Saw in lecture: Can we treat R, L, C as if they're all resistors? For circuits with sinusoidal voltages and currents yes! Can use NVA without pesky $\phi$'s!

We treat a sinusoid as if it's

\[ v(t) = \cos(\omega t) = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t} \]

\[ \tilde{v} = \frac{1}{2} e^{j\theta} \]

\[ v(t) = 2 \cos(\omega t - \frac{\phi}{2}) = e^{j\omega t} e^{-j\frac{\phi}{2}} + e^{-j\omega t} e^{j\frac{\phi}{2}} \]

\[ \tilde{v} = 1 e^{j\theta} \]

Twice as big. Angle shift of $-\frac{\phi}{2}$

Learning objectives

- How to do phasor analysis

Step 1: Write sources as exponentials: $\tilde{X} e^{j\omega t} + \tilde{X} e^{-j\omega t}$.

Step 2: Transform circuits to phasor domain.

(Compute the numerical values of these impedances.)

Step 3: Cast the branch and element equations in the phasor domain. (KVL, KCL, Ohm's)

Step 4: Solve for unknown variables

Step 5: Transform solutions back to time domain

- Inductor impedance derivation (May not hit this one)

$\rightarrow$ If the inductor behaves like a resistor in phasor land, what is its impedance $Z_L$? (Quantity analogous to resistance: $\tilde{V}_L = Z_L \tilde{I}_L$)

\[ \tilde{V}_L(t) = 2 \cos(\omega t - \frac{\phi}{2}) \]

\[ \tilde{V}_L = 1 e^{j\theta} \]

\[ V_r(t) = A \cos(\omega t + \phi) \]
Given this circuit:

Goal: We seek to obtain a solution for \( i(t) \) with the sinusoidal voltage source \( v_S(t) \).

(a) **Step 1: Write sources as exponentials:** \( \tilde{X}e^{j\omega t} + \overline{\tilde{X}}e^{-j\omega t} \).

All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. **Convert \( v_S(t) \) into a exponential and write down its phasor representation \( \tilde{V}_S \).**

Note that \( v_S(t) \) is given in terms of a sine wave, not a cosine wave.

\[
v_S(t) = 12 \sin \left( \omega t - \frac{\pi}{4} \right) = 12 \left( \frac{e^{j\omega t} - e^{-j\omega t}}{2} \right)
\]

**Questions**

\( \tilde{V}_S = \cos \omega t + j \sin \omega t \) **Purpose of phasor?**

A: Capture a sinusoidal \( V \) or \( I \) so that we can do NVA and compute other sinusoidal \( V \) or \( I \)’s w/o solving a DE.

Q: Could we use \( \cos (\omega t - \frac{\pi}{4}) = \sin (\omega t) \)?

A: Yes!

Voltage Phasor: \( \tilde{V}_S = 6e^{j\frac{3\pi}{4}} V \) **unit?**
Voltage Phasor: \[ \tilde{V}_s = 6e^{-j\frac{3\pi}{4}} \text{ V} \]  

(b) **Step 2: Transform circuits to phasor domain.** The voltage source \( v_s(t) \) is represented by its phasor \( \tilde{V}_s \). Similarly, \( v_R(t) \) has phasor \( \tilde{V}_R \), and \( v_C(t) \) has phasor \( \tilde{V}_C \).

The current \( i(t) \) is related to its phasor counterpart \( \tilde{I} \) by

\[ i(t) = \tilde{I}e^{j\omega t} + \tilde{I}e^{-j\omega t}. \]  

Using the numbers given in the problem statement (\( \omega = 1 \times 10^3 \text{ rad/s} \), \( R = \sqrt{3} \text{ k}\Omega \), and \( C = 1 \mu\text{F} \)), compute the numerical values of these impedances.

**Figure 2:** Circuit in “phasor domain”

\[ Z_R = \sqrt{3} \text{ k}\Omega \]

\[ Z_C = \frac{j \cdot 1}{\frac{1 \text{ krad/s}}{s}} \left( \frac{1 \mu\text{F}}{0.1} \right) \left[ \frac{s}{0.1} \right] \]

\[ = \frac{j \cdot 10^6}{10^3} \Omega \]

\[ = 1000 \Omega \]

Labeled circuit:

\[ \sqrt{3} \text{ k}\Omega \]

\[ 6e^{\frac{-3\pi}{4}j\pi} \text{ V} \]

**Questions**

1. Are all \( Z \)'s in units of ohms?  
   - Yes

2. Why are \( Z_R \) and \( Z_C \) (later) \( Z_L \) and \( Z_L \) dependent?  
   - Yes

3. Responds differently to changing voltages/curves of different speeds/rates (derivative of current/derivative of voltage in \( L \) & \( C \)  

4. How are \( Z_R \) and \( Z_C \) (later) \( Z_L \) and \( Z_L \) dependent?  
   - Yes
Take as given that KCL, KVL applies (needs proof though)

(d) **Step 3: Cast the branch and element equations in the phasor domain.** (KVL, KCL, ohm’s)

The previous subpart gave us a concrete relation we can use in the phasor domain to relate the voltages of the circuit elements. Specifically, we **know that** \( \tilde{V}_S = \tilde{V}_R + \tilde{V}_C \).

Now, we must **substitute in the voltage phasors corresponding to these terms**, using the element impedances given in Step 2. At this point, feel free to leave the terms symbolic; in the next part, we will substitute in numbers.

**KVL method**

(unknowns: current)

\[
\tilde{V}_S = \tilde{V}_R + \tilde{V}_C \\
= \tilde{Z}_R \tilde{I}_R + \tilde{Z}_C \tilde{I}_C \]

(KCL @ \( \tilde{V}_C \) node: \( \tilde{I}_R = \tilde{I}_C = \tilde{I} \))

\[
\tilde{V}_S = (\tilde{Z}_R + \tilde{Z}_C) \tilde{I} \\
\tilde{I} = \tilde{I}_C = \tilde{I}_R = \frac{\tilde{V}_S}{\tilde{Z}_R + \tilde{Z}_C} = \frac{6e^{-\frac{3}{4}j\omega}}{\tilde{Z}_R + \tilde{Z}_C} \\
\]

**KCL method (NVA)**

(unknowns: node voltages)

KCL @ \( \tilde{V}_C \) node:

\[
\tilde{I}_R = \tilde{I}_C \\
\tilde{V}_S - \tilde{V}_C = \frac{\tilde{V}_C - 0V}{\tilde{Z}_R} \\
\tilde{V}_S = \frac{\tilde{V}_C}{\tilde{Z}_R} + \frac{\tilde{V}_C}{\tilde{Z}_C} \]

(usual)

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**Note 5 § 7.2 “Prove that KVL holds in phasor domain!”**

(c) **[Practice]** As an intermediate step to use in the next subpart, **show that the fact that the first equation holds for all \( t \) implies the second equation:**

\[
v_S(t) = v_R(t) + v_C(t) \tag{6}
\]

\[
\tilde{V}_S = \tilde{V}_R + \tilde{V}_C \tag{7}
\]
KVL equation
\[ V_s = V_R + V_C \]

KCL equation
\[ \frac{\tilde{V}_s - \tilde{V}_C}{\tau_R} = \frac{\tilde{V}_C - 0V}{\tau_C} \]

(c) Step 4: Solve for unknown variables

Solve the equation you derived in Step 3 for \( \tilde{I} \) and \( \tilde{V}_C \). What is the polar form of \( \tilde{I} \) and \( \tilde{V}_C \)? The polar form is given by \( Ae^{j\theta} \), where \( A \) is a positive real number.

Hints:
- \( \frac{\sqrt{3}}{2} - \frac{i}{2} = e^{-\frac{3\pi}{6}} \)

\[ \tilde{I} = \frac{6e^{-\frac{3\pi}{4}}}{j} \]

\[ (\tilde{I} = \tilde{I}_R, \tilde{I}_C) \]

\[ \tilde{V}_C = \tilde{Z}_C \cdot \tilde{I}_C = \tilde{Z}_C \tilde{I} \]

\[ = j \cdot 1 \cdot \ln(3mA e^{-\frac{3\pi}{4}j}) \]

\[ = e^{\frac{j\pi}{2}} 3V e^{-\frac{3\pi}{4}j} \]

\[ = (3V e^{-\frac{3\pi}{4}j}) \]

Polar form of \( \tilde{V}_C \):
\[ \tilde{V}_C = 3V e^{-\frac{3\pi}{4}j} \]

Polar form of \( \tilde{I}_C \):
\[ \tilde{I}_C = 3mA e^{-\frac{3\pi}{4}j} \]

Questions:

Q: Why units (mA, V) before complex exponential?
A: I can carry them as a factor and multiplication is commutative...

Q: Is this data a LPF (lowpass filter)? Is that why our voltage on C has a lower amplitude?
A: Yes!

Q: Does what we did only apply to sinusoidal signals?
A; Yes, and to sums of sinusoidal signals.
Polar form of $\vec{V}_c$: $\vec{V}_c = 3V e^{-\frac{13}{12}j\pi}$

Polar form of $\vec{I}_c$: $\vec{I}_c = 3mA e^{-\frac{\pi}{2}j\pi}$

(f) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the relation between a sinusoidal function and its phasor counterpart. **What is $i(t)$ and $v_c(t)$?** What is the phase difference between $i(t)$ and $v_c(t)$?

$$V_c(t) = 3V e^{-\frac{13}{12}j\pi} e^{jwt} + 3V e^{\frac{13}{12}j\pi} e^{-jwt}$$

$$= 3V \left( e^{j(wt - \frac{13}{12}\pi)} + e^{-j(wt - \frac{13}{12}\pi)} \right)$$

$$= 3V \cdot 2 \cos \left( wt - \frac{13}{12}\pi \right)$$

$$w = \frac{1 \text{ rad}}{s}$$

$$= 6V \cos \left( \frac{1 \text{ rad}}{s} t - \frac{13}{12}\pi \right)$$

$$i(t) = 6mA \cos \left( \frac{1 \text{ rad}}{s} t - \frac{\pi}{12} \right)$$

$$\left( -\frac{13}{12}\pi + \frac{\pi}{12} \right) = -\frac{\pi}{2}$$

**Questions**

(g) Now, suppose that instead of wherever we analyzed the phasor as $\vec{X}$ (the coefficient associated with the $e^{jwt}$ term), we had instead selected to work with $\vec{X}$, or we solved using both $\vec{X}$ and $\vec{Y}$. **How would our answer or problem-solving procedure have changed?**

The answer at the end doesn't change.

We'd need $\hat{Z}$ for impedances but otherwise the process (translate, NVM, translate) is the same.
2. Inductor Impedance

Given the voltage-current relationship of an inductor \( v(t) = L \frac{di(t)}{dt} \), we want to show that its complex impedance is \( Z_L(j\omega) = j\omega L \). We will perform this analysis in steps.

A sample inductor circuit is in fig. 3.

![Figure 3: A simple inductor circuit.](image)

(a) Suppose that the input current source in fig. 3 has value \( i(t) = I_0 e^{st} \), where \( I_0 \) is some (not necessarily real) constant. **What is the corresponding s-phasor for the current?**

\[
\tilde{I} = I_0 e^{st}
\]

(b) Now, using the governing voltage-current equation for an inductor, **derive the time-domain inductor voltage using the current expression and solve for the corresponding voltage s-phasor.**

\[
\begin{align*}
\text{Find } \tilde{V} & \text{ given that: } \quad L \frac{dI_L}{dt} = V_L \\
V_L(t) &= L \frac{d}{dt} (I_0 e^{st}) \\
&= LI_0 s e^{st} \\
V_L(t) &\Rightarrow \tilde{V} = LI_0 s
\end{align*}
\]

(c) Using the voltage and current s-phasors, **solve for the s-impedance of the inductor** \( Z_L(s) \). (This is the ratio between these phasor quantities).

\[
\frac{\tilde{V}}{\tilde{I}} = Z_L(s) = \frac{LI_0 s}{I_0} = \left[ \frac{Ls}{1} \right]
\]
(d) Now, suppose that our current source value was instead \( i(t) = I_0 \cos(\omega t + \phi) \), where \( \omega \) is the frequency of the cosine wave and \( \phi \) is the phase-offset. \( \phi = 0 \) corresponds to the standard cosine centered at \( t = 0 \).

Using Euler’s formula, represent \( i(t) \) as the sum of two complex exponentials. Using this, find the new phasor \( \tilde{i} \) associated with the complex exponential \( e^{j\omega t} \):

\[
i_0 \cos(\omega t + \phi) = i_0 \left( \frac{1}{2} e^{j(\omega t + \phi)} + \frac{1}{2} e^{-j(\omega t + \phi)} \right)
\]

\[
\tilde{i} = \frac{i_0}{2} e^{j\phi}
\]

(e) Same as before, use \( i(t) \) to derive \( v(t) \) and find the new phasor \( \tilde{V} \) associated with the complex exponential \( e^{j\omega t} \):

\[
v_L = L \frac{dI_L}{dt} = L i_0 \left( -\sin(\omega t + \phi) \omega \right)
\]

\[
= -L i_0 \omega \sin(\omega t + \phi)
\]

\[
= -i L i_0 \omega \left( \frac{1}{2} e^{j(\omega t + \phi)} + \frac{1}{2} e^{-j(\omega t + \phi)} \right)
\]

\[
\Rightarrow \quad \tilde{v} = \frac{-L i_0 \omega}{2} e^{j\phi}
\]

(f) Once again, using the voltage and current phasors, solve for the impedance of the inductor \( Z_L(s) \). Is this the same quantity that we found in the earlier subpart, as expected?

\[
Z_L(j\omega) = \frac{\tilde{v}}{\tilde{i}} = \frac{L i_0 \omega}{2} e^{j\phi} = j\omega L
\]

---

Got to have in section ↓ read solutions for rest. 
notes below
Now, let’s see how we could have used the first result (for a single complex exponential) and taken a shortcut for the generic sinusoid using superposition. By pattern-matching the expansion of \( i(t) = I_0 \frac{1}{2} \left( e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)} \right) \) to the single \( s \)-exponential at the very start, we find that there are 2 components:

i. Component 1: \( i_1(t) = \left( \frac{I_0}{2} e^{j\phi} \right) e^{j\omega t} \)

ii. Component 2: \( i_2(t) = \left( \frac{I_0}{2} e^{-j\phi} \right) e^{-j\omega t} \)

(g) Now, **evaluate your expression for** \( Z_L(s) \) (from the single exponential case) at \( s = j\omega \), and \( s = -j\omega \). What do you notice?

\[
Z_L(s = j\omega) = (s = j\omega) Z_L(s = -j\omega) = (s = -j\omega) Z_L
\]

\[
= j\omega L
\]

\[
\text{conjugates!}
\]

(h) Using the current components given above, **solve for the voltage phasors** \( \tilde{V}_1 \) and \( \tilde{V}_2 \) as the **product of the associated current phasors** \( \tilde{I}_1 \) and \( \tilde{I}_2 \), and the **corresponding impedances**. What do you notice about the current phasors? What do you notice about the voltage phasors? How can we explain the relationships between these results?

\[
i_1(t) \rightarrow \tilde{I}_1 = \frac{I_0}{2} e^{j\phi} \quad \rightarrow \quad \tilde{V}_1 = Z_L(j\omega)\tilde{I}_1 = j\omega L \cdot \frac{I_0}{2} e^{j\phi}
\]

\[
i_2(t) \rightarrow \tilde{I}_2 = \frac{I_0}{2} e^{-j\phi} \quad \rightarrow \quad \tilde{V}_2 = Z_L(-j\omega)\tilde{I}_2 = -j\omega L \cdot \frac{I_0}{2} e^{-j\phi}
\]

Conjugates: \( \tilde{V}_1 = \tilde{V}_2 \) ! \( \tilde{V}_1 = j\omega L \cdot \frac{I_0}{2} e^{j\phi} \)

\[
\tilde{V}_1 = (j\omega L \cdot \frac{I_0}{2} e^{j\phi})
\]

\[
= j\omega L \cdot \frac{I_0}{2} e^{j\phi}
\]

\[
= -j\omega L \cdot \frac{I_0}{2} e^{-j\phi} = \tilde{V}_2
\]

\( \tilde{I}_1 \) and \( \tilde{I}_2 \) were conjugates.

\( Z_L(j\omega) \) and \( Z_L(-j\omega) \) were conjugates.

The products therefore must be conjugates! \( (ab) = \bar{a} \cdot \bar{b} \)