \frac{1}{j\omega C}\right) \tag{27}$$

(e) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\widetilde{I}$ and $\widetilde{V}_C$. What is the polar form of $\widetilde{I}$ and $\widetilde{V}_C$? The polar form is given by $Ae^{j\theta}$, where A is a positive real number.

Hints:

- $\frac{\sqrt{3}}{2} - \frac{j}{2} = e^{-j\frac{\pi}{6}}$

Solution: Many of the simplifications here stem from results derived in Note j.

$$6e^{-j\frac{3\pi}{4}} = \widetilde{I}\left(R + \frac{1}{j\omega C}\right) \tag{28}$$

$$\implies \widetilde{I} = \frac{6e^{-j\frac{3\pi}{4}}}{R + \frac{1}{j\omega C}} \tag{29}$$

$$\widetilde{V}_C = \widetilde{I}Z_C \tag{30}$$

To derive the polar form, we plug in for the values of the impedances as solved for in Step 2, and then simplify. Since the voltage depends on the current, we'll solve for the current phasor first:

$$\widetilde{I} = \frac{6e^{-j\frac{3\pi}{4}}}{R + \frac{1}{j\omega C}} = \frac{6e^{-j\frac{3\pi}{4}}}{\sqrt{3}\cdot 1000 - j1000} \tag{31}$$

$$\implies \left|\widetilde{I}\right| = \frac{\left|6e^{-j\frac{3\pi}{4}}\right|}{\left|\sqrt{3}\cdot 1000 - j1000\right|} \tag{32}$$

$$= \frac{6}{\sqrt{\sqrt{3}^2\cdot 1000^2 + 1000^2}} \tag{33}$$

$$= \frac{6}{\sqrt{3\cdot 1000^2 + 1000^2}} = \frac{6}{\sqrt{4\cdot 10^6}} = \frac{6}{2000} \tag{34}$$

$$= 3\cdot 10^{-3}\mathrm{A} \tag{35}$$

We can similarly derive the phase of $\widetilde{I}$:

$$\measuredangle\widetilde{I} = \measuredangle 6e^{-j\frac{3\pi}{4}} - \measuredangle\left(\sqrt{3}\cdot 1000 - j1000\right) \tag{36}$$

$$= -\frac{3\pi}{4} - \measuredangle\mathrm{atan2}\left(-1000, \sqrt{3}\cdot 1000\right) \tag{37}$$

$$= -\frac{3\pi}{4} - \measuredangle\mathrm{atan2}\left(-1, \sqrt{3}\right) \tag{38}$$

$$= -\frac{3\pi}{4} + \frac{\pi}{6} \tag{39}$$

$$= -\frac{7\pi}{12} \tag{40}$$

Putting the two elements of the polar form together, we have that:

$$\widetilde{I} = \left|\widetilde{I}\right| \mathrm{e}^{\mathrm{j}\measuredangle \widetilde{I}} \tag{41}$$
$$= 3 \cdot 10^{-3} \mathrm{e}^{-\mathrm{j}\frac{7\pi}{12}} \mathrm{A} \tag{42}$$
$$= 3\mathrm{e}^{-\mathrm{j}\frac{7\pi}{12}} \mathrm{mA} \tag{43}$$

For the voltage, we now have:

$$\widetilde{V}_C = \widetilde{I} Z_C(\mathrm{j}\omega) \tag{44}$$
$$\implies \left|\widetilde{V}_C\right| = \left|\widetilde{I}\right| \left|Z_C(\mathrm{j}\omega)\right|, \measuredangle \widetilde{V}_C = \measuredangle \widetilde{I} + \measuredangle Z_C(\mathrm{j}\omega) \tag{45}$$

The magnitude first is:

$$\left|\widetilde{V}_C\right| = \left|\widetilde{I}\right| \left|Z_C(\mathrm{j}\omega)\right| \tag{46}$$
$$\left|\widetilde{V}_C\right| = \left|\widetilde{I}\right| \left|-\mathrm{j}1000\right| \tag{47}$$
$$= 3 \cdot 10^{-3} \cdot 1000 \tag{48}$$
$$= 3 \tag{49}$$

Now, the phase:

$$\measuredangle \widetilde{V}_C = \measuredangle \widetilde{I} + \measuredangle Z_C \mathrm{j}\omega) \tag{50}$$
$$= -\frac{7\pi}{12} - \frac{\pi}{2} \tag{51}$$
$$= -\frac{13\pi}{12} \equiv \frac{11\pi}{12} \tag{52}$$

So, we have that:

$$\widetilde{V}_C = \left|\widetilde{V}_C\right| \mathrm{e}^{\mathrm{j}\measuredangle \widetilde{V}_C} \tag{53}$$
$$= 3\mathrm{e}^{-\mathrm{j}\frac{13\pi}{12}} \mathrm{V} \tag{54}$$

(f) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the relation between a sinusoidal function and its phasor counterpart. **What is $i(t)$ and $v_C(t)$? What is the phase difference between $i(t)$ and $v_C(t)$?**

Solution:

$$i(t) = \widetilde{I}\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{I}}\mathrm{e}^{-\mathrm{j}\omega t} = 3\mathrm{e}^{-\mathrm{j}\frac{7\pi}{12}}\mathrm{e}^{\mathrm{j}\omega t} + 3\mathrm{e}^{\mathrm{j}\frac{7\pi}{12}}\mathrm{e}^{-\mathrm{j}\omega t} = 6\cos\left(\omega t - \frac{7\pi}{12}\right)\mathrm{mA} \tag{55}$$

$$v_C(t) = \widetilde{V}_C\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}_C}\mathrm{e}^{-\mathrm{j}\omega t} = 3\mathrm{e}^{-\mathrm{j}\frac{13\pi}{12}}\mathrm{e}^{\mathrm{j}\omega t} + 3\mathrm{e}^{\mathrm{j}\frac{13\pi}{12}}\mathrm{e}^{-\mathrm{j}\omega t} = 6\cos\left(\omega t - \frac{13\pi}{12}\right)\mathrm{V} \tag{56}$$

The phase difference between $i(t)$ and $v_C(t)$ is $\measuredangle \widetilde{I} - \measuredangle \widetilde{V}_C = -\frac{7\pi}{12} - \left(-\frac{13\pi}{12}\right) = \frac{\pi}{2}$.

(g) Now, suppose that instead of wherever we analyzed the phasor as $\widetilde{X}$ (the coefficient associated with the $e^{j\omega t}$ term), we had instead selected to work with $\overline{\widetilde{X}}$, or we solved using both $\widetilde{X}$ and $\overline{\widetilde{X}}$. **How would our answer or problem-solving procedure have changed?**

Solution: All of our answers would have come out to be the exact same, due to the complex cojugacy inherent in the phasor representation. Specifically, we only need *a single term/coefficient* to completely analyze a circuit in the phasor domain, and this is only possible because of the complex conjugacy (which manifests itself a result of how our circuit is operating with real quantities, like a sine wave input.)

If we wanted, we could keep track of just the conjugate, or even perform calculations for both $\widetilde{X}$ and $\overline{\widetilde{X}}$. However, due to the complex conjugacy, the relationship between these variables is known definitively, and we therefore do not lose information by dealing with only 1 term. This is what allows us to take the more streamlined approach.

2. Inductor Impedance

Given the voltage-current relationship of an inductor $v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$, we want to show that its complex impedance is $Z_L(j\omega) = \mathrm{j}\omega L$. We will perform this analysis in steps.

A sample inductor circuit is in fig. 3.

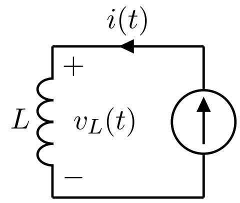

Figure 3: A simple inductor circuit.

(a) Suppose that the input current source in fig. 3 has value $i(t) = I_0 \mathrm{e}^{st}$, where I_0 is some (not necessarily real) constant. **What is the corresponding s-phasor for the current?**

Solution: By definition, the s-phasor of a time-domain signal is the coefficient of the time-dependent exponential. Here, since our input is $i(t) = I_0 \mathrm{e}^{st}$, the phasor $\widetilde{I} = I_0$.

(b) Now, using the governing voltage-current equation for an inductor, **derive the time-domain inductor voltage using the current expression** and **solve for the corresponding voltage s-phasor.**

Solution: By the inductor equation,

$$v_L(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \tag{57}$$

$$= L\frac{\mathrm{d}}{\mathrm{d}t}\left(I_0 \mathrm{e}^{st}\right) \tag{58}$$

$$= I_0 L s \mathrm{e}^{st} \tag{59}$$

Noting again that the phasor of a time-dependent term is the coefficient of the time-dependent exponential, we find that $v_L(t)$ has the phasor

$$\widetilde{V}_L = sLI_0. \tag{60}$$

(c) Using the voltage and current s-phasors, **solve for the s-impedance of the inductor $Z_L(s)$.** (This is the ratio between these phasor quantities).

Solution: The impedance of an inductor is an abstraction to model the inductor as a resistor in the phasor domain. Specifically, just as we take the ratio of V_R to I_R to find a resistor's value, we perform the analogous operation in the phasor domain for the inductor. This impedance is denoted $Z_L(s)$ for some complex number s.

$$Z_L(s) = \frac{\widetilde{V}}{\widetilde{I}} = sL \tag{61}$$

Note that the impedance here is a function of s, but not of I_0. That is, the behavior of the element is not dependent on the specific magnitude of the current, it only depends on the input current's frequency (captured in s).

(d) Now, suppose that our current source value was instead $i(t) = I_0 \cos(\omega t + \phi)$, where ω is the frequency of the cosine wave and ϕ is the phase-offset. $\phi = 0$ corresponds to the standard cosine centered at $t = 0$.

Using Euler's formula, represent $i(t)$ as the sum of two complex exponentials. Using this, **Find the new phasor $\widetilde{I}$ associated with the complex exponential $\mathrm{e}^{\mathrm{j}\omega t}$.**

Solution: We can apply Euler's directly to the given equation, noting that $\cos(x) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}x} + \mathrm{e}^{-\mathrm{j}x}\right)$.

$$i(t) = I_0 \cos(\omega t + \phi) \tag{62}$$
$$= I_0 \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}(\omega t+\phi)} + \mathrm{e}^{-\mathrm{j}(\omega t+\phi)}\right) \tag{63}$$
$$= \frac{I_0}{2}\left(\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\phi} + \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{e}^{-\mathrm{j}\phi}\right) \tag{64}$$
$$= \left(\frac{I_0}{2}\mathrm{e}^{\mathrm{j}\phi}\right)\mathrm{e}^{\mathrm{j}\omega t} + \left(\frac{I_0}{2}\mathrm{e}^{-\mathrm{j}\phi}\right)\mathrm{e}^{-\mathrm{j}\omega t} \tag{65}$$

So, $i(t)$ has the phasor

$$\widetilde{I} = \frac{I_0}{2}\mathrm{e}^{\mathrm{j}\phi}. \tag{66}$$

(e) Same as before, **use $i(t)$ to derive** $v(t)$ and **find the new phasor $\widetilde{V}$ associated with the complex exponential $\mathrm{e}^{\mathrm{j}\omega t}$.**

Solution: Differentiating and simplifying,

$$v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \tag{67}$$
$$= L\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{I_0}{2}\mathrm{e}^{\mathrm{j}\phi}\right)\mathrm{e}^{\mathrm{j}\omega t} + \left(\frac{I_0}{2}\mathrm{e}^{-\mathrm{j}\phi}\right)\mathrm{e}^{-\mathrm{j}\omega t}\right) \tag{68}$$
$$= \left(L\frac{I_0}{2}\mathrm{e}^{\mathrm{j}\phi}\mathrm{j}\omega\right)\mathrm{e}^{\mathrm{j}\omega t} + \left(-L\frac{I_0}{2}\mathrm{e}^{-\mathrm{j}\phi}\mathrm{j}\omega\right)\mathrm{e}^{-\mathrm{j}\omega t} \tag{69}$$

From the coefficient of $\mathrm{e}^{\mathrm{j}\omega t}$ (and using eq. (1)), we find that:

$$\widetilde{V} = \frac{I_0}{2}\mathrm{j}\omega L \mathrm{e}^{\mathrm{j}\phi} \tag{70}$$

(f) Once again, using the voltage and current phasors, **solve for the impedance of the inductor $Z_L(s)$.** Is this the same quantity that we found in the earlier subpart, as expected?

Solution: Indeed, we compute that $Z_L(s) = \frac{\widetilde{V}}{\widetilde{I}} = \mathrm{j}\omega L$, which matches with our previous result. We must note also that here, $s = \mathrm{j}\omega$ by definition.

Now, let's see how we could have used the first result (for a single complex exponential) and taken a shortcut for the generic sinusoid using superposition. By pattern-matching the expansion of $i(t) = I_0\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}(\omega t+\phi)} + \mathrm{e}^{-\mathrm{j}(\omega t+\phi)}\right)$ to the single s-exponential at the very start, we find that there are 2 components:

i. Component 1: $i_1(t) = \left(\frac{I_0}{2}\mathrm{e}^{\mathrm{j}\phi}\right)\mathrm{e}^{\mathrm{j}\omega t}$

ii. Component 2: $i_2(t) = \left(\frac{I_0}{2}\mathrm{e}^{-\mathrm{j}\phi}\right)\mathrm{e}^{-\mathrm{j}\omega t}$

(g) Now, **evaluate your expression for $Z_L(s)$ (from the single exponential case) at $s = \mathrm{j}\omega$, and $s = -\mathrm{j}\omega$.** What do you notice?

Solution: We can see that:

$$Z_L(s)\big|_{s=\mathrm{j}\omega} = \mathrm{j}\omega L \tag{71}$$

$$Z_L(s)\big|_{s=-\mathrm{j}\omega} = -\mathrm{j}\omega L \tag{72}$$

These answers are complex conjugates, and the s-values correspond to the 2 exponential components present in our generic cosine wave from item (d).

(h) Using the current components given above, **solve for the voltage phasors $\widetilde{V}_1$ and $\widetilde{V}_2$ as the product of the associated current phasors $\widetilde{I}_1$ and $\widetilde{I}_2$, and the corresponding impedances**. What do you notice about the current phasors? What do you notice about the voltage phasors? How can we explain the relationships between these results?

Solution: The current phasor associated with the first current component, $i_1(t)$, is $\widetilde{I}_1 = \frac{I_0}{2}\mathrm{e}^{\mathrm{j}\phi}$. The corresponding voltage is $v_1(t)$, and since our calculated impedance was $Z_L(\mathrm{j}\omega) = \mathrm{j}\omega L$, the voltage phasor is then $\widetilde{V}_1 = \mathrm{j}\omega L\widetilde{I}_1$.

Similarly, we can compute that $\widetilde{I}_2 = \frac{I_0}{2}\mathrm{e}^{-\mathrm{j}\phi}$ and $\widetilde{V}_2 = -\mathrm{j}\omega L\widetilde{I}_2$.

Going forward, when solving such problems with a sinudoisal voltage or current input, there's a key shortcut we can take. The above analysis allows us to apply the shortcut while maintaining understanding of what's actually happening.

We can see that $\widetilde{I}_2 = \overline{\widetilde{I}_1}$ (that is, **the current phasors are complex conjugates**). This will always be true for real inputs signals, like the generic cosine we had. Similarly, the impedances $Z_L(\mathrm{j}\omega) = \mathrm{j}\omega L$ and $Z_L(-\mathrm{j}\omega) = -\mathrm{j}\omega L$ are also complex conjugates, and the combination of these ultimately means that $\widetilde{V}_2 = \overline{\widetilde{V}_1}$ by the properties of complex numbers. Therefore, we don't need to explicitly compute both $\widetilde{V}_1$ and $\widetilde{V}_2$; once we calculate one, we can conjugate it to arrive at the other.

Fundamentally, this is why even a sinudoidal signal which consists of two separate complex exponentials can be represented with a single phasor quantity. Only one piece of information is required (and this is the coefficient of the $\mathrm{e}^{\mathrm{j}\omega t}$ term) because the coefficient of the $\mathrm{e}^{-\mathrm{j}\omega t}$ term will certainly be the complex conjugate for real signals. This is the same takeaway as presented at the end of the first problem.

Contributors:

- Yen-Sheng Ho.
- Regina Eckert.
- Jaijeet Roychowdhury.
- Sidney Buchbinder.
- Druv Pai.
- Neelesh Ramachandran.