EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet — Discussion 3B

The following notes are useful for this discussion: Note 3

1. Changing Coordinates and Systems of Differential Equations, II

In the previous discussion we analyzed and solved a pair of differential equations where the variables of interest were coupled (i.e atleast one equation depends on more than variable).

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{1}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t). \tag{2}$$

We solved this system by using a coordinate transformation that gave us a decoupled system of equations. In the last discussion we were simply handed this transformation, but in this discussion we will construct the transformation for ourselves.

We will focus our explorations on the voltages across the capacitors in the following circuit.

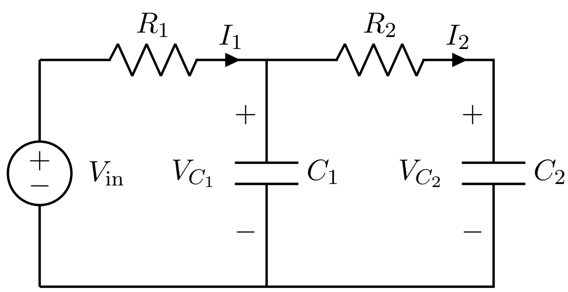

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

(a) **Write the system of differential equations governing the voltages across the capacitors** V_{C_1}, V_{C_2}. Use the following values: $C_1 = 1\mu\text{F}, C_2 = \frac{1}{3}\mu\text{F}, R_1 = \frac{1}{3}\text{M}\Omega, R_2 = \frac{1}{2}\text{M}\Omega$.

Solution: Start by solving for the currents and voltages across the capacitors:

$$V_{C_2} = V_{C_1} - I_2 R_2, \quad I_2 = C_2 \frac{\mathrm{d}}{\mathrm{d}t} V_{C_2} \tag{3}$$

$$V_{\text{in}} - I_1 R_1 = V_{C_1}, \quad I_1 = I_2 + C_1 \frac{\mathrm{d}}{\mathrm{d}t} V_{C_1} \tag{4}$$

Yields,

$$I_1 = \frac{V_{\text{in}}}{R_1} - \frac{V_{C_1}}{R_1}, \quad I_2 = \frac{V_{C_1}}{R_2} - \frac{V_{C_2}}{R_2} \tag{5}$$

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Now, we can plug into the formula for current across a capacitor:

$$\begin{aligned}
\frac{\mathrm{d}V_{C_1}}{\mathrm{d}t} &= \frac{1}{C_1}(I_1 - I_2) && (6)\\
&= \frac{1}{C_1}\left(\frac{V_{\text{in}}}{R_1} - \frac{V_{C_1}}{R_1} - \frac{V_{C_1}}{R_2} + \frac{V_{C_2}}{R_2}\right) && (7)\\
&= -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_{C_1} + \frac{V_{C_2}}{R_2C_1} + \frac{V_{\text{in}}}{R_1C_1} && (8)\\
\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t} &= \frac{1}{C_2}(I_2) && (9)\\
&= \frac{V_{C_1}}{R_2C_2} - \frac{V_{C_2}}{R_2C_2} && (10)
\end{aligned}$$

Now group the terms into a matrix with the values given above,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{\text{in}}(t) \tag{11}$$

Plugging in the values for R, C yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{\text{in}}(t) \tag{12}$$

(b) Suppose also that V_{in} was at $7\,\text{V}$ for a long time, and then transitioned to be $0\,\text{V}$ at time $t = 0$. This "new" system of differential equations (valid for $t \geq 0$)

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{13}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t) \tag{14}$$

with initial conditions $z_1(0) = 7$ and $z_2(0) = 7$.

Write out the differential equations and initial conditions in matrix/vector form.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} \tag{16}$$

We will define the differential matrix as A_z, where

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = A_z\vec{z}(t) \tag{17}$$

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$$A_z = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \tag{18}$$

(c) **Find the eigenvalues $\lambda_1,\ \lambda_2$ and eigenspaces for the matrix corresponding to the differential equation matrix above.**

Solution: Eigenvalues λ and eigenvectors v of matrix A are given by

$$A_z v = \lambda v \tag{19}$$

In order to find the eigenvalues, we take the determinant:

$$\det\left(A_z - \lambda I\right) = 0 \tag{20}$$

$$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = 0 \tag{21}$$

We can solve this using a 2×2 determinant form seen in 16A, (or by Gaussian elimination)

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc, \tag{22}$$

$$(-5-\lambda)(-6-\lambda) - 12 = 0 \tag{23}$$

$$30 + 11\lambda + \lambda^2 - 12 = 0 \tag{24}$$

$$\lambda^2 + 11\lambda + 18 = 0 \tag{25}$$

$$(\lambda+9)(\lambda+2) = 0 \tag{26}$$

Giving:

$$\lambda = -9, -2 \tag{27}$$

The eigenspace associated with $\lambda_1 = -9$ is given by:

$$\begin{bmatrix} -5+9 & 2 \\ 6 & -6+9 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

$$\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{30}$$

The eigenspace associated with $\lambda_2 = -2$ is given by:

$$\begin{bmatrix} -5+2 & 2 \\ 6 & -6+2 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{31}$$

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$$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{32}$$

$$\vec{v}_{\lambda_2} = \beta \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{33}$$

(d) **Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t),\ z_{\lambda_2}(t)$.** (These variables represent eigenbasis-aligned coordinates.)

Solution:

$$\vec{z} = \vec{v}_{\lambda_1} y_{\lambda_1} + \vec{v}_{\lambda_2} y_{\lambda_2}$$
$$\vec{z} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_{\lambda_1} \\ y_{\lambda_2} \end{bmatrix}$$

We can define the change-of-coordinates matrix from the eigenbasis to our original basis as:

$$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \tag{34}$$

Changing coordinates to the eigenbasis:

$$\begin{bmatrix} y_{\lambda_1} \\ y_{\lambda_2} \end{bmatrix} = V^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
$$A_{y_\lambda} = V^{-1} A_z V = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 9 & -4 \\ -18 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}$$

That is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} \tag{35}$$

(e) **Solve the differential equation for $y_{\lambda_i}(t)$ in the eigenbasis.**
Don't forget about the initial conditions!

Solution: First we get the initial condition:

$$\vec{y}_\lambda(0) = V^{-1} \vec{z}(0) = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{36}$$

Then we solve based on the form of the problem and our previous differential equation experience:

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$$\vec{y}_\lambda(t) = \begin{bmatrix} K_1 e^{-9t} \\ K_2 e^{-2t} \end{bmatrix} \tag{37}$$

Plugging in for the initial condition gives:

$$\vec{y}_\lambda(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} \tag{38}$$

(f) **Convert your solution back into the original coordinates to find $z_i(t)$.**

Solution:

$$\vec{z}(t) = V\vec{y}_\lambda(t) = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix} \tag{39}$$

(g) In part (b) of the discussion, we make a simplifying assumption V_{in} transitions from $7\,\text{V}$ to $0\,\text{V}$ at $t = 0$. We now consider the setting, where the voltage V_{in} transitions from $0\,\text{V}$ to $7\,\text{V}$ at $t = 0$, i.e we have $V_{\text{in}}(t) = 7\,\text{V}$ for $t \geq 0$ **Find the solution for $z_i(t)$ under these assumptions.**

Solution: From eq. (12), we have the general form of the differential equation (valid for $t \geq 0$):

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{\text{in}}(t) \tag{40}$$

In this subpart, we have $V_{\text{in}}(t) = 7$, with the initial conditions $V_{C_1}(0) = 0$ and $V_{C_2}(0) = 0$. This system of coupled differential equations in terms of $z_i(t)$ becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{42}$$

Concisely, we rewrite this in matrix-vector form as :

$$\frac{\mathrm{d}\vec{z}}{\mathrm{d}t} = A_z\vec{z} + \vec{b_z} \tag{43}$$

Performing a change of basis as in item (d), we use $\vec{z} = V\vec{y}$, such that :

$$\frac{\mathrm{d}}{\mathrm{d}t} V\vec{y} = A_z V\vec{y} + \vec{b_z} \tag{44}$$

$$V\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = A_z V\vec{y} + \vec{b_z} \tag{45}$$

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = V^{-1}A_z V\vec{y} + V^{-1}\vec{b_z} \tag{46}$$

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where $V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$. Simplifying, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} \frac{-3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 21 \\ 0 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} -9 \\ 6 \end{bmatrix} \tag{48}$$

This is now a system of uncoupled non-homogeneous differential equations,

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -9y_1(t) - 9 \tag{49}$$

$$\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} = -2y_2(t) + 6 \tag{50}$$

with initial conditions

$$\vec{y}(0) = V^{-1}\vec{z}(0)\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{51}$$

With change of variables, we solve eq. (50) to recover the solution:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{-9t} - 1 \\ -3e^{-2t} + 3 \end{bmatrix} \tag{52}$$

Finally, substituting back into the original coordinates, $\vec{z}_t = V\vec{y}(t)$, i.e

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} e^{-9t} - 1 \\ -3e^{-2t} + 3 \end{bmatrix} \tag{53}$$

$$= \begin{bmatrix} 7 - e^{-9t} - 6e^{-2t} \\ 7 + 2e^{-9t} - 9e^{-2t} \end{bmatrix} \tag{54}$$

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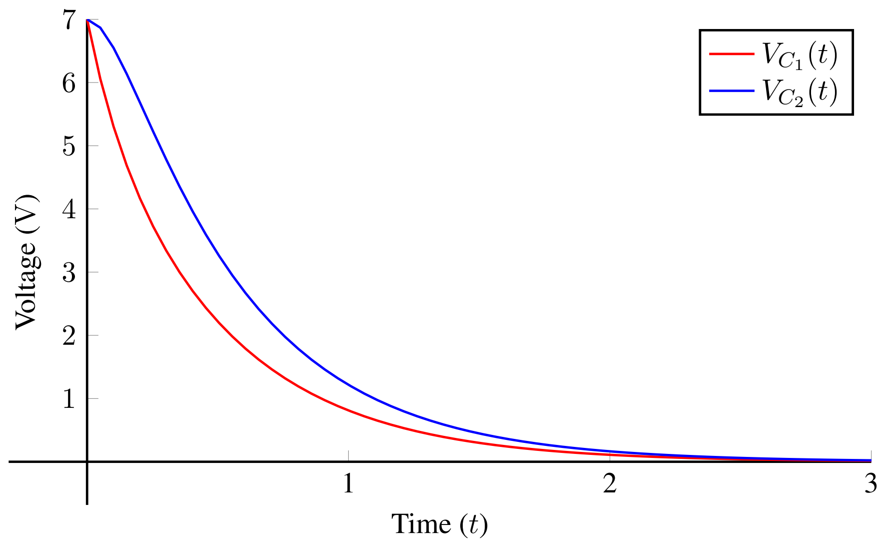

Figure 2: Initial Conditions: $V_{C_1}(0) = 7\,\text{V}$ and $V_{C_2}(0) = 7\,\text{V}$. Homogeneous Case Solution.

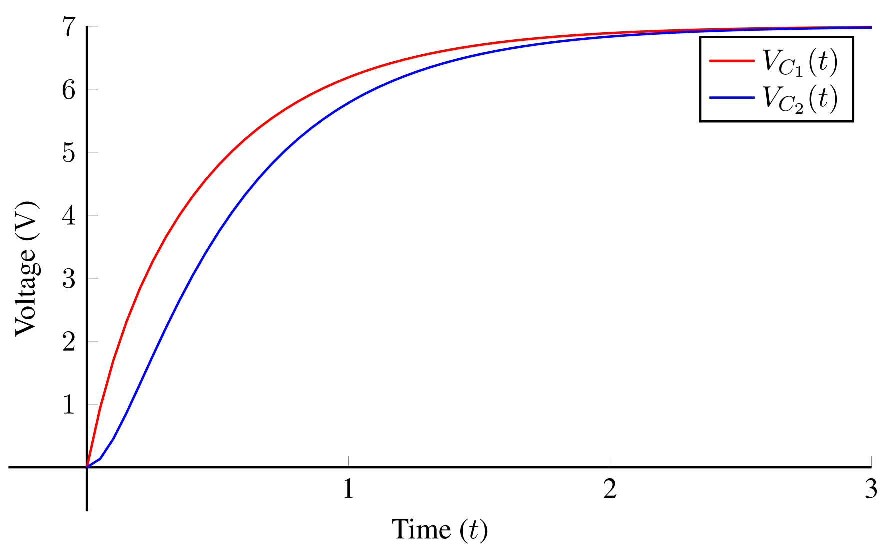

Figure 3: Initial Conditions: $V_{C_1}(0) = 0\,\text{V}$ and $V_{C_2}(0) = 0\,\text{V}$. Non-homogeneous Case Solution.