EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 1B

For this discussion, Note 1 is helpful for the differential equations, and Note j covers the complex numbers fundamentals.

1. RC Circuits: Solving the Differential Equations

Recall that in the last discussion, we were tasked with analyzing an example RC circuit (in fig. 1) and using element equations (governing equations for resistors and capacitors) to formulate a differential equation. This equation describes the time-varying behavior of this circuit. Specifically, we had the following differential equation:

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = V(t) \tag{1}$$

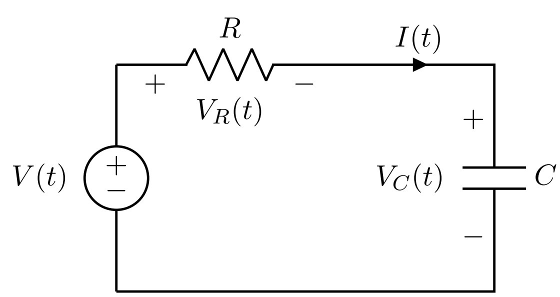

Figure 1: Sample RC Circuit

Our goal is to now solve this differential equation for the voltage across the capacitor, $V_C(t)$.

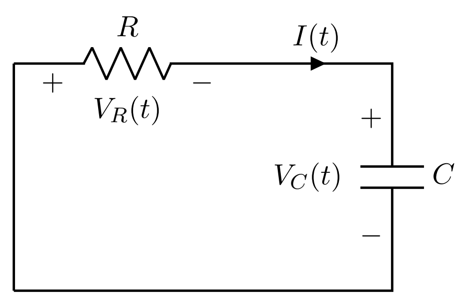

Figure 2: RC Circuit for part (a). Note that the voltage source has been turned off (0 V) for this subpart, and the initial voltage on the capacitor is V_{DD}.

(a) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$ (voltage source is turned off). Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Solution: Because $V(t) = 0$, the differential equation that we derived in the previous discussion (given in eq. (1)) simplifies to

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = 0 \tag{2}$$

Dividing both sides of the equation by RC, we arrive at

$$\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + \frac{1}{RC}V_C(t) = 0 \tag{3}$$

Moving the second term to the right-hand side, we have

$$\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} = -\frac{1}{RC}V_C(t) \tag{4}$$

From the form of equation, we are told that we are looking for some function $V_C(t)$ such that when we take its derivative, we get the same function $V_C(t)$ multiplied by a scalar $-\frac{1}{RC}$.
Because the derivative of the function is equal to a scaled version of the function itself, we believe that the solution $V_C(t)$ might be of the form Ae^{bt}, where A and b are both constants.
Following the steps outlined in lecture, we must first solve for the scalar A using the initial condition. Here, we have

$$\begin{align} V_{DD} &= V_C(0) \tag{5}\\ &= Ae^{b(0)} \tag{6}\\ &= A \tag{7} \end{align}$$

Now that we know the value of A, we can write that $V_C(t) = V_{DD}e^{bt}$. The last task is to find b. We have already used the initial condition, so we must be able to find b from the differential equation. Plugging in our expression for $V_C(t)$ into the differential equation, we find

$$\begin{aligned} \frac{\mathrm{d}V_C(t)}{\mathrm{d}t} &= -\frac{1}{RC}V_C(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}V_{DD}e^{bt} &= bV_{DD}e^{bt} = bV_C(t)\\ &= -\frac{1}{RC}V_C(t)\\ \implies b &= -\frac{1}{RC}. \end{aligned}$$

In this case, we see the value of the remaining constant $b = -\frac{1}{RC}$, and our overall solution is

$$V_C(t) = V_{DD}e^{-\frac{1}{RC}t} \tag{8}$$

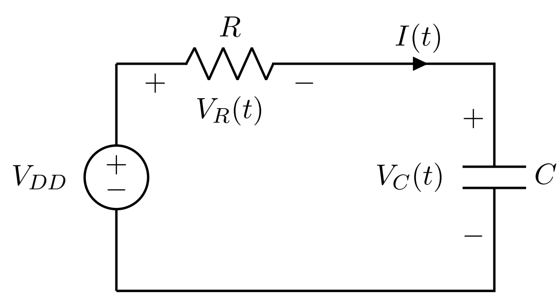

Figure 3: Circuit for part (b)

(b) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit for all $t \geq 0$. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Solution: Substituting $V(t) = V_{DD}$ into our solution from eq. (1):

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = V_{DD} \tag{9}$$

We want to arrange this equation to be in a form that we know how to solve:

$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{V_{DD} - V_C(t)}{RC} \tag{10}$$

This is not quite in the form we have seen before, since the term on the right is not equal to the term being differentiated. There is an additional (constant) term. In general, we don't like constants in the differential equation. The derivative of a constant is zero, so we can wrap the constant into the function being differentiated using a change of variables, like so. Importantly, this change of variables allows us to *transform* a problem for which we don't know the exact solution into a form for which we do have a solution.

Let's define a new variable $\widetilde{V}_C(t) = V_C(t) - V_{DD}$. Note that $\frac{\mathrm{d}\widetilde{V}_C(t)}{\mathrm{d}t} = \frac{\mathrm{d}V_C(t)}{\mathrm{d}t}$, and that $\widetilde{V}_C(0) = V_C(0) - V_{DD} = -V_{DD}$. We can substitute these into our differential equation and obtain

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) - V_{DD} = 0 \tag{11}$$

$$RC\frac{\mathrm{d}\widetilde{V}_C(t)}{\mathrm{d}t} + \widetilde{V}_C(t) = 0 \tag{12}$$

$$\frac{\mathrm{d}\widetilde{V}_C(t)}{\mathrm{d}t} = -\frac{1}{RC}\widetilde{V}_C(t) \tag{13}$$

In this equation, we have now removed V_{DD} from the left hand because of how we defined $\widetilde{V}_C(t)$. And so we get back almost the same differential equation as in the previous part, this time for $\widetilde{V}_C(t)$, with the only difference being that the initial condition changed! And so, we can use that solution to get

$$\widetilde{V}_C(t) = \widetilde{V}_C(0)e^{-\frac{1}{RC}t} = -V_{DD}e^{-\frac{1}{RC}t}. \tag{14}$$

Finally, we need the solution in terms of $V_C(t)$ and not $\widetilde{V}_C(t)$, so we back-substitute:

$$\begin{aligned} V_C(t) &= V_{DD} + \widetilde{V}_C(t) \\ &= V_{DD} - V_{DD}e^{-\frac{1}{RC}t} \\ &= V_{DD}(1 - e^{-\frac{1}{RC}t}). \end{aligned}$$

(c) We now want to combine the principles from the previous two subparts to understand the voltage waveform when a switch occurs at some time t. Specifically, suppose that at $t = 0$, $V(t) = 0\,\mathrm{V}$, $V_C(0) = V_{DD}$. Then, at some $t = t_{\text{switch}}$, the voltage source is turned on $V(t) = V_{DD}$ for $t \geq t_{\text{switch}}$. We want to find the equation for the overall capacitor voltage as a function of time (for times before and after t_{switch}).

Solution: There is a critical realization to make here; the final condition of the first curve (that is, the value of $V_C(t)$ at $t = t_{\text{switch}}$) serves as the initial condition for the second part of the curve. This second phase begins the moment the switch happens (instantaneously). Since we have a general way to find the solution for each phase individually given an initial condition, we can combine the results.

We proceed by first finding the solution to the first curve (for all time $t \geq 0$, not worrying about the switching for now). Noticing this is the same as the answer to part (a), so we write that:

$$V_{C,1}(t) = V_{DD} e^{-\frac{1}{RC}t} \tag{15}$$

Now, the "stopping" point of this curve (that is, the point when this waveform stops and $V_C(t)$ is governed by the second phase's waveform) is at $t = t_{\text{switch}}$. Specifically, at $V_C(t_{\text{switch}}) = V_{DD} e^{-\frac{t_{\text{switch}}}{RC}}$. This is a constant value; every term in this equation is known, and the time-dependence has been removed by plugging in the specific time we want to evaluate the first solution at.

We are now equipped to deal with the second (and last) part of this problem; finding out how the capacitor voltage changes for $t \geq t_{\text{switch}}$. At that point, we will have a piecewise equation (2 parts) that describes the voltage for all $t \geq 0$. Specifically, our initial condition for the second waveform is $V_C(t_{\text{switch}}) = V_{DD} e^{-\frac{t_{\text{switch}}}{RC}}$. Then, the solution we have for a generic initial condition, from part (b), can be applied.

The second part of the problem becomes easier to solve independently when we consider the following; in reality, the initial condition on time axis t occurs at t_{switch}, and this is where $V_C(t_{\text{switch}})$ will be used. However, we can also perform a change-of-variables on the *time axis* in order to create a new axis, t', which has value $t' = 0$ at $t = t_{\text{switch}}$. We are effectively re-centering the axis in order to re-use as much of the solution to (b) as we can.

Applying this principle, we find that as before, $\widetilde{V}_C(0) = V_C(0) - V_{DD}$. Substituting the constant value that we know $V_C(0)$ has at $t' = 0$ ($t = t_{\text{switch}}$), we see that $\widetilde{V}_C(0) = V_{DD} e^{-\frac{t_{\text{switch}}}{RC}} - V_{DD}$. We will not worry about factoring out V_{DD} for now.

Next, we apply the rest of the solution that we had in (b), while making substitutions both for the time-axis change ($t' = t - t_{\text{switch}}$) and the substitution of $\widetilde{V}_C(t')$ for $V_C(t')$:

$$\widetilde{V}_C(t') = \left(V_{DD} e^{-\frac{t_{\text{switch}}}{RC}} - V_{DD}\right) e^{-\frac{t'}{RC}} \tag{16}$$

$$V_C(t') = V_{DD} + \left(V_{DD} e^{-\frac{t_{\text{switch}}}{RC}} - V_{DD}\right) e^{-\frac{t'}{RC}} \tag{17}$$

$$V_C(t') = V_{DD} + V_{DD} e^{-\frac{t_{\text{switch}}}{RC}} e^{-\frac{t'}{RC}} - V_{DD} e^{-\frac{t'}{RC}} \tag{18}$$

$$V_C(t) = V_{DD} + V_{DD} e^{-\frac{t_{\text{switch}}}{RC}} e^{-\frac{t - t_{\text{switch}}}{RC}} - V_{DD} e^{-\frac{t - t_{\text{switch}}}{RC}} \tag{19}$$

We ultimately find that:

$$V_{C,2}(t) = V_{DD} + V_{DD} e^{-\frac{1}{RC}t} - V_{DD} e^{\frac{-t + t_{\text{switch}}}{RC}} \tag{20}$$

$$= V_{DD}\left(1 + e^{-\frac{1}{RC}t} - e^{\frac{-t + t_{\text{switch}}}{RC}}\right) \tag{21}$$

And that's how to solve what happens when an RC circuit switches!

See a visual demo here: https://www.desmos.com/calculator/gvjm36oo6j

2. Complex Algebra (Review)

(a) Express the following values in polar forms: -1, j, $-j$, $\sqrt{j}$, and $\sqrt{-j}$. Recall $j^2 = -1$, and the complex conjugate of a complex number is denoted with a bar over the variable. The complex conjugate is defined as follows: for a complex number $z = x + jy$, the complex conjugate $\overline{z} = x - jy$.

Solution: Here, we review some basic properties of complex numbers and its rectangular and polar form: $z = x + jy = |z|e^{j\theta}$, where $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ and $\measuredangle z = \theta = \text{atan2}(y, x)$. We can also write $x = |z|\cos(\theta)$, $y = |z|\sin(\theta)$.

A complex number can be represented in the following forms:

$$z = a + jb = r\cos(\theta) + jr\sin(\theta) = re^{j\theta}, \tag{22}$$

where, $r = \sqrt{a^2 + b^2}$,$\measuredangle z = \text{atan2}(b, a)$ and a, b are real numbers.

$$-1 = j^2 = e^{j\pi} = e^{-j\pi} \tag{23}$$

$$j = e^{j\frac{\pi}{2}} = \sqrt{-1} \tag{24}$$

$$-j = -e^{j\frac{\pi}{2}} =^{-j\frac{\pi}{2}} \tag{25}$$

$$\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}} \tag{26}$$

$$\sqrt{-j} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = e^{-j\frac{\pi}{4}} = \frac{1-j}{\sqrt{2}} \tag{27}$$

(b) Represent $\sin(\theta)$ and $\cos(\theta)$ using complex exponentials. (*Hint:* Use Euler's identity $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.)

Solution: Note that we can use the fact that $\cos(x)$ is an even function, and $\sin(x)$ is an odd function. This gives us that:

$$\begin{aligned} e^{j\theta} &= \cos(\theta) + j\sin(\theta) \\ e^{-j\theta} &= \cos(-\theta) + j\sin(-\theta) \\ &= \cos(\theta) - j\sin(\theta) \end{aligned}$$

Solving this system of equations for $\cos(\theta)$ and $\sin(\theta)$ gives:

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

For the next parts, let $a = 1 - j\sqrt{3}$ and $b = \sqrt{3} + j$.

(c) Show the number a in complex plane, marking the distance from origin and angle with real axis.

Solution: The location of a in the complex plane is shown in Figure 4. The only two pieces of information we need are the magnitude and the phase, which is the polar coordinates interpretation. We could also use the (perhaps more familiar) x and y Cartesian coordinates.

(d) Show that multiplying a with j is equivalent to rotating the magnitude of the complex number by $\frac{\pi}{2}$ or $90°$ in the complex plane.

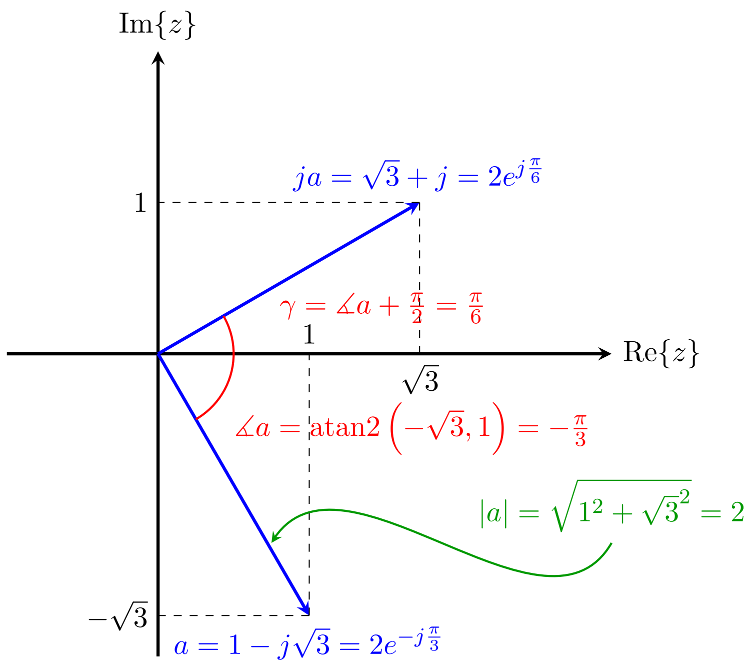

Figure 4: Complex numbers a and its rotated version b represented as vectors in the complex plane.

Solution: Multiplying a by j:

$$ja = e^{j\pi/2} \cdot 2e^{-j\pi/3} = 2e^{j\pi/6} = \sqrt{3} + j$$

The rotation is demonstrated in the same complex plane plot (Figure 4), with a new angle $\gamma = \measuredangle a + \frac{\pi}{2}$.

(e) **(Practice)** For complex number $z = x + jy$ show that $|z| = \sqrt{z\overline{z}}$, where $\overline{z}$ is the complex conjugate of z.

Solution: We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\overline{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z| \tag{28}$$

(f) **(Practice)** Express a and b in polar form.

Solution: Following the definitions in part a):

$$|a| = 2$$
$$|b| = 2$$
$$\measuredangle a = -\frac{\pi}{3}$$
$$\measuredangle b = \frac{\pi}{6}$$

Hence:

$$a = 2e^{-j\frac{\pi}{3}} \qquad b = 2e^{j\frac{\pi}{6}}$$

(g) **(Practice)** Find ab, $a\overline{b}$, $\frac{a}{b}$, $a + \overline{a}$, $a - \overline{a}$, $\overline{ab}$, $\overline{a}\overline{b}$, and $\sqrt{b}$.

Solution: We can evaluate these sequentially using the rules of complex algebra:

$$\begin{aligned}
ab &= 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j \\
a\overline{b} &= 4 \cdot e^{-j\frac{\pi}{2}} = -4j \\
\frac{a}{b} &= e^{-j\frac{\pi}{2}} = -j \\
a + \overline{a} &= 2 \\
a - \overline{a} &= -2j\sqrt{3} \\
\overline{ab} &= 2\sqrt{3} + 2j \\
\overline{a}\overline{b} &= (1 + j\sqrt{3})(\sqrt{3} - j) = \sqrt{3} + \sqrt{3} + j(3-1) = 2\sqrt{3} + 2j \\
\sqrt{b} &= \sqrt{2}e^{j\frac{\pi}{12}}
\end{aligned}$$

Note the following: $a + \overline{a}$ is a purely real number. $a - \overline{a}$ is a purely imaginary number. And, $\overline{ab} = \overline{a}\overline{b}$.

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