Discussion 12B

1. Computing the SVD: A "Tall" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}. \tag{1}$$

Here, we expect $U \in \mathbb{R}^{3\times 3}$, $\Sigma \in \mathbb{R}^{3\times 2}$, and $V \in \mathbb{R}^{2\times 2}$ (recall that U and V must be square since they are orthonormal matrices).

In this problem, we will walk through the SVD algorithm, prove some important theorems about the SVD matrices and column/null spaces, and consider an alternate way to approach the SVD.

(a) Let's start by trying to write A as an outer product in the form of $\sigma\vec{u}\vec{v}^\top$ where both $\vec{u}$ and $\vec{v}^\top$ have unit norm. *(HINT: Are the columns of A linearly independent or dependent? What does that tell us about how we can represent them?)*

(b) In this part, we will walk through Algorithm 7 in Note 16. This algorithm applies for a general matrix $A \in \mathbb{R}^{m\times n}$.

i. **Find $r := \text{rank}(A)$. Compute $A^\top A$ and diagonalize it using the spectral theorem (i.e. find V and Λ).**

ii. **Unpack $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ and unpack $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix}$.**

iii. **Find $\Sigma_r := \Lambda_r^{1/2}$ and then find $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$.**

iv. **Find $U_r := AV_r\Sigma_r^{-1}$, where $U_r \in \mathbb{R}^{3\times 1}$ and then extend the basis defined by columns of U_r to find $U \in \mathbb{R}^{3\times 3}$.**
(HINT: How can we extend a basis, and why is that needed here?)

v. **Use the previous parts to write the full SVD of A.**

vi. **If we were to calculate the SVD of our matrix using a calculator, are we *guaranteed* to always get the same SVD? Why or why not?**

(c) We now want to create the SVD of $A^\top$. **What are the relationships between the matrices composing A and the matrices composing $A^\top$?**

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