EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 3A

The following notes are useful for this discussion: Note 3 (sections 1 and 2)

1. Changing Coordinates and Systems of Differential Equations, I

Suppose we have the pair of differential equations (valid for $t \geq 0$)

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -9x_1(t) \tag{1}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -2x_2(t) \tag{2}$$

with initial conditions $x_1(0) = -1$ and $x_2(0) = 3$.

(a) **Solve for $x_1(t)$ and $x_2(t)$ for $t \geq 0$.**

Solution: From experience solving differential equations with the given form (where the derivative of the variable is a scaled version of the variable itself), we know the solution to these differential equations have the following form:

$$x_1(t) = K_1 e^{-9t} \tag{3}$$

$$x_2(t) = K_2 e^{-2t} \tag{4}$$

Plugging in for the given initial conditions:

$$\begin{aligned} x_1(0) &= K_1 e^0 && (5)\\ &= -1 && (6)\\ \implies K_1 &= -1 && (7)\\ x_2(0) &= K_2 e^0 && (8)\\ &= 3 && (9)\\ \implies K_2 &= 3 && (10) \end{aligned}$$

This yields:

$$x_1(t) = -e^{-9t} \tag{11}$$

$$x_2(t) = 3e^{-2t} \tag{12}$$

Now, suppose we are actually interested in a different set of variables with the following differential equations:

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{13}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t). \tag{14}$$

(b) **Write out the above system of differential equations in matrix form**. Can we solve this system in a similar way as we did above?

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}\right) = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{15}$$

Let's call the transition matrix A for convenience. That is,

$$A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}. \tag{16}$$

We cannot solve this system directly. This is because these equations are *coupled*; specifically, there is an interdependence between these two equations. In part (a), the equations were completely separate from each other, and the solution to one did not depend on the specifics of the other. This had allowed us to use the techniques we knew, but to solve the equation we have in this subpart, we need a new technique.

(c) Consider that in our frustration with the previous system of differential equations (which we cannot directly solve), we start hearing voices[1]. These voices whisper to us that that we should try writing out $\vec{z}(t)$ in terms of new variables, $y_1(t)$ and $y_2(t)$, as follows:

$$z_1(t) = -y_1(t) + 2y_2(t) \tag{17}$$

$$z_2(t) = 2y_1(t) + 3y_2(t). \tag{18}$$

Write out this transformation in matrix form ($\vec{z} = V\vec{y}$). What is V?

Solution:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{19}$$

We can call the forward change-of-coordinates matrix V:

$$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \tag{20}$$

To transform back from $\vec{z}$ to $\vec{y}$, we can use the inverse of V ($\vec{y} = V^{-1}\vec{z}$.)

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{21}$$

For each of the parts (d) - (f), solve the questions two ways: 1. using direct substitution, and 2. using matricies and vectors .

(d) Suppose that the following initial conditions are given: $\vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$. **How do these initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$?**

1. Solve this with direct substitution (For direct substitution, start with the governing equations in eqs. (17) to (18). Then, plug in known information and rearrange terms as needed to find the desired

[1] Friendly voices, so let's assume they're correct :)

quantities):

Solution: Plugging in the known initial values of $z_i(t)$ yields the following system of equations:

$$z_1(0) = 7 = -y_1(0) + 2y_2(0) \tag{22}$$
$$z_2(0) = 7 = 2y_1(0) + 3y_2(0) \tag{23}$$

Solving this system for the unknowns (initial conditions of $y_i(0)$ gives:

$$y_1(0) = -1 \tag{24}$$
$$y_2(0) = 3 \tag{25}$$

2. Solve this with matrices and vectors (Recall that $\vec{y} = V^{-1}\vec{z}$):

Solution: We can transform from $\vec{z}$ back to $\vec{y}$ using V^{-1}:

$$V^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{27}$$

(e) In eqs. (17) to (18), we are given $z_i(t)$ in terms of $y_i(t)$. Now, **write the corresponding equations for $y_i(t)$ in terms of $z_i(t)$**. Can we solve this system of differential equations?

1. Solve this with direct substitution:

Solution: We can solve our system of equations relating $z_i(t)$ and $y_i(t)$ (eqs. (17) to (18)) for $y_i(t)$:

$$y_1(t) = -\frac{3}{7}z_1(t) + \frac{2}{7}z_2(t) \tag{28}$$
$$y_2(t) = \frac{2}{7}z_1(t) + \frac{1}{7}z_2(t). \tag{29}$$

Taking the derivative of these two equations gives:

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -\frac{3}{7}\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} + \frac{2}{7}\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} \tag{30}$$
$$\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} = \frac{2}{7}\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} + \frac{1}{7}\frac{\mathrm{d}z_2(t)}{\mathrm{d}t}. \tag{31}$$

Plugging in the values of $\frac{\mathrm{d}}{\mathrm{d}t}z_i(t)$ from the original differential equations in eqs. (13) to (14) gives:

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -\frac{3}{7}(-5z_1(t) + 2z_2(t)) + \frac{2}{7}(6z_1(t) - 6z_2(t)) \tag{32}$$
$$= \frac{27}{7}z_1(t) - \frac{18}{7}z_2(t) \tag{33}$$
$$\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} = \frac{2}{7}(-5z_1(t) + 2z_2(t)) + \frac{1}{7}(6z_1(t) - 6z_2(t)) \tag{34}$$
$$= -\frac{4}{7}z_1(t) - \frac{2}{7}z_2(t). \tag{35}$$

Now, we convert the right-hand side to terms solely in terms of $y_i(t)$ using eqs. (17) to (18):

$$\begin{aligned}
\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} &= \frac{27}{7}z_1(t) - \frac{18}{7}z_2(t) && (36)\\
&= \frac{27}{7}\big(-y_1(t) + 2y_2(t)\big) - \frac{18}{7}\big(2y_1(t) + 3y_2(t)\big) && (37)\\
\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} &= -\frac{4}{7}z_1(t) - \frac{2}{7}z_2(t). && (38)\\
&= -\frac{4}{7}\big(-y_1(t) + 2y_2(t)\big) - \frac{2}{7}\big(2y_1(t) + 3y_2(t)\big). && (39)
\end{aligned}$$

Simplifying the right hand sides (noticing cancellation of terms) and rewriting in terms of $y_i(t)$ gives the differential equations we seek:

$$\begin{aligned}
\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} &= -9y_1(t) && (40)\\
\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} &= -2y_2(t). && (41)
\end{aligned}$$

Since these equations are decoupled, we can solve them as we did in part (a)! In fact, these equations and initial conditions are identical to the $x_i(t)$ differential equations we solved in the part (a).

2. Solve this with matrices and vectors:

Solution: At this point we have the following equations:

$$\begin{aligned}
\frac{\mathrm{d}\vec{z}}{\mathrm{d}t} &= A\vec{z} && (42)\\
\vec{z} &= V\vec{y} && (43)
\end{aligned}$$

We want to find the matrix A_y such that:

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = A_y\vec{y} \quad (44)$$

By plugging eq. (43) into eq. (44), and expanding the result

$$\begin{aligned}
\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(V^{-1}\vec{z}\Big) && (45)\\
&= V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{z}) && (46)\\
&= V^{-1}A\vec{z} && (47)\\
&= V^{-1}AV\vec{y} && (48)
\end{aligned}$$

we find that $A_y = V^{-1}AV$ and get our differential equation for $\vec{y}$.

$$\begin{aligned}
\begin{bmatrix} \frac{\mathrm{d}y_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}y_2(t)}{\mathrm{d}t} \end{bmatrix} &= \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} && (49)\\
&= \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} && (50)
\end{aligned}$$

(f) **What are the solutions for $z_i(t)$?**

1. Solve this with direct substitution:

Solution: The differential equations and initial conditions are the same for $\vec{x}$ and $\vec{y}$. Thus we know that

$$y_1(t) = x_1(t) = -e^{-9t} \tag{51}$$
$$y_2(t) = x_2(t) = 3e^{-2t} \tag{52}$$

Plugging this into the equation for $\vec{z}$ gives:

$$z_1(t) = -y_1(t) + 2y_2(t) = e^{-9t} + 6e^{-2t} \tag{53}$$
$$z_2(t) = 2y_1(t) + 3y_2(t) = -2e^{-9t} + 9e^{-2t} \tag{54}$$

2. Solve this with matrices and vectors:

Solution: The solution for $z_i(t)$ is:

$$\vec{z} = Vy \tag{55}$$
$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{56}$$
$$= \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} \tag{57}$$
$$= \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix} \tag{58}$$

You can verify that the initial conditions for $z_i(t)$ and differential equation for $\frac{\mathrm{d}}{\mathrm{d}t} z_i(t)$ hold by plugging in for these solutions of $z_1(t)$ and $z_2(t)$.

It is worth recapping what the problem-solving process has looked like until now. We know that when our differential equations are uncoupled, we know the techniques to solve each equation. When the equations are coupled, it seems that we can only use our original single-equation techniques after we change into a convenient basis. Once we do so (and get solutions in the convenient basis), we can invert the change-of-basis and go back to the original basis in which the solution was requested.

Now, why is the specific basis we chose convenient? This is the topic of the next discussion.

The diagram below explains the conceptual picture.

$$\begin{tikzcd}
\vec{z} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{\mathrm{d}\vec{z}}{\mathrm{d}t} \arrow[d, bend right, "V^{-1}"'] \\
\vec{y} \arrow[r, "A_y = V^{-1}AV"'] \arrow[u, bend right, "V"'] & \frac{\mathrm{d}\vec{y}}{\mathrm{d}t} \arrow[u, bend right, "V"']
\end{tikzcd}$$

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