1. Minimum Energy Control & Spectral Theorem

In controllability/reachability analysis, we try to solve the linear system:

\[ C_{i^*} \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i^* - 1] \end{bmatrix} = \vec{x}^* - A^{i^*} \vec{x}_0 \]  

(1)

for the vector quantities \( \vec{u}[0], \ldots, \vec{u}[i^* - 1] \). Cleaning up notation, let us fix \( i^* \), let \( C := C_{i^*} \), let \( \vec{z} := \vec{x}^* - A^{i^*} \vec{x}_0 \), and let \( \vec{u} := \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i^* - 1] \end{bmatrix} \). Then this linear system becomes

\[ C\vec{u} = \vec{z} \]  

(2)

In the real world, we would like to use this framework to control mechanical systems, often expending the minimum energy possible.

But what do we mean by the "energy" of the control? In this context, we will use the squared norm of the input vector \( \|\vec{u}\|^2 = \vec{u}_1^2 + \cdots + \vec{u}_n^2 \) as the model for the cost to applying a set of controls to our system.

Why do we use this definition? This formula for the cost model is closely connected to many physical scenarios relating to energy. Consider a few examples below:

- \( E_{\text{capacitor}} = \frac{1}{2} CV^2 \)
- \( E_{\text{spring}} = \frac{1}{2} kx^2 \)
- \( E_{\text{kinetic}} = \frac{1}{2} mv^2 \)

And so we find that the definition we use is a natural one.

**Optional EECS16A Refresher:** Recall the following vector spaces:

The range (or column space) of a matrix \( A \) refers to the following vector space \( \text{Col}(A) = \{ A\vec{x} : \vec{x} \in \mathbb{R}^n \} \). It is the vector space consisting of all possible linear combinations of the columns of \( A \).

Then, there is the null space of \( A \), which refers to the following vector space \( \text{Null}(A) = \{ \vec{x} : A\vec{x} = 0 \} \).
(a) Suppose we would like to solve for the minimum energy (or minimum norm) solution of the linear system $C\vec{u} = z$. This problem can be expressed as the following optimization problem:

$$\arg\min_{\vec{u}} \|\vec{u}\|^2 = \arg\min_{u[i]} \sum_{i=0}^{\ell-1} u[i]^2$$ \hspace{1cm} (3)

subject to $C\vec{u} = \vec{z}$ \hspace{1cm} (4)

Symmetric matrices can make calculations and reasoning about the properties of a matrix much easier. Suppose $C$ is a real, symmetric matrix. **Rewrite $C$ in terms of its spectral decomposition** (take $Q$ to be the orthonormal basis of eigenvectors of $C$ and $\Lambda$ to be the diagonal matrix of the eigenvalues).

**Solution:** We recall from Spectral Theorem that a symmetric matrix can be written as:

$$C = QAQ^T$$ \hspace{1cm} (5)

(b) One way to look at minimum energy control is through lens of our vector spaces. How might we do this?

Well, Spectral Theorem tells us that $Q$ is an orthonormal basis of $\mathbb{R}^n$. If $\text{Rank}(C) = r$, then $Q$ can be written as the block matrix $[Q_r \quad Q_{n-r}]$ where $Q_r$ forms an orthonormal basis for $\text{Col}(C)$ and $Q_{n-r}$ similarly forms one for $\text{Null}(C)$.

Let’s perform an orthonormal basis change:

$$\vec{u} = Q\vec{u}$$ \hspace{1cm} (6)

Using our new basis, rewrite $\vec{u}$ **in terms of $Q_r$ and $Q_{n-r}$**.

(HINT: Consider breaking up $Q$ and $\vec{u}$ into a block matrix and partitioned vector respectively.) **Solution:**

From the information given in the problem, we can write the orthonormal basis change as a block matrix multiplied to a partitioned vector as follows:

$$\vec{u} = Q\vec{u}$$ \hspace{1cm} (7)

$$\begin{bmatrix} Q_r & Q_{n-r} \end{bmatrix} \begin{bmatrix} \vec{u}_{\text{Col}(C)} \\ \vec{u}_{\text{Null}(C)} \end{bmatrix}$$ \hspace{1cm} (8)

$$= Q_r\vec{u}_{\text{Col}(C)} + Q_{n-r}\vec{u}_{\text{Null}(C)}$$ \hspace{1cm} (9)

Where $\vec{u}_{\text{Col}(C)}$ are the entries in $\vec{u}$ that correspond to the $Q_r$ columns and $\vec{u}_{\text{Null}(C)}$ are those that correspond with $Q_{n-r}$.

(c) Ultimately, the objective we are trying to minimize is still $\|\vec{u}\|^2$. Use your findings from part (b) to show that $\|\vec{u}\|^2 = \|\vec{u}_{\text{Col}(C)}\|^2 + \|\vec{u}_{\text{Null}(C)}\|^2$.

(HINT: Given some arbitrary orthonormal matrix $U$ and arbitrary vector $\vec{p}$, how are $\|\vec{p}\|$ and $\|U\vec{p}\|$ related?)

**Solution:** We start by using what we found in part (b) to find $\|\vec{u}\|^2$:
But recall that \( Q \)

Let’s clarify a bit as to why ignores or zeros out columns of the controllability matrix intuitively? Essentially, the minimum energy solution doesn’t affect the squared norm of the input by \( \|Q \| \) optimization objective) as follows:

\[
\begin{align*}
\|\bar{\vec{u}}\|^2 &= \|Q \bar{\vec{u}}_{\text{Col}(C)} + Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)}\|^2 \\
&= (Q \bar{\vec{u}}_{\text{Col}(C)} + Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)}) \quad (10)
\end{align*}
\]

\[
\begin{align*}
&= (Q \bar{\vec{u}}_{\text{Col}(C)} + Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)})^T (Q \bar{\vec{u}}_{\text{Col}(C)} + Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)}) \\
&= (Q \bar{\vec{u}}_{\text{Col}(C)})^T (Q \bar{\vec{u}}_{\text{Col}(C)}) + 2 (Q \bar{\vec{u}}_{\text{Col}(C)})^T (Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)}) + (Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)})^T (Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)}) \\
&= \bar{\vec{u}}_{\text{Col}(C)}^T Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)} + \bar{\vec{u}}_{\text{Null}(C)}^T Q_{n-r} \bar{\vec{u}}_{\text{Null}(C)} \\
&= \|\bar{\vec{u}}_{\text{Col}(C)}\|^2 + \|\bar{\vec{u}}_{\text{Null}(C)}\|^2 \\
&= (14)
\end{align*}
\]

Where we use orthonormality to conclude that \( Q_{n-r} = I_{r \times (n-r)} \), \( Q_{n-r} = I_{(n-r) \times (n-r)} \), and \( Q_{n-r} \).

Thus, \( \|\bar{\vec{u}}\|^2 = \|\bar{\vec{u}}_{\text{Col}(C)}\|^2 + \|\bar{\vec{u}}_{\text{Null}(C)}\|^2 \).

(d) Putting everything from the last two parts together, let us now solve for the solution to the minimum energy problem (written once more for convenience):

\[
\begin{align*}
\arg\min_{\bar{\vec{u}}} \|\bar{\vec{u}}\|^2 &= \arg\min_{\bar{\vec{u}}} \sum_{i=0}^{\ell-1} u[i]^2 \\
\text{s.t. } C\bar{\vec{u}} &= \bar{\vec{z}}
\end{align*}
\]

\( (17) \)

Solve for the optimal minimum energy input \( \bar{\vec{u}}^* \) in its simplest form in terms of \( \bar{\vec{u}}_{\text{Col}(C)} \) and/or \( \bar{\vec{u}}_{\text{Null}(C)} \). Explain what your result means intuitively.

(HINT: Which of \( \bar{\vec{u}}_{\text{Col}(C)} \) or \( \bar{\vec{u}}_{\text{Null}(C)} \) doesn’t affect \( C\bar{\vec{u}} \) (try to think about your vector space definitions)? What should we do to that value if we want to minimize the squared norm of the input?)

Solution: Remember that we can rewrite the squared norm of the input (the energy, which is our optimization objective) as follows:

\[
\begin{align*}
\|\bar{\vec{u}}\|^2 &= \|\bar{\vec{u}}_{\text{Col}(C)}\|^2 + \|\bar{\vec{u}}_{\text{Null}(C)}\|^2 \\
&= \|\bar{\vec{u}}_{\text{Col}(C)}\|^2 + \|\bar{\vec{u}}_{\text{Null}(C)}\|^2 \\
&= (19)
\end{align*}
\]

Based on how we defined \( \bar{\vec{u}}_{\text{Null}(C)} \), we know it has no effect on \( C\bar{\vec{u}} \). Thus, we should minimize its effect on the squared norm of the input by setting it to zero. Thus, our solution is as follows:

\[
\begin{align*}
\min \|\bar{\vec{u}}\|^2 &= \min (\|\bar{\vec{u}}_{\text{Col}(C)}\|^2 + \|\bar{\vec{u}}_{\text{Null}(C)}\|^2) = \|\bar{\vec{u}}_{\text{Col}(C)}\|^2 \\
&= (20)
\end{align*}
\]

Where the optimal min. energy solution \( \bar{\vec{u}}^* = \bar{\vec{u}}_{\text{Col}(C)} \) such that \( \bar{\vec{u}}_{\text{Null}(C)} = \bar{\vec{0}} \). What does this mean intuitively? Essentially, the min. energy solution ignores or zeros out columns of the controllability matrix that don’t help us get closer to our desired state.

Let’s clarify a bit as to why \( \bar{\vec{u}}_{\text{Null}(C)} \) doesn’t affect \( C\bar{\vec{u}} \). We can rewrite \( C\bar{\vec{u}} \) as follows:

\[
C\bar{\vec{u}} = CQ\bar{\vec{u}}_{\text{Col}(C)} + CQ_{n-r} \bar{\vec{u}}_{\text{Null}(C)}
\]

But recall that \( Q_{n-r} \) lives in \( \text{Null}(C) \)! Thus, \( CQ_{n-r} \bar{\vec{u}}_{\text{Null}(C)} = \bar{\vec{0}} \) and doesn’t affect \( C\bar{\vec{u}} \).
(e) Now, let’s do a numerical example. Consider the following linear discrete time system

\[
\mathbf{x}[i + 1] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}[i] + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u[i]
\] (22)

The controllability matrix for this system is:

\[
C = \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\] (23)

Notice that \( \text{Col}(C) = \text{span}\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \} \) and \( \text{Null}(C) = \text{span}\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \} \).

The minimum norm solution is \( \mathbf{u}_{\text{min}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \). We will compare this to another arbitrary solution \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \).

i Write both \( \mathbf{u}_{\text{min}} \) and \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \) as a linear combination of the column and null space span vectors. Compare the coefficients of the null space span vector. Solution: This can be done through simple inspection.

For \( \mathbf{u}_{\text{min}} \):

\[
\mathbf{u}_{\text{min}} = 2 \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\] (24)

For \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \):

\[
\begin{bmatrix} 3 \\ -1 \end{bmatrix} = 2 \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\] (25)

We notice that the coefficient of the null space vector for the minimum energy/norm solution is zero, aligning with what we found earlier in the problem.

ii Compare the norms of the two solutions. Verify that \( \|\mathbf{u}_{\text{min}}\| \) is smaller. Solution: We compare \( \mathbf{u}_{\text{min}} \) to \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \):

\[
\|\mathbf{u}_{\text{min}}\| = \sqrt{2^2 + (-2)^2} = \sqrt{8}
\] (26)

\[
\|\begin{bmatrix} 3 \\ -1 \end{bmatrix}\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}
\] (29)
Thus, $\|\vec{r}_{\text{min}}\| < \left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|$