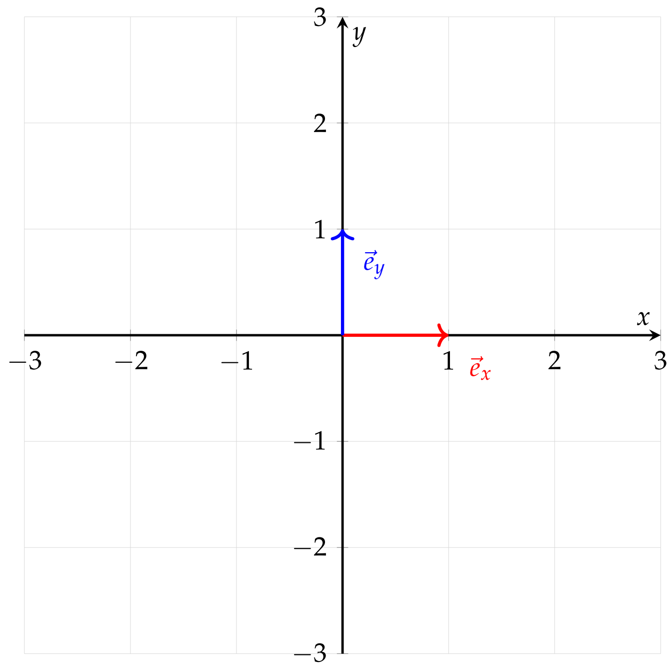
Discussion 12A

1. Geometric Interpretation of the SVD

(a) When we plot the transformation given by a specific matrix, we think about how the matrix transforms the standard basis vectors. In 2D, let $\vec{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The vectors $\vec{e}_x$ and $\vec{e}_y$ are shown below

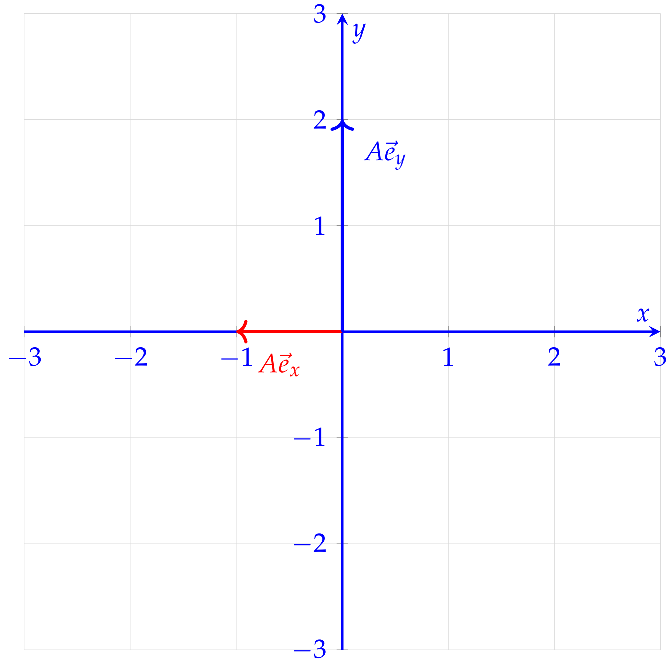


Consider the following matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad (1)$$

How would A transform $\vec{e}_x$ and $\vec{e}_y$? Plot the result.

Solution: We have that $A\vec{e}_x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $A\vec{e}_y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Plotting these, we have



(b) Let's take a look at a special 2×2 matrix.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

Show that this matrix is orthonormal. This matrix is called a rotation matrix and will rotate any vector counterclockwise by θ degrees.

Solution: We can also show orthonormality by showing that the columns have unit norm and that they are orthogonal. We can also show that this matrix is orthonormal by showing that $RR^\top = I_{2\times 2}$ and $R^\top R = I_{2\times 2}$.

$$\|\vec{r}_1\| = \left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \tag{3}$$

$$\|\vec{r}_2\| = \left\| \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1 \tag{4}$$

$$\langle \vec{r}_1, \vec{r}_2 \rangle = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0 \tag{5}$$

(c) Now let's consider how this transformation looks in the lens of the SVD. You are given the following matrix A:

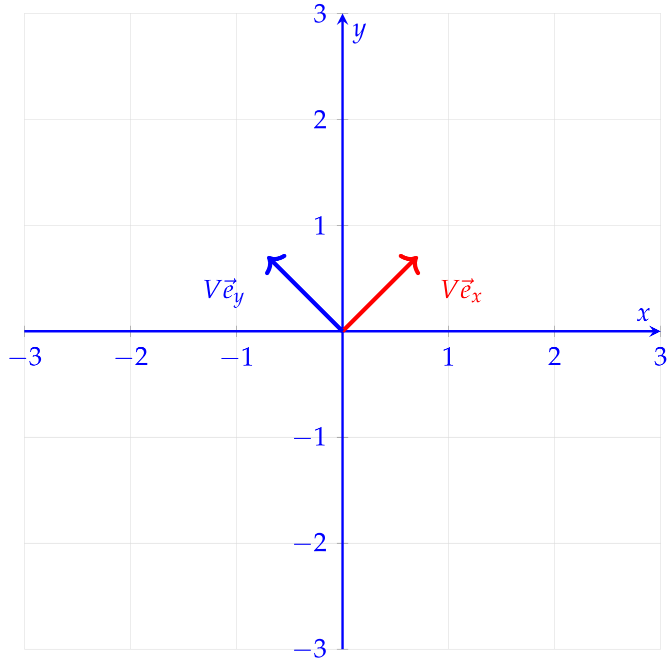
$$A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} \tag{6}$$

Recall that the SVD of this matrix is given by $A = U\Sigma V^\top$. Assume you are told that

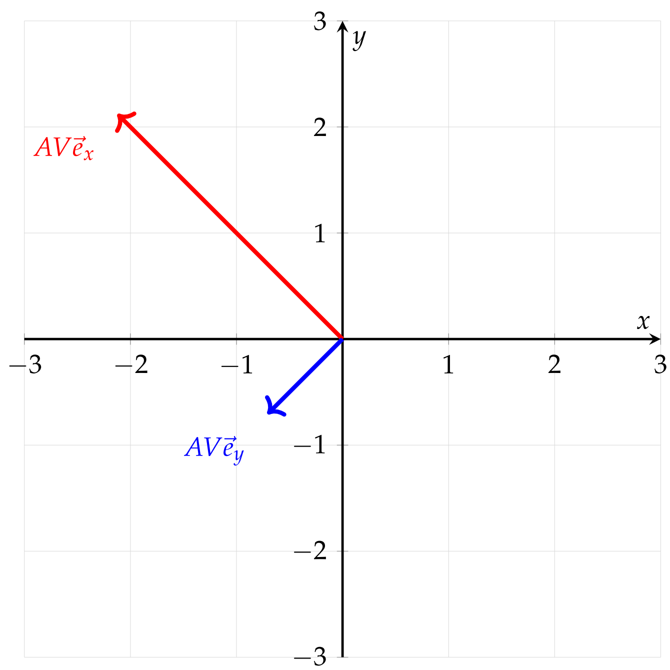
$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{7}$$

We will try to deduce U and Σ graphically, and the confirm our results numerically. **Plot the transformation given by V by showing how it affects $\vec{e}_x$ and $\vec{e}_y$ via left multiplication.** *(HINT: Try writing V as a rotation matrix with a specific θ.)*

Solution: We notice that V is a rotation matrix with $\theta = 45°$. Hence, it will rotate $\vec{e}_x$ and $\vec{e}_y$ by $45°$ counterclockwise.



(d) Suppose you are told that the transformation of AV on $\vec{e}_x$ and $\vec{e}_y$ looks like



Write this transformation AV in terms of U and Σ. Recall that the U matrix is an orthonormal matrix so it will correspond to any rotations or reflections, and the Σ matrix is a diagonal matrix and will perform any scaling operations. **Based on this fact and the plot of the transformation above, write down a guess for what U and Σ might be.**

Solution: We notice that $AV = U\Sigma$ by right multiplying our SVD by V. Now, it is reasonable to assume that, since $AV\vec{e}_x$ appears 3 times as long as $AV\vec{e}_y$, then $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. Furthermore, it appears as if the vectors have been rotated by 135° so it is likely that U is a rotation matrix with $\theta = 135^\circ$, i.e., $U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

(e) **Based on the given V matrix, compute the SVD.** Does your answer match your hypothesis from the previous part?

Solution: We can compute Σ and U as follows:

$$A\vec{v}_1 = \sigma_1\vec{u}_1 \tag{8}$$

$$A\vec{v}_2 = \sigma_2\vec{u}_2 \tag{9}$$

More explicitly,

$$A\vec{v}_1 = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix} \tag{10}$$

$$A\vec{v}_2 = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{11}$$

We can set $\sigma_1 = \|A\vec{v}_1\| = 3$ and $\sigma_2 = \|A\vec{v}_2\| = 1$. These choices yield $\vec{u}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. Hence,

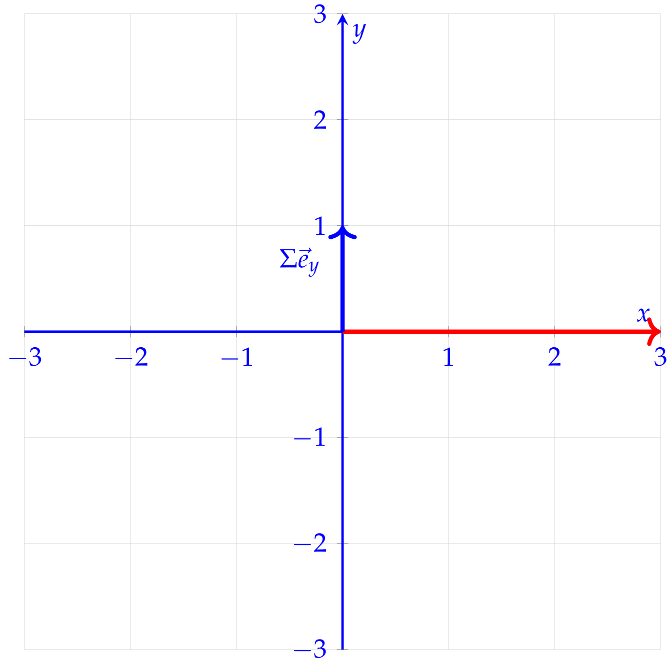
$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

$$U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{13}$$

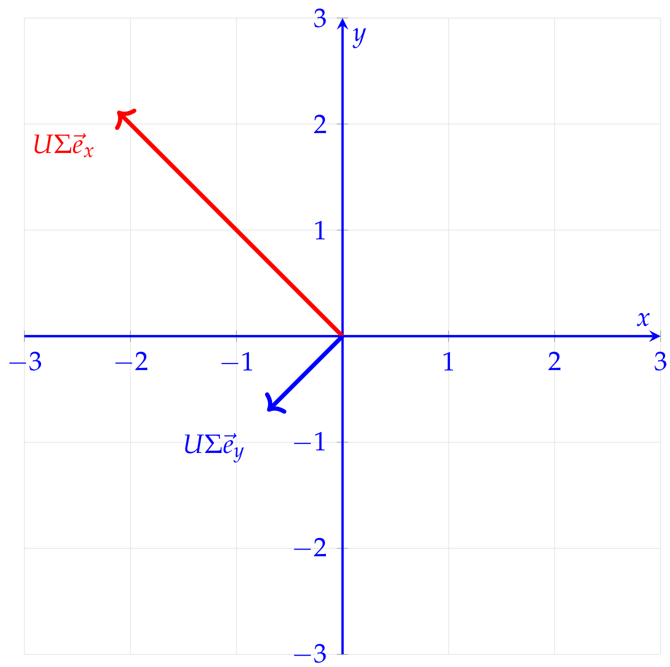
We notice that U is a rotation matrix with $\theta = 135^\circ$, and indeed, this matches with the U and Σ we hypothesized in the previous part.

(f) **Using your answer for U and Σ from the previous part, plot the transformation of Σ on $\vec{e}_x$ and $\vec{e}_y$. From here, plot the transformation of $U\Sigma$ on $\vec{e}_x$ and $\vec{e}_y$.** Does the final plot resemble the transformation shown by AV?

Solution: We notice that $\Sigma\vec{e}_x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\Sigma\vec{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, we get the following plot:



Now, we noticed that U is a rotation matrix with $\theta = 135°$ so this will rotate the graph above by 135°. This yields



which exactly matches what was given above.

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