1. Computing the SVD: A “Tall” Matrix Example

Define the matrix

\[ A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}. \]  

Here, we expect \( U \in \mathbb{R}^{3 \times 3} \), \( \Sigma \in \mathbb{R}^{3 \times 2} \), and \( V \in \mathbb{R}^{2 \times 2} \) (recall that \( U \) and \( V \) must be square since they are orthonormal matrices).

In this problem, we will walk through the SVD algorithm, prove some important theorems about the SVD matrices and column/null spaces, and consider an alternate way to approach the SVD.

(a) Let’s start by trying to write \( A \) as an outer product in the form of \( \sigma \vec{u} \vec{v}^T \) where both \( \vec{u} \) and \( \vec{v}^T \) have unit norm. \((HINT: Are the columns of \( A \) linearly independent or dependent? What does that tell us about how we can represent them?)\)

(b) In this part, we will walk through Algorithm 7 in Note 16. This algorithm applies for a general matrix \( A \in \mathbb{R}^{m \times n} \).

i. **Find** \( r := \text{rank}(A) \). **Compute** \( A^T A \) and diagonalize it using the spectral theorem (i.e. find \( V \) and \( \Lambda \)).

ii. **Unpack** \( V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix} \) and unpack \( \Lambda := \begin{bmatrix} \Lambda_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix} \).

iii. **Find** \( \Sigma_r := \Lambda_r^{1/2} \) and then find \( \Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} \).

iv. **Find** \( U_r := AV_r \Sigma_r^{-1} \), where \( U_r \in \mathbb{R}^{3 \times 1} \) and then extend the basis defined by columns of \( U_r \) to find \( U \in \mathbb{R}^{3 \times 3} \). \((HINT: How can we extend a basis, and why is that needed here?)\)
v. Use the previous parts to write the full SVD of $A$.

vi. If we were to calculate the SVD of our matrix using a calculator, are we guaranteed to always get the same SVD? Why or why not?

(c) We now want to create the SVD of $A^T$. What are the relationships between the matrices composing $A$ and the matrices composing $A^T$?
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