EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet **Discussion 4A**

The following notes are useful for this discussion: Sections 2,3 from Note 3 and Section 1 in Note 4.

1. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations below:

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = \lambda z_1(t) \tag{1}$$
$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = \overline{\lambda} z_2(t) \tag{2}$$

with initial conditions $z_1(0) = c_0$ and $z_2(0) = \overline{c_0}$. Note, λ and c_0 are complex numbers and $\overline{\lambda}$ and $\overline{c_0}$ are their complex conjugates.

(a) First, assume that $\lambda = j$ in the equations for $z_1(t)$ and $z_2(t)$ above. **Solve for $z_1(t)$ and $z_2(t)$. Are the solutions complex conjugates?**

Solution: Using the initial conditions, and keeping our complex algebra principles in mind, we write:

$$z_1(t) = z_1(0)\mathrm{e}^{\lambda t} = c_0\mathrm{e}^{\mathrm{j}t} \tag{3}$$
$$z_2(t) = z_2(0)\mathrm{e}^{\overline{\lambda} t} = \overline{c}_0\mathrm{e}^{-\mathrm{j}t} \tag{4}$$

We observe that $\overline{z_1(t)} = \overline{c_0\mathrm{e}^{\mathrm{j}t}} = \overline{c}_0\mathrm{e}^{-\mathrm{j}t}$. So, we conclude that $z_1(t), z_2(t)$ are complex conjugates.

(b) Suppose now that we have the following different variables related to the original ones:

$$y_1(t) = az_1(t) + \overline{a}z_2(t) \tag{5}$$
$$y_2(t) = bz_1(t) + \overline{b}z_2(t) \tag{6}$$

where a and b are complex numbers and $\overline{a}$ and $\overline{b}$ are their complex conjugates. These numbers can be written in terms of their real and imaginary components:

$$a = a_r + ja_i, \qquad \overline{a} = a_r - ja_i, \tag{7}$$
$$b = b_r + jb_i, \qquad \overline{b} = b_r - jb_i, \tag{8}$$

where a_r, a_i, b_r, b_i are all real numbers. For all following subparts, assume that $\lambda = j$ unless specified.

How do the initial conditions for $\vec{z}(t)$ translate into the initial conditions for $\vec{y}(t)$? Are these purely real, purely imaginary, or complex numbers?

Solution: Just plugging $t = 0$ into the equation for $\vec{y}(t)$ using our known initial conditions for $\vec{z}(t)$, we find that:

$$y_1(0) = ac_0 + \overline{ac_0} \tag{9}$$

$$y_2(0) = bc_0 + \overline{bc_0} \tag{10}$$

These numbers are purely real, as we can see by expanding them or by directly noticing that these form sums of complex numbers with their respective conjugate.

(c) We noticed earlier that $z_1(t)$ and $z_2(t)$ are complex conjugates of each other. **What does this say about $y_1(t)$ and $y_2(t)$? (Are they purely real, purely imaginary, or complex?)**

Solution: Note that

$$y_1(t) = az_1(t) + \overline{a}z_2(t) = az_1(t) + \overline{a}\overline{z}_1(t) \tag{11}$$

Since we're adding a quantity to it's own complex conjugate, $y_1(t)$ is purely real. Similarly,

$$y_2(t) = bz_1(t) + \overline{b}z_2(t) = bz_1(t) + \overline{b}\overline{z}_1(t) \tag{12}$$

and we see that $y_2(t)$ is purely real.

(d) **Write out the change of variables in matrix-vector form $\vec{y} = V\vec{z}$.**

Solution: We just write things in vector/matrix form as:

$$\vec{y} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \vec{z} \tag{13}$$

$$\vec{y} = V\vec{z} \tag{14}$$

This tells us what V is.

(e) *(Practice)*: **Find an expression for the determinant of V. Further, simplify $a\overline{b} + \overline{a}b$, $a\overline{a}$ where a, b are complex numbers.**.

Solution: Thereby, computing the determinant of V, we have:

$$\det(V) = a\overline{b} - \overline{a}b \tag{15}$$

$$a\overline{b} - \overline{a}b = (a_r + \mathrm{j}a_i)(b_r - \mathrm{j}b_i) - (a_r - \mathrm{j}a_i)(b_r + \mathrm{j}b_i) \tag{16}$$

$$= (a_r b_r + a_i b_i - \mathrm{j}b_i a_r + \mathrm{j}a_i b_r) - (a_r b_r + a_i b_i - \mathrm{j}a_i b_r + \mathrm{j}a_r b_i) \tag{17}$$

$$= -2\mathrm{j}a_r b_i + 2\mathrm{j}a_i b_r \tag{18}$$

Note that this coefficient coming from the determinant part of the inverse in front of the matrix in V^{-1} is purely imaginary — this can be seen directly from the fact that it is a complex number minus its conjugate. (The real parts cancel away.)

Simplifying $a\overline{b} + \overline{a}b$, we have

$$a\overline{b} + \overline{a}b = (a_r + \mathrm{j}a_i)(b_r - \mathrm{j}b_i) + (a_r - \mathrm{j}a_i)(b_r + \mathrm{j}b_i) \tag{19}$$

$$= (a_r b_r + a_i b_i - \mathrm{j}b_i a_r + \mathrm{j}a_i b_r) + (a_r b_r + a_i b_i - \mathrm{j}a_i b_r + \mathrm{j}a_r b_i) \tag{20}$$

$$= 2(a_r b_r + a_i b_i) \tag{21}$$

Also, for $a\overline{a}$

$$a\overline{a} = (a_r + \mathrm{j}a_i)(a_r - \mathrm{j}a_i) \tag{22}$$

$$= (a_r^2 - \mathrm{j}a_r a_i + \mathrm{j}a_r a_i + a_i^2) \tag{23}$$

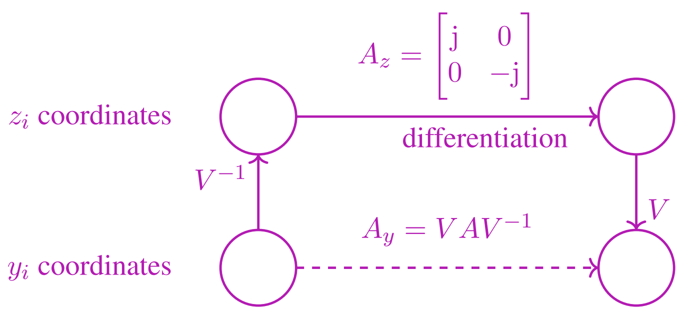
$$= a_r^2 + a_i^2 \tag{24}$$

(f) **Write out the system of differential equations for $\frac{\mathrm{d}}{\mathrm{d}t} y_i(t)$ and $y_i(t)$.**

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \begin{bmatrix} \mathrm{j} & 0 \\ 0 & -\mathrm{j} \end{bmatrix} \vec{z}(t) \tag{25}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{26}$$



$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = A_y \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{27}$$

$$= VA_zV^{-1} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{28}$$

$$\implies VA_zV^{-1} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \begin{bmatrix} \mathrm{j} & 0 \\ 0 & -\mathrm{j} \end{bmatrix} \frac{1}{\det(V)} \begin{bmatrix} \overline{b} & -\overline{a} \\ -b & a \end{bmatrix} \tag{29}$$

$$= \frac{1}{\det(V)} \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \begin{bmatrix} +j\overline{b} & -\mathrm{j}\overline{a} \\ +\mathrm{j}b & -\mathrm{j}a \end{bmatrix} \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{\det(V)} \begin{bmatrix} +\mathrm{j}(a\overline{b} + \overline{a}b) & -2\mathrm{j}a\overline{a} \\ 2\mathrm{j}b\overline{b} & -\mathrm{j}(a\overline{b} + \overline{a}b) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{31}$$

Now, we could leave the final expression as is, but we can also simplify further and expand this out into the $a = a_r + \mathrm{j}a_i$ form: Using results from previous subpart, we have:

$$a\overline{b} + \overline{a}b = 2(a_rb_r + a_ib_i) \tag{32}$$

$$a\overline{a} = a_r^2 + a_i^2 = |a|^2 \tag{33}$$

$$b\overline{b} = b_r^2 + b_i^2 = |b|^2 \tag{34}$$

Note that all of these numbers are purely real since they are either magnitudes squared or the sum of a complex number with its conjugate. We can now substitute into our last expression, and after simplifying, we arrive at:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{(2a_r b_i - 2a_i b_r)} \begin{bmatrix} 2(a_r b_r + a_i b_i) & -2|a|^2 \\ 2|b|^2 & -2(a_r b_r + a_i b_i) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{35}$$

We notice that our differential equation matrix A_y is purely real.

(g) *(Practice)* **Repeat the problem for general complex λ, i.e find solution for $\vec{z}(t)$ and $\vec{y}(t)$, as well as the system of differential equations governing $\vec{y}(t)$.**

Solution: Starting with the differential equation, we have

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = \lambda z_1(t) \tag{36}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = \overline{\lambda} z_2(t) \tag{37}$$

Solving the uncoupled system, we get

$$z_t(t) = c_0 \mathrm{e}^{\lambda t} \tag{38}$$

$$z_2(t) = \overline{c_0} \mathrm{e}^{\overline{\lambda} t} \tag{39}$$

Additionally, from the transform of variables $\vec{y} = V\vec{z}$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \begin{bmatrix} c_0 \mathrm{e}^{\lambda t} \\ \overline{c_0} \mathrm{e}^{\overline{\lambda} t} \end{bmatrix} \tag{40}$$

$$= \begin{bmatrix} a c_0 \mathrm{e}^{\lambda t} + \overline{a c_0} \mathrm{e}^{\overline{\lambda} t} \\ b c_0 \mathrm{e}^{\lambda t} + \overline{b c_0} \mathrm{e}^{\overline{\lambda} t} \end{bmatrix} \tag{41}$$

Note that even for general λ, the solutions $z_1(t), z_2(t)$ are conjugates. Additionally, the solutions $y_1(t), y_2(t)$ are purely real.

(h) Above, we were already given the system in nice decoupled coordinates $\vec{z}$. In general, problems will present in the more coupled form of $\vec{y}$ above. We know how to discover nice coordinates for ourselves. **Find the eigenvalues $\lambda_1,\ \lambda_2$ for the differential equation matrix for $\vec{y}(t)$ above. Verify that the eigenvalues are** $(\mathrm{j}, -\mathrm{j})$.

Solution: The eigenvalues of just the matrix (without the scalar factor) are:

$$M = \begin{bmatrix} 2(a_r b_r + a_i b_i) & -2|a|^2 \\ 2|b|^2 & -2(a_r b_r + a_i b_i) \end{bmatrix} \tag{42}$$

are given by forming the characteristic polynomial of $M - \lambda I$:

$$\left(2\left(a_r b_r + a_i b_i\right) - \lambda\right)\left(-2(a_r b_r + a_i b_i) - \lambda\right) + 4|a|^2|b|^2 = 0 \tag{43}$$

$$\left(\lambda^2 - 4\left(a_r b_r + a_i b_i\right)^2\right) + 4\left(a_r^2 + a_i^2\right)\left(b_r^2 + b_i^2\right) = 0 \tag{44}$$

$$\left(\lambda^2 - 4\left(a_r^2 b_r^2 + 2a_r b_r a_i b_i + a_i^2 b_i^2\right)\right) + 4\left(a_r^2 b_r^2 + a_r^2 b_i^2 + a_i^2 b_r^2 + a_i^2 b_i^2\right) = 0 \tag{45}$$

$$\lambda^2 + 4\left(a_r^2 b_i^2 - 2a_r b_r a_i b_i + a_i^2 b_r^2\right) = 0 \tag{46}$$

$$\lambda^2 + 4\left(a_r b_i - a_i b_r\right)^2 = 0 \tag{47}$$
$$\lambda = \pm \mathrm{j} \cdot 2\left(a_r b_i - a_i b_r\right) \tag{48}$$

The eigenvalues of

$$A_y = \frac{1}{2(a_r b_i - a_i b_r)} \begin{bmatrix} 2(a_r b_r + a_i b_i) & -2|a|^2 \\ 2|b|^2 & -2(a_r b_r + a_i b_i) \end{bmatrix} \tag{49}$$

are therefore given by:

$$\lambda = \pm 2\mathrm{j}(a_r b_i - a_i b_r) \frac{1}{2(a_r b_i - a_i b_r)} = \pm \mathrm{j} \tag{50}$$

(i) *(Practice):* **Find the associated eigenspaces for these eigenvalues.**

Solution: The eigenspace associated with $\lambda_1 = \mathrm{j}$ is given by:

$$\frac{1}{(2a_r b_i - 2a_i b_r)} \begin{bmatrix} 2(a_r b_r + a_i b_i) - \mathrm{j} & -2|a|^2 \\ 2|b|^2 & -2(a_r b_r + a_i b_i) - \mathrm{j} \end{bmatrix} \vec{y}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{51}$$
$$\vec{y}_{\lambda_1} = \alpha \begin{bmatrix} a \\ b \end{bmatrix} \tag{52}$$

The eigenspace associated with $\lambda_2 = -\mathrm{j}$ is given by:

$$\frac{1}{(2a_r b_i - 2a_i b_r)} \begin{bmatrix} 2(a_r b_r + a_i b_i) + \mathrm{j} & -2|a|^2 \\ 2|b|^2 & -2(a_r b_r + a_i b_i) + \mathrm{j} \end{bmatrix} \vec{y}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{53}$$
$$\vec{y}_{\lambda_2} = \beta \begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix} \tag{54}$$

(j) *(Practice):* **Change variables into the eigenbasis to re-express the differential equations in terms of new variables $y_{\lambda_1}(t)$ and $y_{\lambda_2}(t)$.** (The variables should be in eigenbasis aligned coordinates.)

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} = \begin{bmatrix} j & 0 \\ 0 & -\mathrm{j} \end{bmatrix} \begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} \tag{55}$$

(k) *(Practice)*: **Solve the differential equation for $y_{\lambda_i}(t)$ in the eigenbasis.**

Solution:

$$y_{\lambda_1}(t) = K_1 \mathrm{e}^{\mathrm{j}t} \tag{56}$$
$$y_{\lambda_2}(t) = K_2 \mathrm{e}^{-\mathrm{j}t} \tag{57}$$
$$y_{\lambda_1}(0) = K_1 \mathrm{e}^{\mathrm{j}t} = c_0 \tag{58}$$
$$y_{\lambda_2}(0) = K_2 \mathrm{e}^{-\mathrm{j}t} = \bar{c}_0 \tag{59}$$
$$y_{\lambda_1}(t) = c_0 \mathrm{e}^{jt} \tag{60}$$

$$y_{\lambda_2}(t) = \overline{c}_0 \mathrm{e}^{-\mathrm{j}t} \tag{61}$$

(l) *(Practice)*: **Convert your solution back to $\vec{y}(t)$ coordinate and find $\vec{y}(t)$**

Solution: The eigenspace associated with $\lambda_1 = j$ is given by:

$$\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} a \\ b \end{bmatrix} \tag{62}$$

The eigenspace associated with $\lambda_2 = -\mathrm{j}$ is given by:

$$\vec{v}_{\lambda_2} = \beta \begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix} \tag{63}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = V \begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} \tag{64}$$

$$y_1(t) = ac_0 \mathrm{e}^{jt} + \overline{a}\overline{c}_0 \mathrm{e}^{-\mathrm{j}t} \tag{65}$$

$$y_2(t) = bc_0 \mathrm{e}^{jt} + \overline{b}\overline{c}_0 \mathrm{e}^{-\mathrm{j}t} \tag{66}$$

Notice that this is in classic phasor form — something plus its complex conjugate.

2. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage is proportional to the derivative of the current across it. That is:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{67}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the counterpart circuit for an inductor:

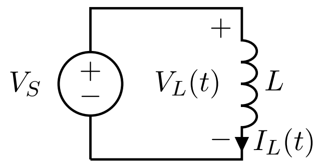


Figure 1: Inductor in series with a voltage source.

(a) **What is the current through an inductor as a function of time? If the inductance is $L = 3\,\mathrm{H}$, what is the current at $t = 6\,\mathrm{s}$?** Assume that the voltage source turns from $0\,\mathrm{V}$ to $5\,\mathrm{V}$ at time $t = 0\,\mathrm{s}$, and there's no current flowing in the circuit before the voltage source turns on i.e $I_L(0) = 0\,\mathrm{A}$.

Solution: We proceed to analyze the given equation. Note that the voltage source is held at a constant value for $t \geq 0$, which allows us to express the derivative of current as a constant:

$$V_L(t) = L\frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{68}$$

$$\frac{V_S}{L} = \frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{69}$$

From here, we can see that the derivative of the current is a constant with respect to time! This immediately indicates that we have a linear relationship between current and time, with a slope set by the derivative. In terms of a general initial condition, the current is:

$$I_L(t) = \frac{V_S}{L}t + I_L(0) \tag{70}$$

So, the current in the inductor keeps growing over time! Inductors store energy in their magnetic field, so the more time that this voltage source feeds the inductor, the higher the current, and the greater the stored energy.

Substituting in the specific values asked for, $I_L(6\,\mathrm{s}) = \frac{5\,\mathrm{V}}{3\,\mathrm{H}} \cdot 6\,\mathrm{s} = 10\,\mathrm{A}$.

(b) Now, we add some resistance in series with the inductor, as in Figure 2.

Solve for the current $I_L(t)$ and voltage $V_L(t)$ in the circuit over time, in terms of R, L, V_S, t. Note that $I_L(0) = 0\,\mathrm{A}$.

Solution: We begin by considering the voltage drop across the resistor, in terms of source voltage and inductor voltage. There's also only a single current in the circuit (the one we're solving for, $I(t)$):

$$V_R(t) = V_S - V_L(t) \tag{71}$$

$$R\,I_L(t) = V_S - L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) \tag{72}$$

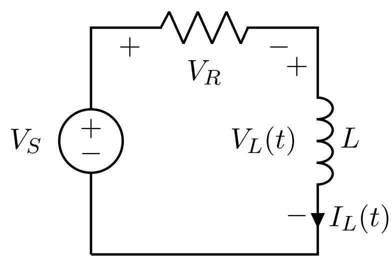


Figure 2: Inductor in series with a voltage source.

$$\frac{\mathrm{d}}{\mathrm{d}t} I_L(t) = -\frac{R}{L} I_L(t) + \frac{V_S}{L} \tag{73}$$

We recognize this as a first-order differential equation! With the practice we have had so far, we could jump straight to the solution for the current $I_L(t)$ since we know the initial condition ($I_L(0) = 0$):

$$I_L(t) = \frac{V_S}{R}\left(1 - \mathrm{e}^{-\frac{R}{L}t}\right) \tag{74}$$

Alternatively, we can derive this solution using a change of variables, $\widetilde{I}_L(t) = I_L(t) - \frac{V_S}{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{I}_L(t) = -\frac{R}{L} I_L(t) \tag{75}$$

Now, we find that $\widetilde{I}_L(t) = c\mathrm{e}^{-\frac{R}{L}t}$, with c to be solved for from the initial condition. Since $I_L(0) = 0$, and $I(t) = \widetilde{I}_L(t) + \frac{V_S}{R} = c\mathrm{e}^{-\frac{R}{L}t} + \frac{V_S}{R}$, we can say that $c = -\frac{V_S}{R}$, and so:

$$I_L(t) = -\frac{V_S}{R}\mathrm{e}^{-\frac{R}{L}t} + \frac{V_S}{R} \tag{76}$$

$$= \frac{V_S}{R}\left(1 - \mathrm{e}^{-\frac{R}{L}t}\right) \tag{77}$$

This is the same current going through the resistor, so we can use this information to conclude that:

$$V_L(t) = V_S \mathrm{e}^{-\frac{R}{L}t} \tag{78}$$

(c) *(Practice):* **Suppose** $R = 500\,\Omega, L = 1\,\mathrm{mH}, V_S = 5\,\mathrm{V}$. **Plot the current through and voltage across the inductor** ($I_L(t), V_L(t)$)**, as these quantities evolve over time.**

Solution: The current begins at $0\,\mathrm{A}$ and over time, the inductor begins to look like a short. In the long-term, the current settles to $\frac{V_S}{R}\mathrm{A} = 1\,\mathrm{mA}$. The voltage begins at $V_S = 5\,\mathrm{V}$ because the inductor initially looks like an open circuit, and this voltage decreases exponentially over time down to zero.

The time constant governing both of these transient curves is $\tau = \frac{L}{R} = 2\,\mu\mathrm{s}$. Using this information, we can sketch the curves for current (Figure 3) and inductor voltage (Figure 4). Notice that it is perfectly fine for the voltage to be discontinuous, but the same is not true for the current.

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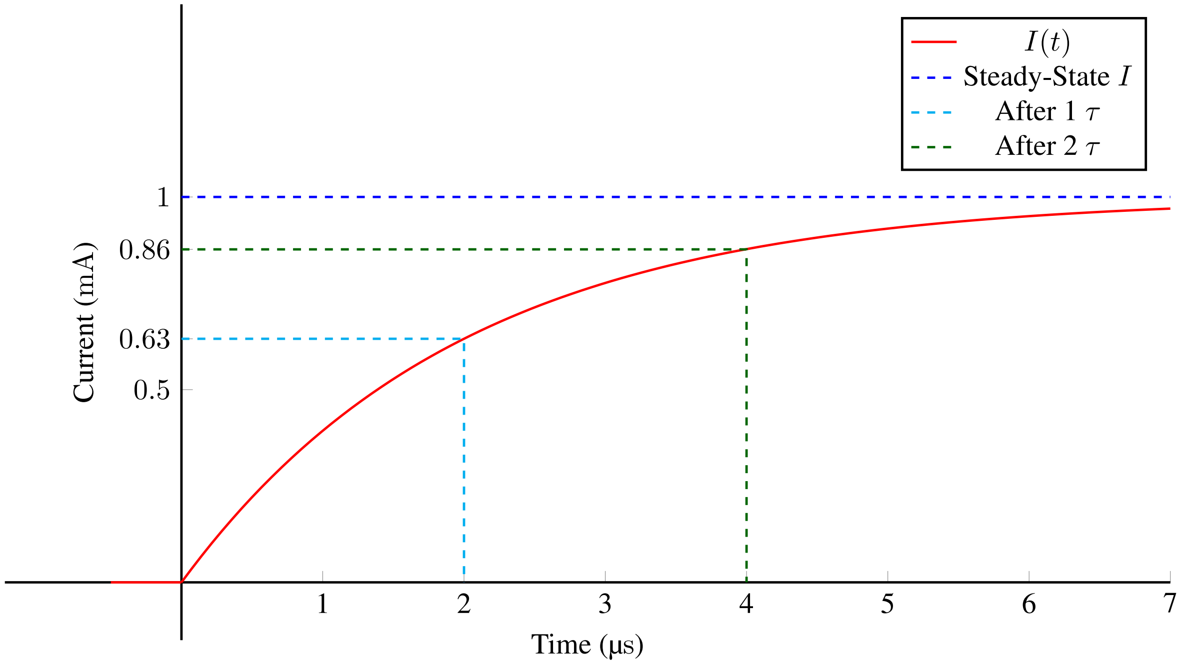

Figure 3: Transient Current in an RL circuit (with initial current $I(0) = 0\,\mathrm{A}$.)

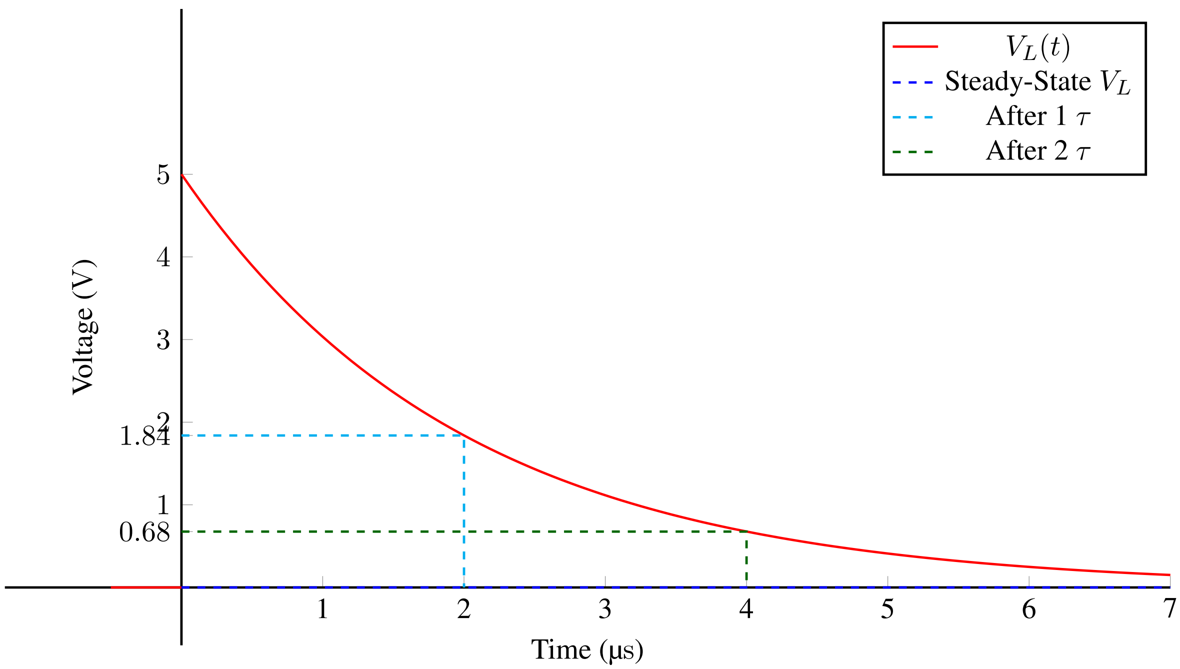

Figure 4: Transient Voltage across the inductor in an RL circuit (with initial current $I(0) = 0\,\mathrm{A}$.)

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