1. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations below:

\[
\frac{d\bar{z}_1(t)}{dt} = \lambda \bar{z}_1(t) \quad (1)
\]
\[
\frac{d\bar{z}_2(t)}{dt} = \bar{\lambda} \bar{z}_2(t) \quad (2)
\]

with initial conditions \( \bar{z}_1(0) = c_0 \) and \( \bar{z}_2(0) = \overline{c_0} \). Note, \( \lambda \) and \( c_0 \) are complex numbers and \( \bar{\lambda} \) and \( \overline{c_0} \) are their complex conjugates.

(a) First, assume that \( \lambda = j \) in the equations for \( \bar{z}_1(t) \) and \( \bar{z}_2(t) \) above. Solve for \( \bar{z}_1(t) \) and \( \bar{z}_2(t) \). Are the solutions complex conjugates?

\[
\bar{z}_1(t) = c_0 e^{jt} \quad \overline{\bar{a}b} = \overline{ab}
\]
\[
\bar{z}_2(t) = \overline{c_0} e^{-jt}
\]

\[
\bar{a} = 1, \quad b = 1 \quad - \quad j = \bar{a} + bj \quad \bar{j} = 0 - 1 \cdot j = -j
\]
(b) Suppose now that we have the following different variables related to the original ones:

\[ y_1(t) = a z_1(t) + \bar{a} z_2(t) \]  
\[ y_2(t) = b z_1(t) + \bar{b} z_2(t) \]

where \(a\) and \(b\) are complex numbers and \(\bar{a}\) and \(\bar{b}\) are their complex conjugates. These numbers can be written in terms of their real and imaginary components:

\[ a = a_r + j a_i, \quad \bar{a} = a_r - j a_i, \]  
\[ b = b_r + j b_i, \quad \bar{b} = b_r - j b_i, \]

where \(a_r, a_i, b_r, b_i\) are all real numbers. For all following subparts, assume that \(\lambda = j\) unless specified.

How do the initial conditions for \(\bar{z}(t)\) translate into the initial conditions for \(\bar{y}(t)\)? Are these purely real, purely imaginary, or complex numbers?

\[ y_1(0) = a \bar{z}_1(0) + \bar{a} \bar{z}_2(0) = a c_0 + \bar{a} \bar{c}_0 \]
\[ y_2(0) = b \bar{z}_1(0) + \bar{b} \bar{z}_2(0) = b c_0 + \bar{b} \bar{c}_0 \]

\[(a_r + a_i j) + (a_r - a_i j) = 2a_r - 0\]
(c) We noticed earlier that \( z_1(t) \) and \( z_2(t) \) are complex conjugates of each other. **What does this say about \( y_1(t) \) and \( y_2(t) \)? (Are they purely real, purely imaginary, or complex?)**

purely real
(d) Write out the change of variables in matrix-vector form \( \vec{y} = V \vec{z} \).

\[
\vec{y} = \begin{bmatrix}
\alpha & \alpha \\
\overline{b} & \overline{b}
\end{bmatrix}
\begin{bmatrix}
Z_1(4) \\
Z_2(4)
\end{bmatrix}
\]
(e) (Practice): Find an expression for the determinant of $V$. Further, simplify $\bar{a} \bar{b} + \bar{a} \bar{b}$, where $a, b$ are complex numbers.

$$\begin{vmatrix} a & \bar{a} \\ \bar{b} & b \end{vmatrix} = a \bar{b} - \bar{ab} = (a_r + a_{ij})(b_r - b_{ij}) - (a_r - a_{ij})(b_r + b_{ij})$$

$$= (a_r b_r - a_r b_{ij} + a_{ij} b_r - a_{ij} b_{ij}) - (a_r b_r + a_r b_{ij} - a_{ij} b_r + a_{ij} b_{ij})$$

$$= -2a_r b_{ij} + 2a_{ij} b_r$$
Write out the system of differential equations for \( \frac{d}{dt} y(t) \) and \( y(t) \).

\[
\frac{d}{dt} \dot{y} = A_y \dot{y} \\
\dot{y} = V \ddot{z} \\
\frac{d}{dt} z_1 = \lambda z_1 \\
\frac{d}{dt} z_2 = \bar{\lambda} z_2 \\
\dot{A}_y = V A_z V^{-1}
\]
\[ A_y = V A_z V^{-1} \]

\[
= \frac{1}{2(a_r b_i - a_i b_r)} \begin{bmatrix}
2(a_r b_r + a_i b_i) & -2(a_r^2 + a_i^2) \\
2(b_r^2 + b_i^2) & -2(a_r b_r + a_i b_i)
\end{bmatrix}
\]
(h) Above, we were already given the system in nice decoupled coordinates \( \vec{x} \). In general, problems will present in the more coupled form of \( \vec{y} \) above. We know how to discover nice coordinates for ourselves.

Find the eigenvalues \( \lambda_1 \), \( \lambda_2 \) for the differential equation matrix for \( \vec{y}(t) \) above. Verify that the eigenvalues are \( \{j, -j\} \).
2. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage is proportional to the derivative of the current across it. That is:

\[ V_{r}(t) = L \frac{dI_{L}(t)}{dt} \quad (7) \]

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the counterpart circuit for an inductor:

![Inductor Circuit Diagram]

**Figure 1**: Inductor in series with a voltage source.

(a) What is the current through an inductor as a function of time? If the inductance is \( L = 3 \text{ H} \), what is the current at \( t = 6 \text{ s} \)? Assume that the voltage source turns from 0 V to 5 V at time \( t = 0 \text{ s} \), and there’s no current flowing in the circuit before the voltage source turns on i.e \( I_L(0) = 0 \text{ A} \).

\[ I_L(t) = \frac{V_s}{L} t + I_0 \]

\[ V_L(t) = L \frac{dI_L}{dt} \]

\[ I_L(0) = 0 \]

\[ V_L(t) = V_s \]

\[ \frac{dI_L}{dt} = \frac{V_s}{L} \]

\[ I_L(6) = \frac{5}{3} \times 6 = 10 \text{ A} \]
(b) Now, we add some resistance in series with the inductor, as in Figure 2.

\[ I_L(+) = \frac{V_S}{R} \left(1 - e^{-\frac{R}{L}t}\right) \]

\[ V_L(+) = e^{-\frac{R}{L}t} \]

\[ I_L(0) = 0 \text{ A} \]

\[ V_S = V_R + V_L = I_L \cdot R + L \frac{dI_L}{dt} \]

\[ V_R = I_L \cdot R \]

\[ V_L = L \frac{dI_L}{dt} \]

\[ \Rightarrow \frac{dI_L}{dt} = \frac{V_S - \frac{R}{L} I_L(\cdot)}{L} \]
(c) (Practice): Suppose $R = 500 \, \Omega$, $L = 1 \, \text{mH}$, $V_S = 5 \, \text{V}$. Plot the current through and voltage across the inductor $(I_L(t), V_L(t))$, as these quantities evolve over time.
"Current through inductor becomes constant"

Voltage across it:
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